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# ADDITIVE ENERGY AND THE FALCONER DISTANCE PROBLEM IN FINITE FIELDS

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**Abstract**

We study the number of the vectors determined by two sets in $d$-dimensional vector spaces over finite fields. We observe that the lower bound of cardinality for the set of vectors can be given in view of an additive energy or the decay of the Fourier transform on given sets. As an application of our observation, we find sufficient conditions on sets where the Falconer distance conjecture for finite fields holds in two dimensions. Moreover, we give an alternative proof of the theorem, due to Iosevich and Rudnev, that any Salem set satisfies the Falconer distance conjecture for finite fields.

## 1. Introduction

Let $\mathbb{F}_q^d, d \geq 1$, be a $d$-dimensional vector space over the finite field $\mathbb{F}_q$ with $q$ elements. Given $A, B \subset \mathbb{F}_q^d$, one may ask what is the cardinality of the set $A - B$, where the difference set $A - B$ is defined by

$$A - B = \{x - y \in \mathbb{F}_q^d : x \in A, y \in B\}.$$

It is clear that $|A - B| \geq \max\{|A|, |B|\}$. Here, and throughout the paper, we denote by $|E|$ the cardinality of the set $E$. However, taking $A = B = \mathbb{F}_q^s, 1 \leq s \leq d$, shows that the trivial estimate for $|A - B|$ is sharp in general, because $|A - B| = |\mathbb{F}_q^s| = q^s$. Moreover, if $s = d - 1$, then the size of $A - B$ is much smaller than that of $\mathbb{F}_q^d$, although $|A||B| = q^{2d-2}$ is somewhat big. Therefore, it may be interesting to find some conditions on the sets $A, B \subset \mathbb{F}_q^d$ such that the cardinality of $A - B$ is much bigger than the trivial lower bound, $\max\{|A|, |B|\}$, of $|A - B|$, or the difference set

$A-B$ contains a positive proportion of all vectors in $\mathbb{F}_q^d$, that is $|A-B| \gtrsim |\mathbb{F}_q^d| = q^d$. Here, we recall that for $l, m > 0$, the expression $l \gtrsim m$ or $m \lesssim l$ means that there exists a constant $C > 0$ independent of $q$, the size of the underlying finite field $\mathbb{F}_q$, such that $Cl \geq m$. The problem to consider the size of difference sets is strongly motivated by the Falconer distance problem for finite fields, which was introduced by Iosevich and Rudnev [9]. In this paper, we shall make an effort to find the connection between the size of the difference set $A-B$ and the cardinality of the distance set determined by $A, B \subset \mathbb{F}_q^d$. As one of the main results, we shall give some examples for sets satisfying the Falconer distance conjecture for finite fields.

First, let us review the Falconer distance problem for the Euclidean case and the finite field case. In the Euclidean setting, the Falconer distance problem is to determine the Hausdorff dimensions of compact sets $E, F \subset \mathbb{R}^d, d \geq 2$, such that the Lebesgue measure of the distance set

$$\Delta(E, F) := \{|x-y| : x \in E, y \in F\}$$

is positive. In the case when $E = F$, Falconer [4] first addressed this problem and showed that if the Hausdorff dimension of the compact set $E$ is greater than $(d+1)/2$, then the Lebesgue measure of $\Delta(E, E)$ is positive. He also conjectured that every compact set with the Hausdorff dimension $> d/2$ yields a distance set with a positive Lebesgue measure. This is known as the Falconer distance conjecture which has not been solved in all dimensions. The best known result for this problem is due to Wolff [17] in two dimensions and Erdoğan [3] in all other dimensions. They proved that if the Hausdorff dimension of any compact set $E \subset \mathbb{R}^d$ is greater than $d/2 + 1/3$, then the Lebesgue measure of $\Delta(E, E)$ is positive. These results are a culmination of efforts going back to Falconer [4] in 1985 and Mattila [13] a few years later. The Falconer distance problem on generalized distances was also studied in [1], [6], [7], [8], and [10]. In addition, the chapter 13 in [14] is highly relevant to the topic we study in this paper.

In the Finite field setting, one can also study the Falconer distance problem. Given $A, B \subset \mathbb{F}_q^d, d \geq 2$, the distance set $\Delta(A, B)$ is given by

$$\Delta(A, B) = \{\|x-y\| \in \mathbb{F}_q : x \in A, y \in B\},$$

where $\|\alpha\| = \alpha_1^2 + \cdots + \alpha_d^2$ for $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{F}_q^d$. It is clear that $|\Delta(A, B)| \leq q$, because the distance set is a subset of the finite field with $q$ elements. In this setting, the Falconer distance problem is to determine the minimum value of $|A||B|$ such that $|\Delta(A, B)| \gtrsim q$. In the case when $A = B$, this problem was introduced by Iosevich and Rudnev [9] and they proved that if $A = B$ and $|A| \gtrsim q^{(d+1)/2}$, then $|\Delta(A, B)| \gtrsim q$. It turned out in [5] that if the dimension $d$ is odd, then the theorem due to Alex and Rudnev gives the best possible result on the Falconer distance problem for finite fields. However, if the dimension $d$ is even, then it has been believed that the aforementioned authors' result may be improved to the following conjecture.

**Conjecture 1.1 (Iosevich and Rudnev [9]).** *Let $K \subset \mathbb{F}_q^d$ with $d \geq 2$ even. If $|K| \geq Cq^{\frac{d}{2}}$, with $C > 0$ sufficiently large, then*

$$|\Delta(K, K)| \gtrsim q.$$

This conjecture has not been solved in all dimensions. The exponent $(d+1)/2$ obtained by Iosevich and Rudnev is currently the best known result for all dimensions except dimension two. In two dimensions, this exponent was improved by $4/3$ (see [2] or [11]). We may consider the following general version of Conjecture 1.1:

**Conjecture 1.2.** *Let $A, B \subset \mathbb{F}_q^d$ with $d \geq 2$ even. If $|A||B| \geq Cq^d$, with $C > 0$ large enough, then*

$$|\Delta(A, B)| \gtrsim q.$$

Theorem 2.1 in [15] due to Shparlinski implies that if $A, B \subset \mathbb{F}_q^d, d \geq 2$, and $|A||B| \gtrsim q^{d+1}$, then $|\Delta(A, B)| \gtrsim q$. This was improved by authors [11] who showed that if $|A||B| \gtrsim q^{8/3}$ for $A, B \subset \mathbb{F}_q^2$, then $|\Delta(A, B)| \gtrsim q$. For a variant of the Falconer distance problem for finite fields, see [16] and [12].

### 1.1. Purpose of This Paper

The goal of this paper is to find some sets $A, B \subset \mathbb{F}_q^d, d \geq 2$, for which Conjecture 1.2 holds. In general, it may not be easy to construct such examples, supporting the claim that Conjecture 1.2 holds. A well-known example is due to Iosevich and Rudnev [9] who showed that if $K \subset \mathbb{F}_q^d, d \geq 2$, is a Salem set and $|K| \gtrsim q^{d/2}$, then $|\Delta(K, K)| \gtrsim q$. Here, we recall that we say that $E \subset \mathbb{F}_q^d$ is a Salem set if for every $m \in \mathbb{F}_q^d \setminus \{(0, \ldots, 0)\}$,

$$|\widehat{E}(m)| := \left| q^{-d} \sum_{x \in E} \chi(-x \cdot m) \right| \lesssim \frac{\sqrt{E}}{q^d},$$

where we denote by $\chi$ a nontrivial additive character of $\mathbb{F}_q$. They obtained this example by showing that the formula of $|\Delta(K, K)|$ is closely related to the decay of the Fourier transform on the set $K$. In this paper, we take a new approach to find such examples. First, we shall show that if $A, B \subset \mathbb{F}_q^d, d \geq 2$ and $|A - B| \gtrsim q^d$, then $|\Delta(A, B)| \gtrsim q$. Second, we find certain conditions on the set $A, B \subset \mathbb{F}_q^d$ such that $|A - B| \sim q^d$. Thus, estimating the size of the difference set $A - B$ makes an important role. For example, using our approach we can recover the example by Iosevich and Rudnev. Moreover, we can find a stronger result that if one of $A, B \subset \mathbb{F}_q^d$ is a Salem set and $|A||B| \gtrsim q^d$, then $A - B$ contains a positive proportion of all elements in $\mathbb{F}_q^d$. In particular, our method yields that if one of $A, B \subset \mathbb{F}_q^2$ intersects with $\sim q$ points in an algebraic curve which does not contain any line, and $|A||B| \gtrsim q^2$, then the sets $A, B$ satisfies Conjecture 1.2 in two dimensions.

## 2. Cardinality of Difference Sets

In this section we introduce the formulas for the lower bound of difference sets. Such formulas are closely related to the additive energy

$$\Lambda(A,B) = |\{(x,y,z,w) \in A \times A \times B \times B : x - y + z - w = 0\}|.$$

To see this, for $c \in \mathbb{F}_q^d$, define

$$\begin{aligned} \nu(c) &= |\{(x,w) \in A \times B : x - w = c\}| \\ &= \sum_{\substack{x \in A, w \in B \\ : x - w = c}} 1. \end{aligned}$$

Applying the Cauchy-Schwarz inequality shows that if $A, B \subset \mathbb{F}_q^d, d \geq 2$, then

$$|A|^2|B|^2 = \left( \sum_{c \in A-B} \nu(c) \right)^2 \leq |A - B| \sum_{c \in \mathbb{F}_q^d} \nu^2(c).$$

Now, we observe that

$$\sum_{c \in \mathbb{F}_q^d} \nu^2(c) = \sum_{c \in \mathbb{F}_q^d} \left( \sum_{\substack{x \in A, w \in B \\ : x - w = c}} 1 \right) \left( \sum_{\substack{y \in A, z \in B \\ : y - z = c}} 1 \right) = \sum_{\substack{x, y \in A, w, z \in B \\ : x - w = y - z}} 1 = \Lambda(A,B).$$

It follows that

$$|A - B| \geq \frac{|A|^2|B|^2}{\Lambda(A,B)}. \tag{2.1}$$

Since $\Lambda(A,B) \leq \min\{|A|^2|B|, |A||B|^2\}$, it is clear that

$$|A - B| \geq \max\{|A|, |B|\},$$

which is in fact a trivial lower bound of $|A-B|$. However, if $A$ and $B$ are subspaces with $A = B$, then the trivial lower bound can not be improved. In this case, the difference set $A-B$ has much smaller cardinality than $|A||B|$. It therefore is natural to guess that if $A$ and $B$ do not contain big subspaces, then $|A-B|$ should be large. In this paper, we shall deal with this issue.

Recall that the Fourier transform on the set $E \subset \mathbb{F}_q^d$ is defined by

$$\widehat{E}(m) = \frac{1}{q^d} \sum_{x \in E} \chi(-x \cdot m) \quad \text{for} \quad m \in \mathbb{F}_q^d,$$

where $\chi$ denotes a nontrivial additive character of $\mathbb{F}_q$ and we write $E$ for the characteristic function on the set $E$. Also recall that the orthogonality relation of the nontrivial additive character of $\mathbb{F}_q$ says that

$$\sum_{t \in \mathbb{F}_q} \chi(a \cdot t) = 0 \quad \text{for} \quad a \neq 0.$$

More generally we have

$$\sum_{x \in \mathbb{F}_q^d} \chi(m \cdot x) = 0 \quad \text{for} \quad m \neq (0, \cdots, 0).$$

The lower bound of $|A - B|$ can be also written in terms of the Fourier transforms on $A$ and $B$. To see this, using the definition of the Fourier transform and the orthogonality relation of the nontrivial additive character of $\mathbb{F}_q$, observe that

$$\begin{aligned}
\Lambda(A, B) &= |\{(x, y, z, w) \in A \times A \times B \times B : x - y + z - w = 0\}| \\
&= \sum_{x,y,z,w \in \mathbb{F}_q^d} A(x)A(y)B(z)B(w)\delta_0(x - y + z - w) \\
&= \sum_{x,y,z,w \in \mathbb{F}_q^d} A(x)A(y)B(z)B(w) \left( q^{-d} \sum_{m \in \mathbb{F}_q^d} \chi(m \cdot (x - y + z - w)) \right) \\
&= q^{3d} \sum_{m \in \mathbb{F}_q^d} |\widehat{A}(m)|^2 |\widehat{B}(m)|^2,
\end{aligned}$$

where $\delta_0(\alpha) = 0$ for $\alpha \neq (0, \ldots, 0)$ and $\delta_0(\alpha) = 1$ for $\alpha = (0, \ldots, 0)$. Therefore, the formula (2.1) can be replaced by

$$|A - B| \geq \frac{|A|^2 |B|^2}{q^{3d} \sum_{m \in \mathbb{F}_q^d} |\widehat{A}(m)|^2 |\widehat{B}(m)|^2}. \tag{2.2}$$

This formula indicates that if the Fourier decay on $A$ or $B$ is good, then several kinds of vectors are contained in the difference set $A - B$. For example, if $A$ or $B$ is a Salem set such as the paraboloid or the sphere, then $|A - B|$ can be big and so a lot of distances can be determined by $A, B$.

## 3. Sets in $\mathbb{F}_q^2$ Satisfying the Falconer Distance Conjecture

In view of the sizes of difference sets, we shall find some sets $A, B \subset \mathbb{F}_q^2$ where the Falconer distance conjecture (Conjecture 1.2) holds. A core idea is due to the following fact.

**Lemma 3.1.** *Let $E \subset \mathbb{F}_q^2$. If $|E| \geq cq^2$ for some $0 < c \leq 1$, then we have*

$$|\{\|x\| \in \mathbb{F}_q : x \in E\}| \geq \frac{cq}{2},$$

where $\|x\| = x_1^2 + x_2^2$ for $x = (x_1, x_2) \in \mathbb{F}_q^2$. *Proof.* For each $a \in \mathbb{F}_q$, consider a vertical line $L_a = \{(a,t) \in \mathbb{F}_q^2 : t \in \mathbb{F}_q\}$. Since $|E| \geq cq^2$, it is clear from the pigeonhole principle that there exists a line $L_b$ for some $b \in \mathbb{F}_q$ with $|E \cap L_b| \geq cq$. Thus, Lemma follows from the following observation that for the fixed $b \in \mathbb{F}_q$,

$$|\{b^2 + t^2 \in \mathbb{F}_q : (b,t) \in E \cap L_b\}| \geq \frac{cq}{2}.$$

□

If $|A - B| \gtrsim |A||B| \gtrsim q^2$, then Lemma 3.1 implies that $A, B \subset \mathbb{F}_q^2$ are the sets to satisfy the Falconer conjecture. Thus, the main task is to find sets $A, B$ such that $|A - B|$ is extremely large. The following lemma tells us some properties of sets $A, B$ where the size of $A - B$ can be large.

**Lemma 3.2.** *Let $B \subset \mathbb{F}_q^2$. Suppose that there exists a set $W \subset \mathbb{F}_q^2$ with $|W| \sim 1$ such that*

$$|B \cap (B + c)| \lesssim 1 \quad \text{for all} \quad c \in \mathbb{F}_q^2 \setminus W. \tag{3.1}$$

*Then for any $A \subset \mathbb{F}_q^2$, we have*

$$|A - B| \gtrsim \min(|A||B|, |B|^2).$$

*Proof.* From (2.1), it suffices to show that

$$\Lambda(A,B) = |\{(x,y,z,w) \in A \times A \times B \times B : x - y + z - w = 0\}| \lesssim |A||B| + |A|^2.$$

It follows that

$$\Lambda(A,B) = \sum_{x,y \in A} \left( \sum_{w,z \in B : z - w = y - x} 1 \right) = \sum_{x,y \in A} |B \cap (B + y - x)|$$

$$= \sum_{x,y \in A : y - x \notin W} |B \cap (B + y - x)| + \sum_{x,y \in A : y - x \in W} |B \cap (B + y - x)|$$

$$= \mathrm{I} + \mathrm{II}.$$

From the assumption (3.1), it is clear that $|\mathrm{I}| \lesssim |A|^2$. On the other hand, the value II can be estimated as follows.

$$\mathrm{II} = \sum_{\beta \in W} \sum_{x,y \in A : y - x = \beta} |B \cap (B + \beta)| \leq \sum_{\beta \in W} \sum_{x,y \in A : y - x = \beta} |B|.$$

Whenever we fix $x \in A$ and $\beta \in W$, there is at most one $y \in A$ such that $y - x = \beta$. We therefore see

$$\mathrm{II} \le |W||A||B| \sim |A||B|.$$

Thus, we complete the proof. $\square$

### 3.1. Examples of the Falconer Conjecture Sets in Two Dimensions

First recall that Bezout's theorem says that two algebraic curves of degrees $d_1$ and $d_2$ can not meet in more than $d_1 \cdot d_2$ points unless they have a component in common. As a direct application of Bezout's theorem, it can be shown that subsets of certain algebraic curves in two dimensions satisfy the condition in (3.1). This observation yields the following theorem.

**Theorem 3.3.** *Let $P(x) \in \mathbb{F}_q[x_1, x_2]$ be a polynomial which does not have any liner factor. Define an algebraic variety $V = \{x \in \mathbb{F}_q^2 : P(x) = 0\}$. If $B \subset V$, then for any $A \subset \mathbb{F}_q^2$, we have*

$$|A - B| \gtrsim \min(|A||B|, |B|^2).$$

*Proof.* First recall that we always assume that the degree of the polynomial is $\sim 1$. Thus, if $B \subset V$, then the pigeonhole principle implies that we can choose a subvariety $V'$ of $V$ and a set $B' \subset V'$ with $|B'| \sim |B|$. Therefore, we may assume that $V$ is a variety generated by an irreducible polynomial with degree $k \geq 2$. Applying Bezout's theorem shows that for any $c \in \mathbb{F}_q^2 \setminus \{(0,0)\}$,

$$|V \cap (V + c)| \le k^2 \lesssim 1.$$

Therefore, the proof is complete from Lemma 3.2. $\square$

The following corollary follows immediately from Lemma 3.2 and Lemma 3.1.

**Corollary 3.4.** *Let $B \subset \mathbb{F}_q^2$ with $|B| \gtrsim q$. Suppose that $W \subset \mathbb{F}_q^2$ with $|W| \sim 1$, and $|B \cap (B + c)| \lesssim 1$ for any $c \in \mathbb{F}_q^2 \setminus W$. Then for any $A \subset \mathbb{F}_q^2$ with $|A| \gtrsim q$, we have*

$$|\Delta(A, B)| = |\{||x - y|| \in \mathbb{F}_q : x \in A, y \in B\}| \gtrsim q.$$

Notice that such sets $A, B$ as in this corollary satisfy the Falconer distance conjecture. Moreover, the difference set $A - B$ contains a positive proportion of all elements in $\mathbb{F}_q^2$. As a consequence of Theorem 3.3 and Corollary 3.4, more concrete examples for the Falconer distance conjecture sets can be found.

**Example 3.5.** Let $A = \{(x_1, x_2) : x_1^2 + x_2^2 = a\}$ for some $a \neq 0$ as considered in [9]. Then it can be checked that $|A| \sim q$. From Theorem 3.3, this implies the difference set $A - A$ has cardinality $\sim q^2$ which in turns sharpens the result in [9]. In general, we may choose any polynomial $P \in \mathbb{F}_q[x_1, x_2]$ which does not contain any linear factor, and define a variety $V = \{x \in \mathbb{F}_q : P(x) = 0\}$. If $|V| \gtrsim q$, then choose a subset $B \subset V$ with $|B| \sim q$. Finally, choose any subset $A$ of $\mathbb{F}_q^2$, whose cardinality is $\sim q$. Then the difference set $A - B$ contains the positive proportion of all elements in $\mathbb{F}_q^2$ and so $|\Delta(A,B)| \sim q$. Since $|A||B| \sim q^2$, the sets $A, B$ satisfy the Falconer distance conjecture.

Observe that if both $A$ and $B$ contain many points in some lines $L_1, L_2$, respectively, then we can not proceed such steps as in the above example. For this reason, if sets $A, B$ possess the structures like product sets, then it seems that two sets $A, B$ determine the distance set $\Delta(A,B)$ with a small cardinality.

## 4. Salem Sets and Difference Sets

If the decay of the Fourier transform on $A, B \subset \mathbb{F}_q^d$ is known, then the formula (2.2) can be very useful to measure the lower bound of $|A - B|$. Here, we shall show that if one of $A$ and $B$ is a Salem set, then $|A - B|$ is so big that $A, B$ satisfy the Falconer distance conjecture. We need the following lemma which shows the relation between the Fourier decay of sets and the size of difference sets.

**Lemma 4.1.** *Let $A, B \subset \mathbb{F}_q^d$. Suppose that for every $m \in \mathbb{F}_q^d \setminus \{(0,\ldots,0)\}$,*

$$|\widehat{B}(m)| \lesssim q^{\beta} \quad \text{for some} \quad \beta \in \mathbb{R}. \tag{4.1}$$

*Then we have*

$$|A - B| \gtrsim \min\left(q^d, \frac{|A||B|^2}{q^{2d+2\beta}}\right).$$

*Proof.* The proof is based on the formula (2.2) and discrete Fourier analysis. It follows that

$$q^{3d} \sum_{m \in \mathbb{F}_q^d} |\widehat{A}(m)|^2 |\widehat{B}(m)|^2$$

$$\leq q^{3d} |\widehat{A}(0,\ldots,0)|^2 |\widehat{B}(0,\ldots,0)|^2 + q^{3d}\left(\max_{m \in \mathbb{F}_q^d \setminus (0,\ldots,0)} |\widehat{B}(m)|^2\right) \sum_{m \in \mathbb{F}_q^d} |\widehat{A}(m)|^2$$

$$= \mathrm{I} + \mathrm{II}.$$

By the definition of the Fourier transform, it is clear that $\mathrm{I} = q^{-d}|A|^2|B|^2$. On the other hand, using the assumption (4.1) and the Plancherel theorem, we obtain that $\mathrm{II} \lesssim q^{2d+2\beta}|A|$. Thus, we have

$$q^{3d} \sum_{m \in \mathbb{F}_q^d} |\widehat{A}(m)|^2 |\widehat{B}(m)|^2 \lesssim q^{-d}|A|^2|B|^2 + q^{2d+2\beta}|A|.$$

Thus, Lemma 2.2 can be used to obtain that

$$|A-B| \gtrsim \frac{|A|^2|B|^2}{q^{-d}|A|^2|B|^2 + q^{2d+2\beta}|A|} \gtrsim \min\left(q^d, \frac{|A||B|^2}{q^{2d+2\beta}}\right),$$

which completes the proof. □

As mentioned in introduction, it is known that if $B \subset \mathbb{F}_q^d$ with $|B| \gtrsim q^{d/2}$ is a Salem set, then $|\Delta(B,B)| \gtrsim q$. Namely, the Salem set $B$ satisfies the Falconer distance conjecture. In this case, we can state a strong fact that $B-B$ contains a positive proportion of all elements in $\mathbb{F}_q^d$. More precisely, we have the following theorem.

**Theorem 4.2.** *If $B \subset \mathbb{F}_q^d$ is a Salem set, then for any $A \subset \mathbb{F}_q^d$ with $|A||B| \gtrsim q^d$, we have*

$$|A-B| \gtrsim q^d.$$

*Proof.* Since $B \subset \mathbb{F}_q^d$ is a Salem set, taking $q^\beta = q^{-d}\sqrt{|B|}$ from Lemma 4.1 shows that

$$|A-B| \gtrsim \min\{q^d, |A||B|\}. \tag{4.2}$$

Since $|A||B| \gtrsim q^d$, the proof is complete. □

The following corollary follows immediately from above theorem and Lemma 3.1.

**Corollary 4.3.** *Let $A \subset \mathbb{F}_q^d$ is a Salem set. Then for any $B \subset \mathbb{F}_q^d$ with $|A||B| \gtrsim q^d$, we have*

$$|\Delta(A,B)| \gtrsim q.$$

*In other words, the sets $A, B$ satisfy the Falconer distance conjecture.*

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