ON THE DIOPHANTINE EQUATION $X^2 - KXY + Y^2 + LX = 0$

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Abstract

For any given positive integer l, we prove that there are only finitely many integers k such that the Diophantine equation $x^2 - kxy + y^2 + lx = 0$ has an infinite number of positive integer solutions (x, y). Moreover, we determine all integers k such that the Diophantine equation $x^2 - kxy + y^2 + lx = 0$, $1 \leq l \leq 33$, has an infinite number of positive integer solutions (x, y).

1. Introduction

In [2], Marlewski and Zarzycki proved that the Diophantine equation

$$x^2 - kxy + y^2 + x = 0 \tag{1}$$

has an infinite number of positive integer solutions (x, y) if and only if $k = 3$. Some computer experiments suggest that for many integers k there are infinitely many positive integer solutions, so they hoped that it is possible to characterize positive integer solutions of the equation $x^2 - kxy + y^2 + lx = 0$ in the general case.

Recently, Yuan and Hu [5] showed that the equation

$$x^2 - kxy + y^2 + 2x = 0 \tag{2}$$

has an infinite number of positive integer solutions (x, y) if and only if $k = 3$, 4; and the equation

$$x^2 - kxy + y^2 + 4x = 0 \tag{3}$$

has an infinite number of positive integer solutions (x, y) if and only if $k = 3, 4, 6$.

The main purpose of the present paper is to determine integers k such that the equation

$$x^2 - kxy + y^2 + lx = 0, \tag{4}$$

where l is a given positive integer, has an infinite number of positive integer solutions (x, y).

In this paper, we use a completely different method to deal with this problem. The main result is as follows.

Theorem 1. *For any given positive integer l, there are only finite many integers k such that equation (4) has an infinite number of positive integer solutions (x, y).*

For positive integers l with $1 \le l \le 33$, by the method indicated in the proof of the main theorem, we compute and list all (k, l) such that equation (4) has infinitely many positive integer solutions (x, y) (see the table at the end of Section 4).

2. Lemmas

In this section, we will present the lemmas that will be needed in the proof of the main theorems.

Lemma 2. ([2, Theorem 1]) *If positive integers x, y, k satisfy equation (1), then there exist positive integers c, e such that $x = c^2$, $y = ce$, and $\gcd(c, e) = 1$.*

Lemma 3. *If positive integers x, y, k satisfy equation (4) with $\gcd(x, y, l) = 1$, then there exist positive integers c, e such that $x = c^2$, $y = ce$, and $\gcd(c, e) = 1$.*

Proof. It follows from (4) that if p is a prime number, then $p \,|x$ implies $p \,|y$. In particular, l is prime to p. Let $x = p^{\mu} x_1$ and $y = p^{\nu} y_1$ with $\gcd(p, x_1 y_1) = 1$. Substituting these values of x and y into equation (4) we have

$$p^{2\nu} y_1^2 = p^{\mu}(p^{\nu} k x_1 y_1 - p^{\mu} x_1^2 - l x_1),$$

which implies that $\mu = 2\nu$ since $\gcd(l x_1, p) = 1$. This means that $x = c^2$, $y = ce$, and $\gcd(c, e) = 1$. $\square$

To prove our results, we need some results on continued fractions.

Definition 4. *The fraction*

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots + \frac{1}{a_N}}}}$$

is called a finite continued fraction, and denoted by $[a_0, a_1, \ldots, a_N]$*. It is called a continued fraction when* $N = +\infty$*. For simplicity, usually it is denoted by* $[a_0, a_1, a_2, \ldots, a_N, \ldots]$*. We call* $\alpha_n = [a_n, a_{n+1}, \ldots]$ *the* $(n+1)$*-st complete quotient of the continued fraction* $\alpha = [a_0, a_1, \ldots, a_n, \ldots]$*.*

Lemma 5. ([1, Theorem 10.8.1]) *Let* d *be a positive integer which is not a square. Then the* $(n+1)$*-st complete quotient* α_n *of the continued fraction* $\alpha = \sqrt{d}$ *is of the form*

$$\frac{\sqrt{d} + P_n}{Q_n}, \qquad P_n^2 \equiv d \pmod{Q_n},$$

where P_n *and* Q_n *are positive integers.*

Lemma 6. ([1, Theorem 10.8.2]) *Let* d*,* P_n*, and* Q_n *be as in Lemma 5. Then the quadratic equation*

$$x^2 - dy^2 = (-1)^n Q_n$$

has a positive integer solution (x, y)*. If* $l \neq (-1)^n Q_n$ *and* $|l| < \sqrt{d}$*, then the Diophantine equation*

$$x^2 - dy^2 = l$$

has no integer solutions (x, y)*.*

The proof of the following lemma is well-known, so we omit the proof here.

Lemma 7. *Let* $d > 1$ *be a positive integer which is not a square and* $c \neq 0$ *a given integer. If the Diophantine equation*

$$x^2 - dy^2 = c \tag{5}$$

has a positive integer solution (x, y) *with* $\gcd(x, y) = 1$*, then equation (5) has infinite many positive integer solutions* (x, y) *with* $\gcd(x, y) = 1$*.*

Lemma 8. ([3, Theorem 108a]) *Let* N*,* D *be positive integers and* D *not a square. Suppose that* $x_0 + y_0\sqrt{D}$ *is the fundamental solution of the Pell equation* $x^2 - Dy^2 = 1$ *and the equation*

$$u^2 - Dv^2 = -N, \ u, v \in \mathbb{Z}, \tag{6}$$

where $\gcd(u, v) = 1$*, is solvable. Then (6) has a solution* $u_0 + v_0\sqrt{D}$ *with the following property;*

$$0 < v_0 \leq \frac{y_0\sqrt{N}}{\sqrt{2(x_0 - 1)}}, \quad 0 \leq u_0 \leq \sqrt{\frac{1}{2}(x_0 - 1)N}. \tag{7}$$

Lemma 9. ([4, Theorem 2]) *Let* N*,* D *be odd positive integers with* D *non-square. Suppose that the equation*

$$x^2 - Dy^2 = 4, \quad \gcd(x, y) = 1 \tag{8}$$

is solvable and let $x_0 + y_0\sqrt{D}$ *be the least solution. If the equation*

$$u^2 - Dv^2 = -4N,\ u,\ v \in \mathbb{Z}, \tag{9}$$

where $\gcd(u, v)\,|2$*, is solvable, then (9) has a solution* $u_0 + v_0\sqrt{D}$ *with the following property:*

$$0 < v_0 \leq \frac{y_0\sqrt{N}}{\sqrt{(x_0 - 2)}}, \quad 0 \leq u_0 \leq \sqrt{(x_0 - 2)N}. \tag{10}$$

3. Proof of Theorem 1

Proof. Since $\gcd(x, y, l) = 1$, by Lemma 3 we have $x = c^2$, $y = ce$, $\gcd(c, e) = 1$. Substituting these values of x and y into equation (4), we have $c^2 - kce + e^2 + l = 0$. It follows that

$$(2c - ke)^2 - (k^2 - 4)e^2 = -4l. \tag{11}$$

`Case (i).` $2\,|k$. Let $k = 2k_1$. Then from (11), we have

$$(c - k_1e)^2 - (k_1^2 - 1)e^2 = -l.$$

Since

$$\sqrt{k_1^2 - 1} = [k_1 - 1, \overline{1, 2k_1 - 2}],$$

where ($k_1 \geq 2$, $k_1 \in \mathbb{Z}^+$), we therefore have $Q_{2t-1} = 2k_1 - 2$, $Q_{2t} = 1$, $t > 0$. By Lemma 6, if equation (11) has a positive integer solution (c, e) and $|l| < k_1$, then we have

$$-l = (-1)^{2t-1}Q_{2t-1} = -2k_1 + 2,$$

which is possible only when $l = k - 2$.

`Case (ii).` $2 \nmid k$ and $2\,|e$. Let $e = 2e_1$. Then from (11), we have

$$(c - ke_1)^2 - (k^2 - 4)e_1^2 = -l.$$

Since

$$\sqrt{k^2 - 4} = [k - 1, \overline{1, (k - 3)/2, 2, (k - 3)/2, 1, 2k - 1}],$$

where $k \geq 5$ is odd, a short computation shows that

$$Q_{6t+1} = 2k - 5,\ Q_{6t+2} = 4,\ Q_{6t+3} = k - 2,$$

$$Q_{6t+4} = 4,\ Q_{6t+5} = 2k - 5,\ Q_{6t+6} = 1,\ t \geq 0.$$

By Lemma 6, if equation (11) has a positive integer solution (c, e) and $|l| < k$, then we have

$$-l = (-1)^nQ_n = -2k + 5, 4, -k + 2, 1,$$

which is possible only when $l = k-2, -4, -1$.

(iii) $2 \nmid ke$. Then from (11), we have

$$(2c-ke)^2 - (k^2-4)e^2 = -4l.$$

Since

$$\sqrt{k^2-4} = [k-1, \overline{1, (k-3)/2, 2, (k-3)/2, 1, 2k-1}],$$

where $k \geq 5$ is odd, similarly, we have

$$Q_{6t+1} = 2k-5,\ Q_{6t+2} = 4,\ Q_{6t+3} = k-2,$$

$$Q_{6t+4} = 4,\ Q_{6t+5} = 2k-5,\ Q_{6t+6} = 1,\ t \geq 0.$$

By Lemma 6, if equation (11) has a positive integer solution (c, e) and $4|l| < k$, then we have

$$-4l = (-1)^n Q_n = -2k+5, 4, -k+2, 1,$$

which is possible only when $l = -1$.

To sum up, we derive that if equation (11) has a positive integer solution (c, e), then $k < 4l$, which implies the theorem. $\square$

Remark. If $l = dl_1$, d, $l_1 \in \mathbb{Z}$ and the equation $x^2 - kxy + y^2 + l_1 x = 0$ has infinitely many positive integer solutions (x_n, y_n), $n = 1, 2, \ldots$, then the equation

$$u^2 - kuv + v^2 + lu = 0$$

has infinitely many positive integer solutions (u_n, v_n), $n = 1, 2, \ldots$, where $u_n = dx_n$, $v_n = dy_n$.

In view of the arguments in the proof of Theorem 1, the Remark and Lemmas 6-8, for a given positive integer l, to find all possible integers k such that the equation $x^2 - kxy + y^2 + lx = 0$ has infinitely many positive integer solutions (x, y), we only need to find all $k_1 = k/2$ such that the equation

$$x^2 - (k_1^2 - 1)y^2 = -l,\ \gcd(x, y) = 1$$

has a positive integer solution (x, y) with $1 < k_1 \leq l \leq 33$, $y \leq \sqrt{33/2(x_0-1)} < 5$; and to find all integers k such that the equation

$$x^2 - (k^2-4)y^2 = -4l,\ \gcd(x, y)\,|2$$

has a positive integer solution (x, y) with $1 < k \leq 4l \leq 132$, $y \leq \sqrt{132/(x_0-2)} < 7$, $x_0 \geq 5$. This can be easily done by MATHLAB. The following is the list of the computation.

l	k	l	k
2	$3,4$	18	$3,4,5,7,8,11,20$
3	$3,4,5$	19	$3,9,12,21$
4	$3,4,6$	20	$3,4,5,6,7,10,12,22$
5	$3,5,7$	21	$3,4,5,6,9,13,23$
6	$3,4,5,8$	22	$3,4,7,8,13,24$
7	$3,6,9$	23	$3,4,6,10,11,14,25$
8	$3,4,6,10$	24	$3,4,5,6,8,10,14,26$
9	$3,4,5,7,11$	25	$3,5,7,15,27$
10	$3,4,5,7,12$	26	$3,4,9,12,15,28$
11	$3,4,7,8,13$	27	$3,4,5,7,11,13,16,29$
12	$3,4,5,6,8,14$	28	$3,4,6,8,9,16,30$
13	$3,9,15$	29	$3,7,11,13,17,31$
14	$3,4,6,8,9,16$	30	$3,4,5,7,8,10,12,17,32$
15	$3,4,5,7,8,10,17$	31	$3,6,12,18,33$
16	$3,4,6,10,18$	32	$3,4,6,10,14,18,34$
17	$3,5,9,11,19$	33	$3,4,5,7,8,13,19,35$

4. The Equation $2x^2 - kxy + y^2 + x = 0$

In this section, we consider a variation of the above problem. We ask the same question for the equation

$$2x^2 - kxy + y^2 + x = 0, \tag{12}$$

i.e., for which k, equation (12) has infinitely many positive integer solutions (x, y). To our surprise, it is much difficult to completely solve this question.

Lemma 10. *If positive integers x, y, k satisfy equation* (12)*, then there exist positive integers c, e such that $x = c^2$, $y = ce$, and* $\gcd(c, e) = 1$.

Proof. It follows from (12) that if p is a prime number, then $p \,|x$ implies $p \,|y$. Let $x = p^{\mu} x_1$ and $y = p^{\nu} y_1$ with $\gcd(p, x_1 y_1) = 1$. Substituting these values of x and y into equation (12) we have

$$p^{2\nu} y_1^2 = p^{\mu}(p^{\nu} k x_1 y_1 - p^{\mu} x_1^2 - x_1),$$

which implies that $\mu = 2\nu$ since $\gcd(x_1, p) = 1$. This means that $x = c^2$, $y = ce$, and $\gcd(c, e) = 1$. $\square$

Theorem 11. *Equation (12) has infinitely many positive integer solutions (x, y) if and only if the equation $x^2 - (k^2 - 8)y^2 = -1$ has a positive integer solution.*

Proof. If (x, y) is a positive integer solution of equation (12), by Lemma 10, we have $x = c^2$, $y = ce$, $\gcd(c, e) = 1$. Substituting these values of x and y into equation (12), we have

$$2c^2 - kce + e^2 + 1 = 0. \tag{13}$$

We divide the proof into two cases.

Case 1 $2 \mid k$. Let $k = 2k_1$. Then from (13), we have

$$(e - k_1c)^2 - (k_1^2 - 2)c^2 = -1.$$

Since

$$\sqrt{k_1^2 - 2} = [k_1 - 1, \overline{1, k_1 - 2, 1, 2k_1 - 2}],$$

where $(k_1 \geq 2, k_1 \in \mathbb{Z}^+)$, so we have $Q_{2t} = 2k_1 - 3, Q_{2t} = 1$ or $2, t > 0$. By Lemma 6, equation (12) has no solutions in this case.

Case 2 $2 \nmid k$. From (13) we have $2 \nmid e$. We divide the proof into two subcases.

(i) $2 \mid c$. Let $c = 2c_1$. Then from (13), we have

$$(e - kc_1)^2 - (k^2 - 8)c_1^2 = -1. \tag{14}$$

Therefore, in this case, equation (12) has a positive integer solution (x, y) if and only if the equation $x^2 - (k^2 - 8)y^2 = -1$ has a positive integer solution.

(ii) $2 \nmid kec$. From (13), we have $(2e - kc)^2 - (k^2 - 8)c^2 = -4$, and we derive that $1 - 1 \equiv 4 \pmod 8$ by taking modulo 8, which is a contradiction.

In conclusion, by Lemma 6 and the above arguments, equation (12) has infinitely many positive integer solutions (x, y) if and only if the equation $x^2 - (k^2 - 8)y^2 = -1$ has a positive integer solution. This completes the proof of Theorem 11. □

Finally, we propose a conjecture which is closely related to equation (12). It is generally believed that there are infinitely many primes of the form $u^2 - 8$, so we believe that the following conjecture is true.

Conjecture 12. *There are infinitely many positive integers k such that the equation $x^2 - (k^2 - 8)y^2 = -1$ has a positive integer solution (x, y).*

If the above conjecture is true, then by Theorem 11, there are infinitely many positive integers k such that equation (12) has infinitely many positive integer solutions (x, y).

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