S-DIOPHANTINE QUADRUPLES WITH TWO PRIMES CONGRUENT TO 3 MODULO 4

László Szalay
Institute of Mathematics, University of West Hungary, Sopron, Hungary
laszalay@emk.nyme.hu

Volker Ziegler[1]
Johann Radon Institute for Computational and Applied Mathematics (RICAM)
Austrian Academy of Sciences, Linz, Austria
volker.ziegler@ricam.oeaw.ac.at

Received: 3/12/13, Revised: 11/8/13, Accepted: 11/29/13, Published: 12/13/13

Abstract

Let S be a fixed set of primes and let $a_1, \ldots, a_m$ denote distinct positive integers. We call the m-tuple $(a_1, \ldots, a_m)$ an S-Diophantine tuple if the integers $a_i a_j + 1 = s_{i,j}$ are S-units for all $i \neq j$. In this paper, we show that if $S = \{p, q\}$ and $p, q \equiv 3 \pmod 4$, then no S-Diophantine quadruple exists.

1. Introduction

It is an old problem to find m-tuples $(a_1, \ldots, a_m)$ of distinct positive integers such that

$$a_i a_j + 1 = \square \tag{1}$$

for $i \neq j$. Such m-tuples are called Diophantine m-tuples and have been studied since ancient times by several authors. Most notable is Dujella's result [3] that no Diophantine sextuple exists and that there are only finitely many quintuples. Even more is believed to be true. A folklore conjecture states that there exist no quintuples at all.

Beside Diophantine m-tuples, various variants have also been considered. For instance, Bugeaud and Dujella [1] examined m-tuples, where $\square$ in (1) is replaced by a k-th power, and Dujella and Fuchs [4] investigated a polynomial version. Later Fuchs, Luca and the first author [5] replaced $\square$ by terms of a given binary recurrence sequence, in particular, the Fibonacci sequence [6]. Recently the authors

[1]The second author was supported by the Austrian Science Found (FWF) under the project P 24801-N26

substituted □ by S-units [7]. For complete overview we suggest Dujella's web page on Diophantine tuples [2].

In this paper, we continue our research on S-Diophantine m-tuples. Let S be a fixed set of primes. Then we call the m-tuple $(a_1, \ldots, a_m)$ with positive distinct integers a_i $(1 \leq i \leq m)$ an S-Diophantine m-tuple, if we have $a_i a_j + 1 = s_{i,j}$, $1 \leq i < j \leq n$, are S-units.

In a recent paper [7] the authors showed that if $S = \{p, q\}$ and $C(\xi) < p < q < p^\xi$ for some $\xi > 1$ and for some explicitly computable constant $C(\xi)$, then no S-Diophantine quadruple exists. This result and numerical experiments (see [7, Lemma 9], where we found no quadruples with $1 \leq a < b < c < d \leq 1000$) raise the question of whether or not an S-Diophantine quadruple with $|S| = 2$ exists. In this connection we have the following conjecture.

Conjecture 1.1. There is no pair of primes (p, q) such that a $\{p, q\}$-Diophantine quadruple exists.

Unfortunately, we can prove only the following weaker statement which is the main result of this paper.

Theorem 1.2. *Let $S = \{p, q\}$ with primes $p, q \equiv 3 \pmod 4$. Then no S-Diophantine quadruple exists.*

The proof of Theorem 1.2 is organized as follows. In the next section we provide two auxiliary results which enable us to prove Theorem 1.2 partially in Section 3. Later the proof is completed in the last section of the paper. Here we note that Lemma 2.2 is the only place where we used the assertion p, q congruent to 3 modulo 4, so the technique we applied might be useful in the proof of the conjecture.

2. Auxiliary Results

We start with a very useful lemma (see [7, Lemma 2]) which excludes some divisibility relations for S-Diophantine triples.

Lemma 2.1. *Assume that (a, b, c) is an S-Diophantine triple with $1 \leq a < b < c$. If $ac + 1 = s$ and $bc + 1 = t$, then $s \nmid t$.*

This lemma is exactly Lemma 2 in [7]. The proof is short and, since we intend to keep this paper independent and self-contained, we repeat the proof here.

Proof. Assume that $s \mid t$. Then

$$m = \frac{bc+1}{ac+1} = \frac{b}{a} + \frac{a-b}{a^2c+a} = \frac{b}{a} + \frac{\theta}{a^2} \in \mathbb{Z}$$

holds if $|\theta| < 1$. Therefore m is an integer if and only if $\theta = 0$. Thus, we have $a = b$ which is a contradiction to $a < b$. □

Now we deduce a few restrictions on the exponents appearing in the prime factorization of the S-units $s_{i,j}$.

Lemma 2.2. *Let $S = \{p, q\}$ with $p, q \equiv 3 \pmod 4$ and let (a, b, c) be an S-Diophantine triple. Further assume that*

$$ab + 1 = p^{\alpha_1} q^{\beta_1}, \qquad ac + 1 = p^{\alpha_2} q^{\beta_2}, \qquad bc + 1 = p^{\alpha_3} q^{\beta_3}.$$

Then at least one of $\alpha_1, \alpha_2, \alpha_3$ is zero and at least one of $\beta_1, \beta_2, \beta_3$ is equal to zero.

Proof. Using the notation of the lemma we have

$$(abc)^2 = \left(p^{\alpha_1} q^{\beta_1} - 1\right)\left(p^{\alpha_2} q^{\beta_2} - 1\right)\left(p^{\alpha_3} q^{\beta_3} - 1\right).$$

If α_1, α_2 and α_3 are all positive, then $(abc)^2 \equiv -1 \pmod p$ and we arrive at a contradiction since the Legendre symbol $(-1/p) = -1$. Similarly, at least one of β_1, β_2 and β_3 must be zero. □

3. Proof of Theorem 1.2

For the rest of the paper we assume that $S = \{p, q\}$ and $p, q \equiv 3 \pmod 4$.

Suppose now that (a, b, c, d) is an S-Diophantine quadruple. Therefore there are non-negative integers α_i, β_i, $i = 1, \ldots, 6$, such that

$$\begin{aligned} ab + 1 &= p^{\alpha_1} q^{\beta_1}, & bc + 1 &= p^{\alpha_4} q^{\beta_4}, \\ ac + 1 &= p^{\alpha_2} q^{\beta_2}, & bd + 1 &= p^{\alpha_5} q^{\beta_5}, \\ ad + 1 &= p^{\alpha_3} q^{\beta_3}, & cd + 1 &= p^{\alpha_6} q^{\beta_6} \end{aligned}$$

hold. Since (a, b, c) is an S-Diophantine triple, according to Lemma 2.2, at least one of α_1, α_2 and α_4 is zero. Let us assume for the moment that all of them vanish, i.e., $\alpha_1 = \alpha_2 = \alpha_4 = 0$. Without loss of generality, we may suppose $a < b < c$. Thus, $ac + 1 \mid bc + 1$ and Lemma 2.1 yields a contradiction. Subsequently, at least one of α_1, α_2 and α_4 is non-zero, and the same is true for β_1, β_2 and β_4.

Proposition 3.1. *If exactly one of α_1, α_2 and α_4 is zero, or exactly one of β_1, β_2 and β_4 is zero, then (a, b, c, d) cannot be an S-Diophantine quadruple.*

Proof. By switching p and q if necessary, and by rearranging the quadruple (a, b, c, d) we may assume that $\alpha_1 = 0$ and α_2, α_4 are positive. Note that (b, c, d) is also an S-Diophantine triple. In view of Lemma 2.2, one of α_5 and α_6 must be zero.

Letting $\alpha_5 = 0$, it implies $\alpha_6 = 0$. Indeed, consider the S-Diophantine triple (a, c, d) and the corresponding equations $ac + 1 = p^{\alpha_2} q^{\beta_2}$, $ad + 1 = p^{\alpha_3} q^{\beta_3}$ and $cd + 1 = p^{\alpha_6} q^{\beta_6}$. By Lemma 2.2, one of α_3 and α_6 vanishes. But $\alpha_3 = 0$ leads to a contradiction because it provides $ab+1=q^{\beta_1}$, $ad+1=q^{\beta_3}$ and $bd+1=q^{\beta_5}$, which contradicts Lemma 2.1. Hence $\alpha_5=\alpha_6=0$.

Thus, the following lemma completes the proof of the proposition.

Lemma 3.2. *There exists no S-Diophantine quadruple (a,b,c,d) with $\alpha_1=\alpha_6=0$.*

The proof of this lemma is long and technical; we postpone the proof to the forthcoming section. However, assuming this lemma, the proof of our proposition is complete. $\square$

By virtue of Proposition 3.1, at least two of α_1, α_2 and α_4 are zero, and similarly, at least two of β_1, β_2 and β_4 equal zero. Therefore one pair fulfills $(\alpha_i, \beta_i)=(0,0)$ with $i \in \{1,2,4\}$. But, this is impossible since all of $ab+1$, $ac+1$ and $bc+1$ are at least 3.

Hence, up to the proof of Lemma 3.2, we have proved Theorem 1.2.

4. Proof of Lemma 3.2

In view of the assumptions of Lemma 3.2, we have to study the system

$$\begin{aligned} ab+1&=q^{\beta_1}, & bc+1&=p^{\alpha_4}q^{\beta_4},\\ ac+1&=p^{\alpha_2}q^{\beta_2}, & bd+1&=p^{\alpha_5}q^{\beta_5},\\ ad+1&=p^{\alpha_3}q^{\beta_3}, & cd+1&=q^{\beta_6}. \end{aligned}$$

Consider the triple (a,b,c). By Lemma 2.2, we deduce that either $\beta_2=0$ or $\beta_4=0$. There is no restriction in assuming $\beta_2=0$ (switch a and b as well as the corresponding exponents, if necessary). Thus, we obtain the system

$$\begin{aligned} ab+1&=q^{\beta_1}, & bc+1&=p^{\alpha_4}q^{\beta_4},\\ ac+1&=p^{\alpha_2}, & bd+1&=p^{\alpha_5}q^{\beta_5},\\ ad+1&=p^{\alpha_3}q^{\beta_3}, & cd+1&=q^{\beta_6}. \end{aligned}$$

Subsequently,

$$ab\cdot cd=\left(q^{\beta_1}-1\right)\left(q^{\beta_6}-1\right)=\left(p^{\alpha_2}-1\right)\left(p^{\alpha_5}q^{\beta_5}-1\right)=ac\cdot bd.$$

Assuming $\beta_5>0$ we get

$$1\equiv 1-p^{\alpha_2} \pmod{q}. \tag{2}$$

Note that the positivity of a,b,c and d entails that β_1 and β_6 are also positive integers. However, equation (2) yields the contradiction $q|p^{\alpha_2}$. Thus, we have

$\beta_5 = 0$, and the system

$$\begin{aligned} ab+1 &= q^{\beta_1}, & bc+1 &= p^{\alpha_4}q^{\beta_4},\\ ac+1 &= p^{\alpha_2}, & bd+1 &= p^{\alpha_5},\\ ad+1 &= p^{\alpha_3}q^{\beta_3}, & cd+1 &= q^{\beta_6} \end{aligned}$$

is valid. Consider now the equation

$$ac\cdot bd = (p^{\alpha_2}-1)(p^{\alpha_5}-1) = \left(p^{\alpha_3}q^{\beta_3}-1\right)\left(p^{\alpha_4}q^{\beta_4}-1\right) = ad\cdot bc$$

and its consequence

$$p^{\alpha_2+\alpha_5} - p^{\alpha_2} - p^{\alpha_5} = p^{\alpha_3+\alpha_4}q^{\beta_3+\beta_4} - p^{\alpha_3}q^{\beta_3} - p^{\alpha_4}q^{\beta_4}. \tag{3}$$

By switching a, b and c, d simultaneously, we may assume that $\alpha_5 \geq \alpha_2$. Moreover, the p-adic valuation of the left and right hand side of (3) coincide, hence the least two of $\alpha_2, \alpha_3, \alpha_4$ and α_5 must be equal. In particular, we have the following three cases: $\alpha_2 = \alpha_3 \leq \alpha_4$, $\alpha_2 = \alpha_4 \leq \alpha_3$ and $\alpha_3 = \alpha_4 < \alpha_2$. Note that with $\alpha_2 = \alpha_5$ at least one further exponent is necessarily minimal.

Similarly, we can arrive at the equation

$$q^{\beta_1+\beta_6} - q^{\beta_1} - q^{\beta_6} = p^{\alpha_3+\alpha_4}q^{\beta_3+\beta_4} - p^{\alpha_3}q^{\beta_3} - p^{\alpha_4}q^{\beta_4},$$

where we may assume that $\beta_6 \geq \beta_1$. Thus, the least two of $\beta_1, \beta_3, \beta_4$ and β_6 must coincide. Hence, in total, we have 9 possibilities which will be treated successively (see Table 1).

α	β
$\alpha_2 = \alpha_3 \leq \alpha_4$	$\beta_1 = \beta_3 \leq \beta_4$
	$\beta_1 = \beta_4 \leq \beta_3$
	$\beta_3 = \beta_4 < \beta_1$
$\alpha_2 = \alpha_4 \leq \alpha_3$	$\beta_1 = \beta_3 \leq \beta_4$
	$\beta_1 = \beta_4 \leq \beta_3$
	$\beta_3 = \beta_4 < \beta_1$
$\alpha_3 = \alpha_4 < \alpha_2$	$\beta_1 = \beta_3 \leq \beta_4$
	$\beta_1 = \beta_4 \leq \beta_3$
	$\beta_3 = \beta_4 < \beta_1$

Table 1: List of cases

4.1. The Case $\alpha_2 = \alpha_3 \leq \alpha_4$ and $\beta_1 = \beta_3 \leq \beta_4$

Consider the triple (a, b, c) with

$$ab + 1 = q^{\beta_1}, \qquad ac + 1 = p^{\alpha_2}, \qquad bc + 1 = p^{\alpha_4} q^{\beta_4}.$$

The assumption $\beta_1 \leq \beta_4$ implies $ab < bc$, i.e., $a < c$ immediately. Similarly, $a < b$ is concluded from $\alpha_2 \leq \alpha_4$. Hence either $ab+1 \mid bc+1$ with $a < c < b$ or $ac+1 \mid bc+1$ with $a < b < c$ holds. But both cases contradict Lemma 2.1.

4.2. The Case $\alpha_2 = \alpha_3 \leq \alpha_4$ and $\beta_1 = \beta_4 \leq \beta_3$

We clone the treatment of the previous case. Consider the triple (a, b, c) and deduce $a < c$ and $a < b$. Then either $ab + 1 \mid bc + 1$ with $a < c < b$ or $ac + 1 \mid bc + 1$ with $a < b < c$, and we arrive again at a contradiction.

4.3. The Case $\alpha_2 = \alpha_3 \leq \alpha_4$ and $\beta_3 = \beta_4 < \beta_1$

For simplicity we omit certain subscripts by writing $\beta := \beta_3 = \beta_4$ and $\alpha := \alpha_2 = \alpha_3$. Comparing $ac + 1$ with $bc + 1$ and $ad + 1$ we obtain $a < b$ and $c < d$, and therefore $\alpha_4 < \alpha_5$. Moreover, by the properties of triple (a, b, c) we have $c < b$, otherwise a contradiction to Lemma 2.1 occurs.

Now consider the equation

$$ad \cdot bc = \left(p^{\alpha} q^{\beta} - 1\right)\left(p^{\alpha_4} q^{\beta} - 1\right) = \left(p^{\alpha} - 1\right)\left(p^{\alpha_5} - 1\right) = ac \cdot bd$$

modulo p^{α_4}. We get $p^{\alpha} q^{\beta} - 1 \equiv p^{\alpha} - 1 \pmod{p^{\alpha_4}}$ and then

$$q^{\beta} \equiv 1 \pmod{p^{\alpha_4 - \alpha}}. \tag{4}$$

This yields $p^{\alpha_4 - \alpha} \mid q^{\beta} - 1$, i.e., $p^{\alpha_4 - \alpha} \leq q^{\beta} - 1$.

At this point we distinguish the two cases: $\beta_1 \geq 2\beta$ and $\beta_1 < 2\beta$. Taking

$$ad \cdot bc = \left(p^{\alpha} q^{\beta} - 1\right)\left(p^{\alpha_4} q^{\beta} - 1\right) = \left(q^{\beta_6} - 1\right)\left(q^{\beta_1} - 1\right) = ab \cdot cd$$

modulo $q^{2\beta}$, the congruence

$$q^{\beta}\left(p^{\alpha_4} + p^{\alpha}\right) \equiv 0 \pmod{q^{2\beta}}$$

follows. This yields $q^{\beta} \mid p^{\alpha_4 - \alpha} + 1$. Thus, $p^{\alpha_4 - \alpha} \geq q^{\beta} - 1$, which together with (4) implies $p^{\alpha_4 - \alpha} = q^{\beta} - 1$. Clearly, the two sides have opposite parity.

In case of $2\beta > \beta_1$ consider

$$ad \cdot bc = \left(p^{\alpha} q^{\beta} - 1\right)\left(p^{\alpha_4} q^{\beta} - 1\right) = \left(q^{\beta_1} - 1\right)\left(q^{\beta_6} - 1\right) = ab \cdot cd$$

modulo q^{β_1}. Thus,

$$q^{\beta}\left(p^{\alpha_4} + p^{\alpha}\right) \equiv 0 \pmod{q^{\beta_1}}.$$

Consequently, $q^{\beta_1-\beta} \mid p^{\alpha_4-\alpha}+1$.

A simple calculation shows that

$$\frac{b}{a} = \frac{bc}{ac} = \frac{p^{\alpha_4}q^{\beta}-1}{p^{\alpha}-1} = p^{\alpha_4-\alpha}q^{\beta} + \frac{p^{\alpha_4-\alpha}q^{\beta}-1}{p^{\alpha}-1} > p^{\alpha_4-\alpha}q^{\beta}.$$

If we assume $q^{\beta_1-\beta} \neq p^{\alpha_4-\alpha}+1$, then $2q^{\beta_1-\beta} \leq p^{\alpha_4-\alpha}+1$ follows. Thus, we have

$$\frac{b}{a} > 2q^{\beta_1} - q^{\beta} > \frac{5}{3}q^{\beta_1} > q^{\beta_1} > b,$$

which is a contradiction. Then $q^{\beta_1-\beta} = p^{\alpha_4-\alpha}+1$ holds, and it contradicts the fact that $q^{\beta_1-\beta}$ and $p^{\alpha_4-\alpha}+1$ have opposite parity.

4.4. The Case $\alpha_2 = \alpha_4 \leq \alpha_3$ and $\beta_1 = \beta_3 \leq \beta_4$

Similar to the case 4.1 ($\alpha_2 = \alpha_3 \leq \alpha_4$ and $\beta_1 = \beta_3 \leq \beta_4$), consider the triple (a, b, c) to find a contradiction to Lemma 2.1.

4.5. The Case $\alpha_2 = \alpha_4 \leq \alpha_3$ and $\beta_1 = \beta_4 \leq \beta_3$

Again, similar to the case 4.1, the triple (a, b, c) contradicts Lemma 2.1.

4.6. The Case $\alpha_2 = \alpha_4 \leq \alpha_3$ and $\beta_3 = \beta_4 < \beta_1$

Write $\beta := \beta_3 = \beta_4$ and $\alpha := \alpha_2 = \alpha_4$. Comparing $ac+1$ with $bc+1$ we obtain $a < b$. Since $p^{\alpha_3}q^{\beta}-1 = ad < bd = p^{\alpha_5}-1$ we have $\alpha_3 \leq \alpha_5$. The equations

$$ad \cdot bc = \left(p^{\alpha_3}q^{\beta}-1\right)\left(p^{\alpha}q^{\beta}-1\right) = \left(p^{\alpha}-1\right)\left(p^{\alpha_5}-1\right) = ac \cdot bd$$

modulo p^{α_3} admit $p^{\alpha}q^{\beta}-1 \equiv p^{\alpha}-1 \pmod{p^{\alpha_3}}$. Therefore, $q^{\beta} \equiv 1 \pmod{p^{\alpha_3-\alpha}}$, and hence $p^{\alpha_3-\alpha} \mid q^{\beta}-1$. We also have $c \mid c(b-a) = p^{\alpha}(q^{\beta}-1)$. Thus, $c \mid q^{\beta}-1$ and $c < q^{\beta}$ follow. Since c and p are coprime (note that $ac+1 = p^{\alpha_3}$), $c \mid \frac{q^{\beta}-1}{p^{\alpha_3-\alpha}}$. Clearly, $bc+1 = p^{\alpha}q^{\beta}$ implies $p^{\alpha}q^{\beta} \leq \frac{q^{\beta}-1}{p^{\alpha_3-\alpha}}b+1$ and then

$$b \geq \frac{p^{\alpha_3}q^{\beta}-p^{\alpha_3-\alpha}}{q^{\beta}-1} \geq p^{\alpha_3} - \frac{p^{\alpha_3}}{p^{\alpha}q^{\beta}} \geq p^{\alpha_3}\left(1-\frac{1}{p^{\alpha}}\right).$$

On the other hand, $b \mid b(a-c) = q^{\beta}(q^{\beta_1-\beta}-p^{\alpha})$. Thus, $b \mid q^{\beta_1-\beta}-p^{\alpha}$. The assumption $b < p^{\alpha}$ leads to the contradiction $bc+1 < p^{\alpha}q^{\beta}$. Therefore, we necessarily obtain $q^{\beta_1-\beta} > p^{\alpha}$. Hence $b \leq q^{\beta_1-\beta}-p^{\alpha}$. But $ab+1 = q^{\beta_1} \leq (q^{\beta_1-\beta}-p^{\alpha})a+1$, and we have

$$a \geq \frac{q^{\beta_1}-1}{q^{\beta_1-\beta}-p^{\alpha}}.$$

For the moment, assume that $d > b$. Then we have

$$p^{\alpha_3}q^{\beta} = \ ad+1 > ab > p^{\alpha_3}\left(1-\frac{1}{p^{\alpha}}\right)\frac{q^{\beta_1}-1}{q^{\beta_1-\beta}-p^{\alpha}} = p^{\alpha_3}q^{\beta}\left(1-\frac{1}{p^{\alpha}}\right)\frac{q^{\beta_1}-1}{q^{\beta_1}-p^{\alpha}q^{\beta}} > p^{\alpha_3}q^{\beta}.$$

Indeed,

$$\frac{p^{\alpha}-1}{p^{\alpha}}\cdot\frac{q^{\beta_1}-1}{q^{\beta_1}-p^{\alpha}q^{\beta}} \ > \ 1$$

is implied by $p^{\alpha}+q^{\beta_1} < p^{2\alpha}q^{\beta}$, which is coming from

$$q^{\beta}p^{2\alpha} = (ac+1)(bc+1) > ab+1+ac+1 = q^{\beta_1}+p^{\alpha}.$$

Hence $d < b$. But this relation, together with $c < a$ leads to

$$cd+1 = q^{\beta_6} < q^{\beta_1} = ab+1,$$

which contradicts the assumption $\beta_1 \leq \beta_6$.

4.7. The Case $\boldsymbol{\alpha_3 = \alpha_4 < \alpha_2}$ and $\boldsymbol{\beta_1 = \beta_3 \leq \beta_4}$

By switching p, q and b, c, we arrive at the case $\alpha_2 = \alpha_3 \leq \alpha_4$ and $\beta_3 = \beta_4 < \beta_1$.

4.8. The Case $\boldsymbol{\alpha_3 = \alpha_4 < \alpha_2}$ and $\boldsymbol{\beta_1 = \beta_4 \leq \beta_3}$

This is equivalent to the case $\alpha_2 = \alpha_4 \leq \alpha_3$ and $\beta_3 = \beta_4 < \beta_1$ by exchanging p and q, respectively b and c.

4.9. The Case $\boldsymbol{\alpha_3 = \alpha_4 < \alpha_2}$ and $\boldsymbol{\beta_3 = \beta_4 < \beta_1}$

First suppose $c < a$. From $cd+1 = q^{\beta_6} \geq q^{\beta_1} = ab+1$ we deduce $d > b$. Then $ad+1 = p^{\alpha}q^{\beta} = bc+1$, contradicting $c < a$ and $b < d$.

Now assume that $b < a$. Then $bd+1 = p^{\alpha_5} \geq p^{\alpha_2} = ac+1$, and therefore $d > c$ follows. Thus, we again arrive at a contradiction to $ad+1 = p^{\alpha}q^{\beta} = bc+1$.

Consequently, $a < b$, $a < c$, and

$$b \mid b(c-a) = q^{\beta}(p^{\alpha}-q^{\beta_1-\beta}) \quad \text{and} \quad c \mid c(b-a) = p^{\alpha}(q^{\beta}-p^{\alpha_2-\alpha}).$$

Hence $b < q^{\beta}$ and $c < p^{\alpha}$ follow, and $p^{\alpha}q^{\beta} < bc+1 = p^{\alpha}q^{\beta}$ shows the final contradiction.

References

[1] Y. Bugeaud and A. Dujella. On a problem of Diophantus for higher powers. *Math. Proc. Cambridge Philos. Soc.*, 135(1): 1–10, 2003.

[2] A. Dujella. On `http://web.math.hr/~duje/dtuples.html`.

[3] A. Dujella. There are only finitely many Diophantine quintuples. *J. Reine Angew. Math.*, 566: 183–214, 2004.

[4] A. Dujella and C. Fuchs. Complete solution of the polynomial version of a problem of Diophantus. *J. Number Theory*, 106(2): 326–344, 2004.

[5] C. Fuchs, F. Luca, and L. Szalay. Diophantine triples with values in binary recurrences. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 7(4): 579–608, 2008.

[6] F. Luca and L. Szalay. Fibonacci Diophantine triples. *Glas. Mat. Ser. III*, 43(63)(2): 253–264, 2008.

[7] L. Szalay and V. Ziegler. On an S-unit variant of Diophantine m-tuples. *Publ. Math. Debrecen*, 83(1-2): 97–121, 2013.