CONVOLUTION SUMS INVOLVING LEGENDRE-JACOBI SYMBOL AND DIVISOR FUNCTIONS

Mohamed El Bachraoui
Dept. of Mathematical Sciences, United Arab Emirates University, Al-Ain, United Arab Emirates
melbachraoui@uaeu.ac.ae

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Abstract

We use elementary tools to evaluate a variety of convolution sums which involve the Legendre-Jacobi symbol and the divisor functions.

1. Introduction

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ and let $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. We will use the following definitions.

Definition 1. Let the function σ be defined on $\mathbb{Q}$ as follows: $\sigma(0) = 1$, if $n \in \mathbb{Q} \backslash \mathbb{N}_0$, then $\sigma(n) = 0$, and if $n \in \mathbb{N}$, then

$$\sigma(n) = \sum_{d|n} d.$$

Definition 2. Let $n, r \in \mathbb{N}_0$, let $m \in \mathbb{N}$, and let

$$\sigma_{r,m}(n) = \sum_{\substack{d|n \\ d \equiv r \bmod m}} d.$$

We will be using the basic fact that for all $m, n \in \mathbb{N}$

$$\sigma_{0,m}(n) = m\sigma(n/m). \tag{1}$$

Definition 3. Let the function σ^* be defined on $\mathbb{Q}$ as follows: if $n \in \mathbb{Q} \setminus \mathbb{N}$, then $\sigma^*(n) = 0$ and if $n \in \mathbb{N}$, then

$$\sigma^*(n) = \sum_{d|n,\ \frac{n}{d} \text{ odd}} d.$$

By [10, Theorem 3.4] we have for all $n \in \mathbb{N}$ that

$$\sigma^*(n) = \sigma(n) - \sigma(n/2). \tag{2}$$

For an integer n and an odd prime number p we shall use $\big(n\,\big|\,p\big)$ to denote the Legendre-Jacobi symbol of n with respect to p, see for instance Apostol [2] about properties of $\big(n\,\big|\,p\big)$. In this note we shall evaluate a variety of convolution sums involving the Legendre-Jacobi symbol and the divisor functions. Our main tool is the following theorem.

Theorem 1. [6] *Let $\alpha \in \mathbb{N}$. Let $A_1, A_2, \dots A_\alpha \subseteq \mathbb{N}$ and $\underline{A} = (A_1, A_2, \dots, A_\alpha)$ and let $f_i : A_i \to \mathbb{C}$ for $i = 1, 2, \dots, \alpha$ be arithmetic functions and let $\underline{f} = (f_1, f_2, \dots, f_\alpha)$. If both*

$$F_{\underline{A}}(q) = \prod_{i=1}^{\alpha} \prod_{n \in A_i} (1 - q^n)^{-\frac{f_i(n)}{n}} = \sum_{n=0}^{\infty} p_{\underline{A},\underline{f}}(n) q^n$$

and

$$\sum_{i=1}^{\alpha} \sum_{n \in A_i} \frac{f_i(n)}{n} q^n$$

converge absolutely and represent analytic functions in the unit disk $|q| < 1$, then

$$n p_{\underline{A},\underline{f}}(n) = \sum_{k=1}^{n} \left(p_{\underline{A},\underline{f}}(n-k) \sum_{i=1}^{\alpha} f_{i,A_i}(k) \right),$$

where $p_{\underline{A},\underline{f}}(0) = 1$ and

$$f_{i,A_i}(k) = \sum_{\substack{d|k \\ d \in A_i}} f_i(d).$$

We note that Theorem 1 for the special case $\alpha = 1$, has been given in Apostol [2] and in Robbins [9] to give formulas relating arithmetic functions to sums of divisors functions. The authors' key argument is that generating functions for the arithmetic functions they considered have the form of infinite products ranging over a single set of natural numbers. In our previous work [6] we used Theorem 1 to deal with arithmetic functions whose generating functions involve finitely many infinite products ranging over different sets of natural numbers and to derive a variety of inductive formulas for such functions. Among our results in [6] we were able by means of Theorem 1 to reproduce the following result of Liouville [7].

$$(-1)^n s(n) n = \frac{\sigma(n) - \sigma_{1,2}(n)}{2} + \sum_{k \ge 1} (-1)^{k+1} \Big(\sigma(n - k^2) + \sigma_{1,2}(n - k^2) \Big),$$

where

$$s(n) = \begin{cases} 1, & \text{if } n = m^2 \\ 0, & \text{otherwise.} \end{cases}$$

We note further that the infinite products along with their power series expansions which we will use in this paper are all found as theorems in the recent paper of Alaca, Alaca, and Williams [1]. Besides several new results on infinite products of Ramanujan type in [1], the authors reproduced a variety of other results including some due to Carlitz [5] and Ramanujan [8].

2. The Results

From now on we will suppose that n is a positive integer. While the corollaries in this section are given along with their proofs, proofs for the theorems are provided in the next section.

Theorem 2. *We have*

$$(a)\quad \sum_{j=1}^{n-1}\Big(\sum_{d|j} d^2\left(-3\,\middle|\,d\right)\Big)\left(\sigma(n-j)-\sigma\left(\frac{n-j}{3}\right)\right)=\frac{1}{9}\sigma(n)-\frac{1}{9}\sigma\left(\frac{n}{3}\right)-\frac{n}{9}\sum_{d|n} d^2\left(-3\,\middle|\,d\right).$$

$$(b)\quad \sum_{j=1}^{n-1}\Big(\sum_{d|j} d^2\left(-3\,\middle|\,\frac{j}{d}\right)\Big)\left(\sigma(n-j)-9\sigma\left(\frac{n-j}{3}\right)\right)=\frac{n-1}{3}\sum_{d|n} d^2\left(-3\,\middle|\,\frac{n}{d}\right).$$

Corollary 1. *If p is an odd prime, then*

$$(a)\quad \sum_{j=1}^{p-1}\Big(\sum_{d|j} d^2\left(-3\,\middle|\,d\right)\Big)\left(\sigma(p-j)-\sigma\left(\frac{p-j}{3}\right)\right)=\begin{cases}0, & \textit{if } p=3\\ \frac{1-p^3}{9}, & \textit{if } p\equiv 1 \pmod 3\\ \frac{1+p^3}{9}, & \textit{if } p\equiv -1 \pmod 3).\end{cases}$$

$$(b)\quad \sum_{j=1}^{p-1}\Big(\sum_{d|j} d^2\left(-3\,\middle|\,\frac{j}{d}\right)\Big)\left(\sigma(p-j)-9\sigma\left(\frac{p-j}{3}\right)\right)=\begin{cases}6, & \textit{if } p=3\\ \frac{(p-1)(p^2+1)}{3}, & \textit{if } p\equiv 1 \pmod 3\\ \frac{(p-1)(p^2-1)}{3}, & \textit{if } p\equiv -1 \pmod 3).\end{cases}$$

Proof. The case $p=3$ is easily checked in both parts. Suppose now that $p\neq 3$. Then as

$$\sigma(p)=p+1,\ \sigma(p/3)=0,\ \text{and}\ \left(-3\,\middle|\,p\right)=\begin{cases}1, & \text{if } p\equiv 1 \pmod 3\\ -1, & \text{if } p\equiv -1 \pmod 3),\end{cases}$$

Theorem 2(a) yields

$$9\sum_{j=1}^{p-1}\Big(\sum_{d|j} d^2\left(-3\,\middle|\,d\right)\Big)\left(\sigma(p-j)-\sigma\left(\frac{p-j}{3}\right)\right)=p+1-p\Big(\left(-3\,\middle|\,1\right)+p^2\left(-3\,\middle|\,p\right)\Big)$$
$$=\begin{cases}1-p^3, & \text{if } p\equiv 1 \pmod 3\\ 1+p^3, & \text{if } p\equiv -1 \pmod 3),\end{cases}$$

which after division by 9 gives part (a). Further by Theorem 2 (b) we have

$$\sum_{j=1}^{p-1}\Bigg(\sum_{d|j} d^2\left(-3\,\Big|\,\frac{j}{d}\right)\Bigg)\left(\sigma(p-j)-9\sigma\left(\frac{p-j}{3}\right)\right)=\frac{p-1}{3}\Big(\left(-3\,|\,p\right)+p^2\left(-3\,|\,1\right)\Big)$$

$$=\begin{cases}\frac{(p-1)(1+p^2)}{3}, & \text{if } p\equiv 1 \pmod 3\\ \frac{(p-1)(-1+p^2)}{3}, & \text{if } p\equiv -1 \pmod 3,\end{cases}$$

which proves part (b). □

Theorem 3. *The following identities are true:*

(a) $\displaystyle\sum_{j=1}^{n-1}\Bigg(\sum_{d|j} d^2\left(-4\,|\,d\right)\Bigg)\left(\sigma^*(n-j)+4\sigma^*\left(\frac{n-j}{2}\right)\right)=\frac{\sigma^*(n)}{4}+\sigma^*\left(\frac{n}{2}\right)-\frac{n}{4}\sum_{d|n} d^2\left(-4\,|\,d\right).$

(b) $\displaystyle\sum_{j=1}^{n-1}\Bigg(\sum_{d|j} d^2\left(-4\,\Big|\,\frac{j}{d}\right)\Bigg)\left(\sigma(n-j)-3\sigma\left(\frac{n-j}{2}\right)-4\sigma\left(\frac{n-j}{4}\right)\right)=\frac{n-1}{4}\sum_{d|n} d^2\left(-4\,\Big|\,\frac{n}{d}\right).$

(c) $\displaystyle\sum_{j=1}^{n-1}\Bigg(\sum_{d|j}\left(-4\,|\,d\right)\Bigg)\left(\sigma^*(n-j)-4\sigma^*\left(\frac{n-j}{2}\right)\right)=-\frac{1}{4}\sigma^*(n)+\sigma^*\left(\frac{n}{2}\right)+\frac{n}{4}\sum_{d|n}\left(-4\,|\,d\right).$

Corollary 2. *If p is an odd prime, then ;*

(a) $$\sum_{j=1}^{p-1}\Bigg(\sum_{d|j} d^2\left(-4\,|\,d\right)\Bigg)\left(\sigma^*(p-j)+4\sigma^*\left(\frac{p-j}{2}\right)\right)=\begin{cases}\frac{1-p^3}{4}, & \text{if } p\equiv 1 \pmod 4\\ \frac{1+p^3}{4}, & \text{if } p\equiv -1 \pmod 4.\end{cases}$$

(b) $$\sum_{j=1}^{p-1}\Bigg(\sum_{d|j} d^2\left(-4\,\Big|\,\frac{j}{d}\right)\Bigg)\left(\sigma(p-j)-3\sigma\left(\frac{p-j}{2}\right)-4\sigma\left(\frac{p-j}{4}\right)\right)$$

$$=\begin{cases}\frac{(p-1)(p^2+1)}{4}, & \text{if } p\equiv 1 \pmod 4\\ \frac{(p-1)(p^2-1)}{4}, & \text{if } p\equiv -1 \pmod 4.\end{cases}$$

(c) $$\sum_{j=1}^{p-1}\Bigg(\sum_{d|j}\left(-4\,|\,d\right)\Bigg)\left(\sigma^*(p-j)-4\sigma^*\left(\frac{p-j}{2}\right)\right)=\begin{cases}\frac{(p-1)}{4}, & \text{if } p\equiv 1 \pmod 4\\ \frac{-p-1}{4}, & \text{if } p\equiv -1 \pmod 4.\end{cases}$$

Proof. Suppose that p is an odd prime. Then Theorem 3 (a) implies

$$-4\sum_{j=1}^{p-1}\Bigg(\sum_{d|j} d^2\left(-4\,|\,d\right)\Bigg)\left(\sigma^*(p-j)+\sigma^*\left(\frac{p-j}{2}\right)\right)$$

$$=p\Big(\left(-4\,|\,1\right)+p^2\left(-4\,|\,p\right)\Big)-p-1$$

$$=\begin{cases}p^3-1, & \text{if } p\equiv 1 \pmod 4\\ -p^3-1, & \text{if } p\equiv -1 \pmod 4,\end{cases}$$

which upon division by -4 yields the required formula in part (a). Next by Theorem 3 (b) we get

$$\sum_{j=1}^{p-1}\Big(\sum_{d|j} d^2\left(-4\,\middle|\,\frac{j}{d}\right)\Big)\Big(\sigma(p-j)-3\sigma\left(\frac{p-j}{2}\right)-4\sigma\left(\frac{p-j}{4}\right)\Big)$$
$$=\frac{p-1}{4}\Big(\left(-4\,\middle|\,p\right)+p^2\left(-4\,\middle|\,1\right)\Big)$$
$$=\begin{cases}\frac{(p-1)(1+p^2)}{4}, \text{ if } p\equiv 1 \pmod 4\\ \frac{(p-1)(-1+p^2)}{4}, \text{ if } p\equiv -1 \pmod 4),\end{cases}$$

giving part (b). Further by Theorem 3 (c) we have

$$\sum_{j=1}^{p-1}\Big(\sum_{d|j}\left(-4\,\middle|\,d\right)\Big)\Big(\sigma^*(p-j)-4\sigma^*\Big(\frac{p-j}{2}\Big)\Big)=\frac{p\Big(1+\left(-1\,\middle|\,p\right)\Big)-p-1}{4}$$
$$=\begin{cases}\frac{p-1}{4}, \text{ if } p\equiv 1 \pmod 4\\ \frac{-p-1}{4}, \text{ if } p\equiv -1 \pmod 4),\end{cases}$$

which proves part (c). □

Theorem 4. *We have*

(a) $$\sum_{j=1}^{n-1}\Big(\sum_{d|j} d\left(5\,\middle|\,d\right)\Big)\Big(\sigma(n-j)-\sigma\left(\frac{n-j}{5}\right)\Big)=\frac{1}{5}\sigma(n)-\frac{1}{5}\sigma\left(\frac{n}{5}\right)-\frac{n}{5}\sum_{d|n} d\left(5\,\middle|\,d\right).$$

(b) $$\sum_{j=1}^{n-1}\Big(\sum_{d|j} d\left(5\,\middle|\,\frac{j}{d}\right)\Big)\Big(\sigma(n-j)-25\sigma\left(\frac{n-j}{5}\right)\Big)=(n-1)\sum_{d|n} d\left(5\,\middle|\,\frac{n}{d}\right).$$

Corollary 3. *If p is an odd prime, then*

(a) $$\sum_{j=1}^{p-1}\Big(\sum_{d|j} d\left(5\,\middle|\,d\right)\Big)\Big(\sigma(p-j)-\sigma\left(\frac{p-j}{5}\right)\Big)=\begin{cases}0, \text{ if } p=5\\ \frac{1-p^2}{5}, \text{ if } p\equiv\pm1 \pmod 5\\ \frac{1+p^2}{5}, \text{ if } p\equiv\pm2 \pmod 5).\end{cases}$$

(b) $$\sum_{j=1}^{p-1}\Big(\sum_{d|j} d\left(5\,\middle|\,\frac{j}{d}\right)\Big)\Big(\sigma(p-j)-25\sigma\left(\frac{p-j}{5}\right)\Big)$$
$$=\begin{cases}17, \text{ if } p=5\\ (p-1)(p+5), \text{ if } p\equiv\pm1 \pmod 5\\ (p-1)(p-5), \text{ if } p\equiv\pm2 \pmod 5).\end{cases}$$

Proof. The results are easily checked if $p = 5$. Now suppose that $p \neq 5$ is an odd prime. Then by Theorem 4(a) we find

$$\sum_{j=1}^{p-1}\left(\sum_{d|j} d\,(\,5\,|\,d\,)\right)\left(\sigma(p-j)-\sigma\left(\frac{p-j}{5}\right)\right) = \frac{-p\Big(\,(\,5\,|\,1\,)+p\,(\,5\,|\,p\,)\Big)+p+1}{5}$$
$$= \begin{cases} \frac{1-p^2}{5}, \text{ if } p \equiv \pm 1 \pmod 5 \\ \frac{1+p^2}{5}, \text{ if } p \equiv \pm 2 \pmod 5, \end{cases}$$

which proves part (a). Part (b) follows similarly by Theorem 4(b). □

Theorem 5. *The following formulas hold:*

(a) $$\sum_{j=1}^{n-1}\left(\sum_{d|j} d\,(\,8\,|\,d\,)\right)\left(\sigma^*(n-j)+2\sigma^*\left(\frac{n-j}{2}\right)+8\sigma^*\left(\frac{n-j}{4}\right)\right)$$
$$= \frac{1}{2}\sigma^*(n)+\sigma^*\left(\frac{n}{2}\right)+4\sigma^*\left(\frac{n}{4}\right)-\frac{n}{2}\sum_{d|n} d\,(\,8\,|\,d\,)\,.$$

(b) $$\sum_{j=1}^{n-1}\left(\sum_{d|j} d\,(\,8\,|\,\tfrac{j}{d}\,)\right)\left(\sigma(n-j)-3\sigma\left(\frac{n-j}{2}\right)-2\sigma\left(\frac{n-j}{4}\right)-8\sigma\left(\frac{n-j}{8}\right)\right)$$
$$= \frac{n-1}{2}\sum_{d|n} d\,(\,8\,|\,\tfrac{j}{d}\,)\,.$$

(c) $$\sum_{j=1}^{n-1}\left(\sum_{d|j}(\,-8\,|\,d\,)\right)\left(\sigma^*(n-j)-2\sigma^*\left(\frac{n-j}{2}\right)-8\sigma^*\left(\frac{n-j}{4}\right)\right)$$
$$= -\sigma^*(n)+\sigma^*\left(\frac{n}{2}\right)+8\sigma^*\left(\frac{n}{4}\right)+n\sum_{d|n}(\,-8\,|\,d\,)\,.$$

Corollary 4. *If p is an odd prime, then*

(a) $$\sum_{j=1}^{p-1}\left(\sum_{d|j} d\,(\,8\,|\,d\,)\right)\left(\sigma^*(p-j)+2\sigma^*\left(\frac{p-j}{2}\right)+8\sigma^*\left(\frac{p-j}{4}\right)\right)$$
$$= \begin{cases} \frac{1-p^2}{2}, \textit{ if } p \equiv \pm 1 \pmod 8 \\ \frac{1+p^2}{2}, \textit{ if } p \equiv \pm 3 \pmod 8. \end{cases}$$

(b) $$\sum_{j=1}^{p-1}\left(\sum_{d|j} d\,(\,8\,|\,\tfrac{j}{d}\,)\right)\left(\sigma(p-j)-3\sigma\left(\frac{p-j}{2}\right)-2\sigma\left(\frac{p-j}{4}\right)-8\sigma\left(\frac{p-j}{8}\right)\right)$$
$$= \begin{cases} \frac{p^2-1}{2}, \textit{ if } p \equiv \pm 1 \pmod 8 \\ \frac{(p-1)^2}{2}, \textit{ if } p \equiv \pm 3 \pmod 8. \end{cases}$$

$$(c)\quad \sum_{j=1}^{p-1}\Bigg(\sum_{d|j}\left(\,-8\,\middle|\,d\,\right)\Bigg)\left(\sigma^*(p-j)-2\sigma^*\left(\frac{p-j}{2}\right)-8\sigma^*\left(\frac{p-j}{4}\right)\right)$$

$$=\begin{cases}\frac{p-1}{2}, & \text{if } p\equiv 1,3 \pmod 8\\ \frac{-p-1}{2}, & \text{if } p\equiv -1,-3 \pmod 8).\end{cases}$$

Proof. By Theorem 5(a) we get

$$\sum_{j=1}^{p-1}\Bigg(\sum_{d|j}d\left(\,8\,\middle|\,d\,\right)\Bigg)\left(\sigma^*(p-j)+2\sigma^*\left(\frac{p-j}{2}\right)+8\sigma^*\left(\frac{p-j}{4}\right)\right)$$

$$=\frac{-1}{2}\Big(p\Big(\left(\,8\,\middle|\,1\,\right)+p\left(\,8\,\middle|\,p\,\right)\Big)-p-1\Big)=\begin{cases}\frac{1-p^2}{2}, & \text{if } p\equiv \pm1 \pmod 8\\ \frac{1+p^2}{2}, & \text{if } p\equiv \pm3 \pmod 8),\end{cases}$$

which proves part (a). Parts (b) and (c) follow similarly by Theorem 5, parts (b) and (c). □

Theorem 6. *The following identities are valid:*

$$(a)\quad \sum_{j=1}^{n-1}\Bigg(\sum_{d|j}d\left(\,12\,\middle|\,d\,\right)\Bigg)\Bigg(\sigma(n-j)+3\sigma\left(\frac{n-j}{3}\right)$$

$$+8\sigma\left(\frac{n-j}{4}\right)+12\sigma\left(\frac{n-j}{6}\right)-24\sigma\left(\frac{n-j}{12}\right)\Bigg)$$

$$=\sigma(n)+3\sigma\left(\frac{n}{3}\right)+8\sigma\left(\frac{n}{4}\right)+12\sigma\left(\frac{n}{6}\right)-24\sigma\left(\frac{n}{12}\right)-n\sum_{d|n}d\left(\,12\,\middle|\,d\,\right).$$

$$(b)\quad \sum_{j=1}^{n-1}\Bigg(\sum_{d|j}d\left(\,12\,\middle|\,\frac{j}{d}\,\right)\Bigg)\Bigg(\sigma(n-j)-2\sigma\left(\frac{n-j}{2}\right)$$

$$-3\sigma\left(\frac{n-j}{3}\right)-2\sigma\left(\frac{n-j}{4}\right)-6\sigma\left(\frac{n-j}{12}\right)\Bigg)$$

$$=\frac{n-1}{2}\sum_{d|n}d\left(\,12\,\middle|\,\frac{n}{d}\,\right).$$

Corollary 5. *If p is an odd prime, then*

$$(a)\quad \sum_{j=1}^{p-1}\Bigg(\sum_{d|j}d\left(\,12\,\middle|\,d\,\right)\Bigg)\Bigg(\sigma(p-j)+3\sigma\left(\frac{p-j}{3}\right)$$

$$+8\sigma\left(\frac{p-j}{4}\right)+12\sigma\left(\frac{p-j}{6}\right)-24\sigma\left(\frac{p-j}{12}\right)\Bigg)$$

$$=\begin{cases}4, & \text{if } p=3\\ 1-p^2, & \text{if } p\equiv \pm1 \pmod{12}\\ 1+p^2, & \text{if } p\equiv \pm5 \pmod{12}).\end{cases}$$

$$(b)\quad \sum_{j=1}^{p-1}\Big(\sum_{d|j} d\left(12\,\Big|\,\frac{j}{d}\right)\Big)\Big(\sigma(p-j)-2\sigma\left(\frac{p-j}{2}\right)$$

$$-3\sigma\left(\frac{p-j}{3}\right)-2\sigma\left(\frac{p-j}{4}\right)-6\sigma\left(\frac{p-j}{12}\right)\Big)$$

$$=\begin{cases}2, & \textit{if } p=3\\ \frac{p^2-1}{2}, & \textit{if } p\equiv \pm 1 \pmod{12}\\ \frac{(p-1)^2}{2}, & \textit{if } p\equiv \pm 5 \pmod{12}.\end{cases}$$

Proof. Part (a) follows by Theorem 6(a) and part (b) follows by Theorem 6(b) using the same sort arguments as before. □

Theorem 7. *We have*

$$\sum_{j=1}^{n-1}\Big(\sum_{d|j}\left(-15\,|\,d\right)\Big)\Big(\sigma(n-j)-6\sigma\left(\frac{n-j}{3}\right)-10\sigma\left(\frac{n-j}{5}\right)+15\sigma\left(\frac{n-j}{15}\right)\Big)$$

$$=-\sigma(n)+6\sigma\left(\frac{n}{3}\right)+10\sigma\left(\frac{n}{5}\right)-15\sigma\left(\frac{n}{15}\right)+n\sum_{d|n}\left(-15\,|\,d\right).$$

An immediate consequence is the next corollary.

Corollary 6. *If p is an odd prime, then we have*

$$\sum_{j=1}^{p-1}\Big(\sum_{d|j}\left(-15\,|\,d\right)\Big)\Big(\sigma(p-j)-6\sigma\left(\frac{p-j}{3}\right)-10\sigma\left(\frac{p-j}{5}\right)+15\sigma\left(\frac{p-j}{15}\right)\Big)$$

$$=\begin{cases}5, & \textit{if } p=3\\ 9, & \textit{if } p=5\\ p-1, & \textit{if } p\equiv 1,17,19,23,-7,-11,-13,-29 \pmod{60}\\ -p-1, & \textit{if } p\equiv 7,11,13,29,-1,-17,-19,-23 \pmod{60}.\end{cases}$$

Theorem 8. *We have*

$$\sum_{j=1}^{n-1}\Big(\sum_{d|j}\left(-20\,|\,d\right)\Big)\Big(\sigma(n-j)-2\sigma\left(\frac{p-j}{2}\right)-4\sigma\left(\frac{n-j}{4}\right)$$

$$-5\sigma\left(\frac{n-j}{5}\right)-10\sigma\left(\frac{n-j}{10}\right)+20\sigma\left(\frac{n-j}{20}\right)\Big)$$

$$=-\sigma(n)+2\sigma\left(\frac{n}{2}\right)+4\sigma\left(\frac{n}{4}\right)+5\sigma\left(\frac{n}{5}\right)+10\sigma\left(\frac{n}{10}\right)-20\sigma\left(\frac{n}{20}\right)+n\sum_{d|n}\left(-20\,|\,d\right).$$

An immediate consequence is the next corollary.

Corollary 7. *If p is an odd prime, then*

$$\sum_{j=1}^{p-1}\Bigg(\sum_{d|j}\left(\,-20\,\middle|\,d\,\right)\Bigg)\Bigg(\sigma(p-j)-2\sigma\left(\frac{p-j}{2}\right)-4\sigma\left(\frac{p-j}{4}\right)$$
$$-5\sigma\left(\frac{p-j}{5}\right)-10\sigma\left(\frac{p-j}{10}\right)+20\sigma\left(\frac{p-j}{20}\right)\Bigg)$$
$$=\begin{cases}4, & \text{if } p=5\\ p-1, & \text{if } p\equiv 1,3,7,9 \pmod{20}\\ -p-1, & \text{if } p\equiv -1,-3,-7,-9 \pmod{20}.\end{cases}$$

Theorem 9. *We have*

$$\sum_{j=1}^{n-1}\Bigg(\sum_{d|j}\left(\,-24\,\middle|\,d\,\right)\Bigg)\Bigg(\sigma(n-j)-2\sigma\left(\frac{n-j}{2}\right)-3\sigma\left(\frac{n-j}{3}\right)$$
$$-8\sigma\left(\frac{n-j}{8}\right)-12\sigma\left(\frac{n-j}{12}\right)+24\sigma\left(\frac{n-j}{24}\right)\Bigg)$$
$$=-\sigma(n)+2\sigma\left(\frac{n}{2}\right)+3\sigma\left(\frac{n}{3}\right)+8\sigma\left(\frac{n}{8}\right)-12\sigma\left(\frac{n}{12}\right)-24\sigma\left(\frac{n}{24}\right)+n\sum_{d|n}\left(\,-24\,\middle|\,d\,\right).$$

We have the following consequence.

Corollary 8. *If p is an odd prime, then we have*

$$\sum_{j=1}^{p-1}\Bigg(\sum_{d|j}\left(\,-24\,\middle|\,d\,\right)\Bigg)\Bigg(\sigma(p-j)-2\sigma\left(\frac{p-j}{2}\right)-3\sigma\left(\frac{p-j}{3}\right)$$
$$-8\sigma\left(\frac{p-j}{8}\right)-12\sigma\left(\frac{p-j}{12}\right)+24\sigma\left(\frac{p-j}{24}\right)\Bigg)$$
$$=\begin{cases}2, & \text{if } p=3\\ p-1, & \text{if } p\equiv 1,5,7,11 \pmod{24}\\ -p-1, & \text{if } p\equiv -1,-5,-7,-11 \pmod{24}.\end{cases}$$

3. Proofs of the Theorems

Throughout this section we shall suppose that $q\in\mathbb{C}$ such that $|q|<1$.

Proof of Theorem 2. The following two identities are due to Carlitz [5] and new proofs have been reproduced in Alaca *et al.* [1, Theorems 2.2 and 2.3].

$$\prod_{n=1}^{\infty}(1-q^n)^9(1-q^{3n})^{-3} = \sum_{n=0}^{\infty} a(n)q^n = 1 - 9\sum_{n=1}^{\infty}\Big(\sum_{d\mid n} d^2 \left(-3 \,\middle|\, d\right)\Big)q^n, \qquad (3)$$

where

$$a(0) = 1 \text{ and } a(n) = -9\sum_{d\mid n} d^2 \left(-3 \,\middle|\, d\right).$$

$$\prod_{n=1}^{\infty}(1-q^{3n})^9(1-q^n)^{-3} = \sum_{n=0}^{\infty} b(n)q^n = \sum_{n=1}^{\infty}\Big(\sum_{d\mid n} d^2 \left(-3 \,\middle|\, \frac{n}{d}\right)\Big)q^{n-1}, \qquad (4)$$

where

$$b(0) = 1 \text{ and } b(n) = \sum_{d\mid (n+1)} d^2 \left(-3 \,\middle|\, \frac{n+1}{d}\right).$$

To prove part (a) we apply Theorem 1 to the identity (3) with the sets $A_1 = \mathbb{N}$ and $A_2 = 3\mathbb{N}$ and the functions $f_1(k) = -9k$ on A_1 and $f_2(k) = 3k$ on A_2 to get

$$\begin{aligned} na(n) &= -9\sigma(n) + 3\sigma_{0,3}(n) + \sum_{j=1}^{n-1} a(j)\big(-9\sigma(n-j) + 3\sigma_{0,3}(n-j)\big) \\ &= -9\sigma(n) + 9\sigma(n/3) - 9\sum_{j=1}^{n-1} a(j)\Big(\sigma(n-j) - \sigma\left(\frac{n-j}{3}\right)\Big), \end{aligned} \qquad (5)$$

where the second identity follows by the relation (1). Now use the definition of $a(j)$ and divide by 9 to conclude the desired formula. As to part (b), apply Theorem 1 to the identity (4) for the sets $A_1 = 3\mathbb{N}$ and $A_2 = \mathbb{N}$ and the functions $f_1(k) = -9k$ on A_1 and $f_2(k) = 3k$ on A_2 to get

$$nb(n) = \sum_{j=0}^{n-1} b(j)\big(-9\sigma_{0,3}(n-j) + 3\sigma(n-j)\big) = 3\sum_{j=0}^{n-1} b(j)\Big(\sigma(n-j) - 9\sigma\left(\frac{n-j}{3}\right)\Big). \quad (6)$$

Equivalently,

$$\sum_{j=1}^{n}\Big(\sum_{d\mid j} d^2 \left(-3 \,\middle|\, \frac{j}{d}\right)\Big)\Big(\sigma(n+1-j) - 9\sigma\left(\frac{n+1-j}{3}\right)\Big) = \frac{n}{3}\sum_{d\mid n+1} d^2 \left(-3 \,\middle|\, \frac{n+1}{d}\right),$$

as desired. □

Proof of Theorem 3. The following two identities are due to Carlitz [5], see [1, Theorems 2.4 and 2.5] for new proofs.

$$\prod_{n=1}^{\infty}(1-q^n)^4(1-q^{2n})^6(1-q^{4n})^{-4} = \sum_{n=0}^{\infty} a(n)q^n = 1 - 4\sum_{n=1}^{\infty}\Big(\sum_{d\mid n} d^2 \left(-4 \,\middle|\, d\right)\Big)q^n, \; (7)$$

where

$$a(0) = 1 \text{ and } a(n) = -4\sum_{d|n} d^2\left(-4 \,\middle|\, d\right).$$

$$\prod_{n=1}^{\infty}(1-q^{2n})^6(1-q^{4n})^4(1-q^n)^{-4} = \sum_{n=0}^{\infty} b(n)q^n = \sum_{n=1}^{\infty}\Big(\sum_{d|n} d^2\left(-4 \,\middle|\, \frac{n}{d}\right)\Big)q^{n-1}, \quad (8)$$

where

$$b(0) = 1 \text{ and } b(n) = \sum_{d|(n+1)} d^2\left(-4 \,\middle|\, \frac{n+1}{d}\right).$$

By virtue of Theorem 1 applied to the formula (7) and identity (2) we obtain

$$\begin{aligned} na(n) &= -4\sigma(n) - 12\sigma(n/2) + 16\sigma(n/4) + \\ &\sum_{j=1}^{n-1} a(j)\Big(-4\sigma(n-j) - 12\sigma\left(\frac{n-j}{2}\right) + 16\sigma\left(\frac{n-j}{4}\right)\Big) \\ &= -4\sigma^*(n) - 16\sigma^*(n/2) - 4\sum_{j=1}^{n-1} a(j)\Big(\sigma^*(n-j) + 4\sigma^*\left(\frac{n-j}{2}\right)\Big), \end{aligned} \quad (9)$$

which is equivalent to

$$n\sum_{d|n} d^2\left(-4 \,\middle|\, d\right) = \sigma^*(n) + 4\sigma^*(n/2) - 4\sum_{j=1}^{n-1}\Big(\sum_{d|j} d^2\left(-4 \,\middle|\, d\right)\Big)\Big(\sigma^*(n-j) + 4\sigma^*\left(\frac{n-j}{2}\right)\Big),$$

giving part (a). As to part (b), applying Theorem 1 to the relation (8) and proceeding as before we find

$$\sum_{j=1}^{n}\Big(\sum_{d|j} d^2\left(-4 \,\middle|\, \frac{j}{d}\right)\Big)\Big(\sigma(n+1-j) - 3\sigma\left(\frac{n+1-j}{2}\right) - 4\sigma\left(\frac{n+1-j}{4}\right)\Big)$$

$$= \frac{n}{4}\sum_{d|n+1} d^2\left(-4 \,\middle|\, \frac{n+1}{d}\right),$$

as desired. As to part (c), we have by [1, Theorem 4.5]

$$\prod_{n=1}^{\infty}(1-q^{2n})^{10}(1-q^n)^{-4}(1-q^{4n})^{-4} = 1 + 4\sum_{n=1}^{\infty}\Big(\sum_{d|n}\left(-4 \,\middle|\, d\right)\Big)q^n, \quad (10)$$

which by Theorem 1 and the relation (2) translates into

$$16\sum_{j=1}^{n-1}\Big(\sum_{d|j}\left(-4 \,\middle|\, d\right)\Big)\Big(\sigma^*(n-j) - 4\sigma^*\Big(\frac{n-j}{2}\Big)\Big)$$

$$= -4\sigma^*(n) + 16\sigma^*\left(\frac{n}{2}\right) + 4n\sum_{d|n}\left(-4 \,\middle|\, d\right);$$

yielding the desired identity. □

Proof of Theorem 4. The following two identities are due to Ramanujan, see Alaca *et al.* [1, Theorems 3.2 and 3.3] for proofs based on work by Bailey [3, 4]:

$$\prod_{n=1}^{\infty}(1-q^n)^5(1-q^{5n})^{-1} = 1 - 5\sum_{n=1}^{\infty}\Big(\sum_{d|n} d\left(5 \,\middle|\, d\right)\Big)q^n. \tag{11}$$

$$\prod_{n=1}^{\infty}(1-q^{5n})^5(1-q^n)^{-1} = \sum_{n=1}^{\infty}\Big(\sum_{d|n} d\left(5 \,\middle|\, \frac{n}{d}\right)\Big)q^{n-1}. \tag{12}$$

By Theorem 1 applied to (11) we have

$$-5n\sum_{d|n} d\left(5 \,\middle|\, d\right) = -5\sigma(n)+5\sigma(n/5)+\sum_{j=1}^{n-1}\Big(-5\sum_{d|j} d\left(5 \,\middle|\, d\right)\Big)\Big(-5\sigma(n-j)+5\sigma\left(\frac{n-j}{5}\right)\Big),$$

or equivalently,

$$\sum_{j=1}^{n-1}\Big(\sum_{d|j} d\left(5 \,\middle|\, d\right)\Big)\Big(\sigma(n-j)-\sigma\left(\frac{n-j}{5}\right)\Big) = \frac{1}{5}\sigma(n) - \frac{1}{5}\sigma\left(\frac{n}{5}\right) - \frac{n}{5}\sum_{d|n} d\left(5 \,\middle|\, d\right),$$

which gives part (a). As to part (b), apply Theorem 1 to identity (12) to get

$$n\sum_{d|n+1} d\left(5 \,\middle|\, \frac{n+1}{d}\right) = \sum_{j=1}^{n}\Big(\sum_{d|j} d\left(5 \,\middle|\, \frac{j}{d}\right)\Big)\Big(\sigma(n+1-j) - 25\sigma\left(\frac{n+1-j}{5}\right)\Big),$$

and proceed as in part (a). □

Proof of Theorem 5. We have the following two formulas of Carlitz [5] which appear as Theorem 3.4 and Theorem 3.5 respectively in Alaca *et al.* [1]:

$$\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{2n})(1-q^{4n})^3(1-q^{8n})^{-2} = 1 - 2\sum_{n=1}^{\infty}\Big(\sum_{d|n} d\left(8 \,\middle|\, d\right)\Big)q^n, \tag{13}$$

$$\prod_{n=1}^{\infty}(1-q^{2n})^3(1-q^{4n})(1-q^{8n})^2(1-q^n)^{-2} = \sum_{n=1}^{\infty}\Big(\sum_{d|n} d\left(8 \,\middle|\, \frac{n}{d}\right)\Big)q^{n-1}. \tag{14}$$

Part (a) follows by Theorem 1 applied to (13) and part (b) follows by Theorem 1 applied to (14). To prove part (c) use the same sort of argument with an application of Theorem 1 to the following identity which is Theorem 4.6 in [1]:

$$\prod_{n=1}^{\infty}(1-q^{2n})^3(1-q^{4n})^3(1-q^n)^{-2}(1-q^{8n})^{-2} = 1 + 2\sum_{n=1}^{\infty}\Big(\sum_{d|n}\left(-8 \,\middle|\, d\right)\Big)q^n.$$

□

Proof of Theorem 6. We proceed as in the previous proofs. We have the following two identities by Alaca *et al.* [1, Theorems 3.6 and 3.7]:

$$\prod_{n=1}^{\infty}(1-q^n)(1-q^{3n})(1-q^{4n})^2(1-q^{6n})^2(1-q^{12n})^{-2} = 1 - \sum_{n=1}^{\infty}\Big(\sum_{d|n} d\,\big(\,12\,\big|\,d\,\big)\Big)q^n,$$

$$\prod_{n=1}^{\infty}(1-q^{2n})^2(1-q^{3n})^2(1-q^{4n})(1-q^{12n})(1-q^n)^{-2} = \sum_{n=1}^{\infty}\Big(\sum_{d|n} d\,\big(\,12\,\big|\,\frac{n}{d}\,\big)\Big)q^{n-1}.$$

Then application of Theorem 1 to the first identity yields part (a) and application of Theorem 1 to the second identity yields part (b). □

Proof of Theorem 7. Just as before, apply Theorem 1 to the following result of Alaca *et al.* [1, Theorem 4.2].

$$\prod_{n=1}^{\infty}(1-q^{3n})^2(1-q^{5n})^2(1-q^n)^{-1}(1-q^{15n})^{-1} = 1 + \sum_{n=1}^{\infty}\Big(\sum_{d|n}\big(\,-15\,\big|\,d\,\big)\Big)q^n.$$

□

Proof of Theorem 8. By [1, Theorem 4.3] we have

$$\prod_{n=1}^{\infty}(1-q^{2n})(1-q^{4n})(1-q^{5n})(1-q^{10n})(1-q^n)^{-1}(1-q^{20n})^{-1} = 1+\sum_{n=1}^{\infty}\Big(\sum_{d|n}\big(\,-20\,\big|\,d\,\big)\Big)q^n,$$

which by virtue of Theorem 1 for an odd prime p yields the desired identity. □

Proof of Theorem 9. Combine Theorem 1 with the following formula [1, Theorem 4.4]

$$\prod_{n=1}^{\infty}(1-q^{2n})(1-q^{3n})(1-q^{8n})(1-q^{12n})(1-q^n)^{-1}(1-q^{24n})^{-1} = 1+\sum_{n=1}^{\infty}\Big(\sum_{d|n}\big(\,-24\,\big|\,d\,\big)\Big)q^n.$$

□

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References

[1] A. Alaca, S. Alaca, and K. Williams, *Some infinite products of Ramanujan type*, Canad. Math. Bull. Vol. 52 (4), (2009), 481-492.

[2] T. M. Apostol, *Introduction to Analytic Number Theory*, Undergraduate Texts in Mathematics, Springer, first edition, 1976.

[3] W. N. Bailey, *A note on two of Ramanujan's formulae*, Quart. J. Math. Oxford Ser. (2) **3**(1952), 29-31.

[4] W. N. Bailey, *A further note on two of Ramanujan's formulae*, Quart. J. Math. Oxford Ser. (2) **3**(1952), 158-160.

[5] L. Carlitz, *Note on some partition formulae*, Quart. J. Math. Oxford Ser. (2) **4**(1953), 168-172.

[6] M. El Bachraoui, *Inductive formulas for some arithmetic functions*, International Journal of Number Theory 7, Number 8, (2011), 2261-2268.

[7] J. Liouville, *Sur quelques formules générales qui peuvent être utiles dans la théorie des nombres*, (7th article), J. Math. Pures Appl. 4, (1859), 1-8.

[8] S. Ramanujan, *The lost notebook and other unpublished papers*, Narosa Publishing House, New Delhi, 1988.

[9] N. Robbins, *Some identities connecting partition functions to other number theoretic functions*, Rocky Mountain Journal of Mathematics, Volume 29, Number 1, (1999), 335-345.

[10] K. S. Williams, *Number Theory in the Spirit of Liouville*, Cambridge University Press, New York, First edition, 2011.