ON THE HIGHER-DIMENSIONAL GENERALIZATION OF A PROBLEM OF ROTH

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Abstract

Long ago Roth conjectured that for any k-coloring of the positive integers the equation $x + x' = n, x \neq x'$ has a monochromatic solution in (x, x') for more than cM integers n up to M (where c is an absolute constant independent of k). Later Erdős, Sárközy and T. Sós proved this conjecture with $\frac{1}{2} - \varepsilon$ in place of c. In this paper we will prove a higher-dimensional generalization of this theorem by using a higher-dimensional extension of the well known Hilbert cube-lemma. We will also give bounds for the number of monochromatic solutions in higher dimension.

1. Introduction

K. F. Roth conjectured (see [2] and [6]) that for an arbitrary k-coloring of the positive integers there are more than cM integers $n \leq M$ such that the equation $x + x' = n, x \neq x'$, has a monochromatic solution in (x, x'). In [1] Erdős, Sárközy and T. Sós proved this conjecture in the following form:

Theorem 1. *For every $k \geq 2$ there exists a positive integer $M_0(k)$ such that for any $M \geq M_0(k)$ and an arbitrary k-coloring of the set $\mathbb{N}$, the number of positive integers $n \leq M$ for which there is a monochromatic solution of the equation $x + x' = n$, $x \neq x'$, is greater than $\frac{M}{2} - 3M^{1-2^{-k-1}}$*

The proof of this theorem was based on the density version of Hilbert's cube lemma (for the original coloring version see [7]). Szemerédi proved that if we consider a sequence of positive integers of positive density, then the sequence must contain a so-called Hilbert d-cube or affine d-cube, i.e., for every d a set of the form $u + \sum_{i=1}^{d} \varepsilon_i v_i$, where ε_i=0 or 1 for every i. In [7] Hilbert used the coloring version of this lemma in studying irreducibility of polynomials with integer coefficients. Later, Szemerédi gave the density version of this lemma (see [3] and [10]). This density version of Hilbert's cube lemma is generally called Szemerédi's cube lemma. Erdős, Sárközy and T. Sós used the following quantitative version of this lemma:

Lemma 1. *(Szemerédi's cube lemma): If H is a subset of (1,M) for M large enough and H has at least* $3M^{1-2^{-d}}$ *elements, then H contains a Hilbert-cube.*

Our first goal is to give a higher-dimensional generalization of Lemma 1. With this generalization we will prove the following result:

Theorem 2. *For fixed positive integers* r, s, k, *there is a positive integer* m_0 *with the following property: for any positive integer* $m > m_0$ *and for any* k*-coloring of the elements of the set* $(1,m)^r$ *there are at least* $\left[\frac{m}{s}\right]^r - 3\cdot\left(2^{r-1}m^r\right)^{1-2^{-ks+k-1}}$ *vectors* $\vec{x}$ *in* $(1,m)^r$*, such that one can find pairwise distinct vectors* $\vec{x_1}, \vec{x_2}, \cdots, \vec{x_s}$ *of the same color in* $(1,m)^r$*, whose sum is* $\vec{x}$.

The special case $r = 1$ and $s = 2$ in Theorem 2 gives the result of Theorem 1. After the proof of Theorem 2 we will study the number of solutions of the equation $\vec{x} + \vec{x'} = \vec{n}, \vec{x} \neq \vec{x'}$ for vectors $\vec{n}$ in $(1,m)^r$, where $\vec{x}$ and $\vec{x'}$ are monochromatic. We will get the following result:

Theorem 3. *For every positive real number* α *and* β *with the property* $\alpha^r + \beta^r \leq \frac{1}{2^{2r+1}k}$ *there is a positive integer* $m_{\alpha\beta}$*, such that for every* $m > m_{\alpha\beta}$ *and for every* k*-coloring of* $\mathbb{N}^r$ *the number of elements in* $(1,m)^r$ *having representations as a sum of two monochromatic distinct vectors in more than* $\frac{\beta^r}{2}m^r$ *ways is more than* $\alpha^r m^r$.

One can observe that the result of Theorem 2 is asymptotically independent of the number of the colors, if we fix r and s. In Theorem 3 we want to search for an arbitrary k-coloring a "large number" of vectors with a "large number of representations". Here we will see that these "large numbers" already depend on the number of the colors for fixed r and s. First we study only the case of two summands, later we show a way of studying the case of more summands.

2. The Generalization of Hilbert's Cube Lemma

Similarly to the original definition one can interprete d-cubes in the set of r-dimensional vectors.

Definition 1. If $\vec{u}, \vec{v_1}, \vec{v_2}, \cdots, \vec{v_d}$ are r-dimensional vectors, then the set of the sums in the form $\vec{u} + \sum_{i=1}^{d} \varepsilon_i \vec{v_i}$, where ε_i=0 or 1 for every i, is an affine d-cube or a d-dimensional Hilbert cube.

Lemma 2. *If* H *is a subset of distinct vectors of* $(1,m)^r$ *and the set* H *has at least* $3\cdot\left(2^{r-1}m^r\right)^{1-2^{-d}}$ *elements, then* H *contains an affine* d*-cube.*

Proof. Our proof is a generalization of the proof given in [1]. We will define sets $H_0, H_1, \cdots, H_d$ and r-dimensional vectors $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_d}$ in the following way:

(i) $H_0 = H$

(ii) $H_j \cup \left\{\vec{b} + \vec{v_j} | \vec{b} \in H_j\right\}$ is a subset of H_{j-1} for every $j = 1, 2, \cdots, d$

(iii)$|H_j| \geq |H|^{2^j} \left(3 \cdot 2^{r-1} m^r\right)^{-\left(2^j - 1\right)}$

We construct $H_0, H_1, \cdots, H_d$ and $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_d}$ recursively. Let $H_0 = H$. Assume that $0 \leq j \leq d-1$ and in the case $j > 0$ sets $H_0, H_1, \cdots, H_j$ and vectors $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_j}$ have been defined. Let F be the set of the vectors in $(-(m-1), (m-1))^r$, whose first nonzero coordinate is positive. Denote by $f(H_j, \vec{h})$ the number of solutions of the equation $\vec{b} - \vec{b'} = \vec{h}$, where $\vec{b}, \vec{b'} \in H_j$ and $\vec{h} \in F$.

Let L be the maximum value of the numbers $f(H_j, \vec{h})$, where $\vec{h} \in F$ and $\vec{h} \neq \vec{v_1}, \vec{v_2}, \cdots, \vec{v_j}$. For all r-dimensional vectors $\vec{h}$ we have $f(H_j, \vec{h}) \leq |H_j|$. Clearly $\sum_{\vec{h} \in F} f(H_j, \vec{h}) = \binom{|H_j|}{2}$. We give an estimate for L. We can majorize $f(H_j, \vec{h})$ by $|H_j|$, if $\vec{h} \in \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_j}\}$ and by L otherwise. Thus we get the estimate

$$\binom{|H_j|}{2} \leq j\,|H_j| + \frac{(2m-1)^r - 1}{2} L < j\,|H_j| + 2^{r-1} m^r L.$$

Thus we have $L \geq \frac{1}{2^r m^r}\left(|H_j|^2 - |H_j| - 2j\,|H_j|\right) = \frac{|H_j|}{3 \cdot 2^{r-1} \cdot m^r}\left(\frac{3}{2}\,|H_j| - \frac{3}{2} - 3j\right)$. According to our assumption we have (for m large enough) the estimate

$$|H_j| \geq |H|^{2^j} \left(3 \cdot 2^{r-1} m^r\right)^{-\left(2^j - 1\right)} > \left(3 \cdot \left(2^{r-1} m^r\right)^{1 - 2^{-d}}\right)^{2^j} \left(3 \cdot 2^{r-1} m^r\right)^{-\left(2^j - 1\right)}$$

$$= 3 \cdot \left(2^{r-1} m^r\right)^{1 - 2^{j-d}} > 3 \cdot \left(2^{r-1} m^r\right)^{1 - 2^{-1}} > 3 + 6d > 3 + 6j.$$

So we have $\frac{|H_j|^2}{3 \cdot 2^{r-1} m^r} < L$. By (iii) we get $L > |H|^{2^{j+1}} \left(3 \cdot 2^{r-1} m^r\right)^{-\left(2^{j+1} - 1\right)}$. This means that the vector $\vec{h}$ can play the role of $\vec{v_{j+1}}$ and we are able to define set H_{j+1}, too. Thus indeed we can define sets $H_0, H_1, \cdots, H_d$ and r-dimensional vectors $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_d}$ recursively. □

3. The Proof of the Generalization of Roth's Problem

Here we give a proof of Theorem 2.

Proof. Assume to the contrary that there is an appropriate k-coloring for infinitely many positive integers m, such that the number of vectors $\vec{x}$ with the given property in $(1, m)^r$ is less than $\left[\frac{m}{s}\right]^r - 3 \cdot \left(2^{r-1} m^r\right)^{1 - 2^{-ks+k-1}}$. We will get a contradiction via Lemma 2. Let S be the subset of $(1, m)^r$, in which all the r coordinates of the elements are divisible by s. Let S' denote the set of those elements of S, which do not have a representation in the given form. According to our assumption we have $3 \cdot \left(2^{r-1}m^r\right)^{1-2^{-ks+k-1}} < |S'|$. Now we can apply Lemma 2. In S' one can find an affine $(ks-k+1)$-cube, so that there are r-dimensional vectors $\vec{u}, \vec{v_1}, \vec{v_2}, \cdots, \vec{v_{ks-k+1}}$ in $(-(m-1),(m-1)))^r$, such that all the sums $\vec{u} + \sum_{i=1}^{ks-k+1} \varepsilon_i \vec{v_i}$ are in S',where ε_i=0 or 1 for every i. By the pigeonhole principle there are s vectors in the set $\left\{\frac{1}{s}\vec{u} + \vec{v_i}, i = 1, 2, \cdots, ks-k+1\right\}$ with the same color. We can assume without loss of generality, that these vectors are $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_s}$. By the definition of the Hilbert-cube the vector $\vec{u} + \sum_{i=1}^{s} \vec{v_i} = \sum_{i=1}^{s} \left(\frac{1}{s}\vec{u} + \vec{v_i}\right)$ is an element of S', which contradicts the definition of S'. □

4. On the Number of Representations as the Sum of Two Monochromatic Distinct Vectors

In this section we study the number of representations, if the number of summands is $s = 2$. Our goal is to prove Theorem 3. The key will be the following lemma:

Lemma 3. *If for positive real numbers α and β there exist infinitely many positive integers m, such that there is a k-coloring of $\mathbb{N}^r$, for which at most $\alpha^r m^r$ elements of $(1, m)^r$ have representations as a sum of two monochromatic distinct vectors of $\mathbb{N}^r$ in more than $\frac{\beta^r}{2} m^r$ ways, then $\alpha^r + \beta^r > \frac{1}{2^{2r+1}k}$.*

Proof. Let be m a positive integer with the given property. Let a be the minimal and b the maximal positive integer such that $\frac{a}{m} \geq \alpha$ and $\frac{b}{m} \leq \beta$. In this case at most a^r elements of $(1, m)^r$ have representations as a sum of two monochromatic distinct vectors in more than $\frac{b^r}{2}$ ways. Let $f(\vec{x})$ denote the number of representations of $\vec{x} = (x_1, x_2, \cdots, x_r,)$ as the sum of two distinct vectors in $(1, x_1) \times (1, x_2) \times \cdots \times (1, x_r)$. Clearly $f(\vec{x}) = \left[\frac{\prod_{i=1}^{r}(x_i - 1)}{2}\right]$, because one can order the vectors of this set (except at most one vector) into disjoint pairs such that each pair consists of two distinct vectors with sum $\vec{x}$. For all vectors $\vec{x}$ of the set $(1, m)^r$ let $g(\vec{x})$ denote the number of representations of $\vec{x}$ as the sum of two distinct monochromatic vectors of $(1, m)^r$. Let m_i be the number of vectors with the i-th color in $\left(1, \left[\frac{m}{2}\right]\right)^r$. Clearly $\sum_{i=1}^{k} m_i = \left[\frac{m}{2}\right]^r$ and $\sum_{i=1}^{k} \binom{m_i}{2} \leq \sum_{\vec{x} \in (1,m)^r} g(\vec{x})$. We give an upper estimate for the sum $\sum_{\vec{x} \in (1,m)^r} g(\vec{x})$, too. If one can write the vectors $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_t} (t \leq a^r)$, in more than $\frac{b^r}{2}$ ways as the sum of two monochromatic distinct vectors, then

$$\sum_{i=1}^{t} g(\vec{v_i}) \leq \sum_{\vec{x} \in (m-a+1, m)^r} f(\vec{x}).$$

Hence we have the following upper estimate:

$$\begin{aligned}
\sum_{\vec{x}\in(1,m)^r} g(\vec{x}) &\le \frac{b^r}{2}(m)^r + \sum_{\vec{x}\in(m-a+1,m)^r} f(\vec{x}) \\
&\le \frac{b^r}{2}m^r + \frac{1}{2}\sum_{x_r=m-a+1}^{m}\cdots\sum_{x_1=m-a+1}^{m}(x_1-1)(x_2-1)\cdots(x_r-1) \\
&= \frac{b^r}{2}m^r + \frac{1}{2}\sum_{x_r=m-a}^{m-1}\cdots\sum_{x_1=m-a}^{m-1}x_1x_2\cdots x_r = \frac{b^r}{2}m^r + \frac{1}{2}\left(am-\frac{a^2+a}{2}\right)^r \\
&< \frac{b^r}{2}m^r + (am)^r.
\end{aligned}$$

Using the Cauchy-Schwarz-inequality we get

$$\sum_{i=1}^{k}\binom{m_i}{2} = \frac{1}{2}\sum_{i=1}^{k}m_i^2 - \frac{1}{2}\left[\frac{m}{2}\right]^r \ge \frac{1}{2}k\left(\frac{\left[\frac{m}{2}\right]^r}{k}\right)^2 - \frac{1}{2}\left[\frac{m}{2}\right]^r.$$

By dividing by m^{2r} we get

$$\frac{\frac{1}{2}k\left(\frac{\left[\frac{m}{2}\right]^r}{k}\right)^2 - \frac{1}{2}\left[\frac{m}{2}\right]^r}{m^{2r}} < \frac{\frac{1}{2}b^r m^r + \frac{1}{2}(am)^r}{m^{2r}}.$$

If m tends to infinity, then the left-hand side is asymptotically $\frac{1}{2^{2r+1}k}$ and the right-hand side is asymptotically $\frac{1}{2}\beta^r + \frac{1}{2}\alpha^r$. Clearly we have $\frac{1}{2}\beta^r + \frac{1}{2}\alpha^r < \alpha^r + \beta^r$, and hence $\alpha^r + \beta^r > \frac{1}{2^{2r+1}k}$. □

With Lemma 3 we can easily prove Theorem 3. The proof can be done in the following way:

Proof. Assume, for a contradiction, that there are positive numbers α and β such that $\alpha^r+\beta^r \le \frac{1}{2^{2r+1}k}$ and for infinitely many positive integers m there is a k-coloring of $\mathbb{N}^r$ such that the number of elements in $(1,m)^r$ having representations as a sum of two monochromatic distinct vectors in more than $\frac{\beta^r}{2}m^r$ ways is at most $\alpha^r m^r$. This contradicts Lemma 3. □

Remark. Let k' be the maximal odd integer such that $k \ge k'$. We color the elements of $\mathbb{N}^r$ with at most k colors in the following way: if the sum of the coordinates is 0 or 1 modulo k', then we color this vector by the first color; the other vectors are colored by the other $k'-1$ colors according to the other residue classes. In this case there are at most $3\cdot\left[\frac{m}{k'}\right]m^{r-1}$ vectors in $(1,m)^r$, for which the number of representations as a sum of two distinct monochromatic vectors is at least $2\cdot\left[\frac{m}{k'}\right]m^{r-1}$ (to see this, one should only observe the vectors having the sum of coordinates equal to 0, 1 or 2 modulo k'). Thus Theorem 2 cannot be improved significantly.

5. Further Remarks

We studied in the last section only the case $s = 2$ (i.e., the number of addends is two). We can do calculations in a similar way using the further asymptotic formulae (see [9]). The method we use will not be combinatorially new, one needs only the same technique, but with a little more work. Let us denote by $p(n, s)$ the number of partititons of n into exactly s not necessarily distinct parts and $q(n, s)$ the number of partititons of n into exactly s distinct parts. We formulate the relevant statements of [9] in Lemma 4.

Lemma 4. *$p(n, s) \sim \frac{n^{s-1}}{(1\cdot 2\cdot\dots\cdot(s-1))^2\cdot s}$ and $q(n, s) \sim \frac{n^{s-1}}{(1\cdot 2\cdot\dots\cdot(s-1))^2\cdot s}$ where $n >> 1$ and $s = O(1)$.*

We give only a sketch of how our methods generally work. Let $H(x_1, x_2, \cdots, x_r)$ denote the number of representations of the vector $\vec{x} = (x_1, x_2, \cdots, x_r)$ as a sum of s distinct vectors in $(1, x_1)\times(1, x_2)\times\cdots\times(1, x_r)$. Let m_i be the number of vectors with the i-th color in $\left(1, \left[\frac{m}{s}\right]\right)^r$. It is obvious that $\sum_{i=1}^{k} \binom{m_i}{s} \le \sum_{\vec{x}\in(1,m)^r} T(\vec{x})$, where $T(\vec{x})$ is the number of ways vector $\vec{x}$ can be written as a sum of s monochromatic distinct vectors. A similar argument to that of Section 4 shows that we have to majorize $H(x_1, x_2, \cdots, x_r)$.

It is easy to verify that

$$q(x_1, s)\prod_{i=2}^{r}(1\cdot 2\cdot\dots\cdot s\cdot q(x_i, s)) \le H(x_1, x_2, \cdots, x_r) \le p(x_1, s)\prod_{i=2}^{r}(1\cdot 2\cdot\dots\cdot s\cdot p(x_i, s)).$$

By Lemma 4 we get the conclusion that $H(x_1, x_2, \cdots, x_r)$ is asymptotically equal to

$$\frac{x_1^{s-1}}{(1\cdot 2\cdot\dots\cdot(s-1))^2\cdot s}\prod_{i=2}^{r}\left(1\cdot 2\cdots\cdot s\frac{x_i^{s-1}}{(1\cdot 2\cdot\dots\cdot(s-1))^2\cdot s}\right),$$

if s is fixed and $x_i \gg 1$. With further calculation one can achieve analogous results as formulated in Theorem 3.

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