ON A SEQUENCE OF POLYNOMIALS WITH HYPOTHETICALLY INTEGER COEFFICIENTS

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Received: 3/31/12, Revised: 12/11/12, Accepted: 2/3/13, Published: 2/8/13

Abstract

The first author introduced a sequence of polynomials defined recursively. One of the main results of this study is proof of the integrality of its coefficients.

1. Introduction

In point of fact, there are only a few examples of sequences known where the question of the integrality of the terms is a difficult problem. In 1989, Somos [9] posed a problem on the integrality of sequences depending on parameter $k \geq 4$ which are defined by the recursion

$$a_n = \frac{\sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} a_{n-j} a_{n-(k-j)}}{a_{n-k}}, \;\; n \geq k \geq 4,$$

with the initial conditions $a_i = 1, \;\; i = 1, \ldots, k-1$.
Gale [3] proved the integrality of Somos sequences when $k = 4$ and 5, attributing a proof to Malouf [4]. Hickerson and Stanley (see [6]) independently proved the integrality of the $k = 6$ case in unpublished work and Fomin and Zelevinsky (2002) gave the first published proof. Finally, Lotto (1990) gave an unpublished proof for the $k = 7$ case. These are sequences A006720-A006723 in [8]. It is interesting that, for $k \geq 8$, the property of integrality disappears (see sequence A030127 in [8]). In connection with this, note that in the so-called Göbel's sequence ([11]) defined by the recursion

$$x_n = \frac{1}{n}(1 + \sum_{i=0}^{n-1} x_i^2), \;\; n \geq 1, \;\; x_0 = 1,$$

the first non-integer term is $x_{43} = 5.4093 \times 10^{178485291567}$.

In this paper we study the Shevelev sequence of polynomials $\{P_n(x)\}_{n\geq 1}$ that are defined by the following recursion: $P_1 = 1,\ \ P_2 = 1$, and, for $n \geq 2$,

$$4(2x+n)P_{n+1}(x) = 2(x+n)P_n(x) + \\ (2x+n)P_n(x+1) + (4x+n)l_n(x), \text{ if } n \text{ is odd}, \tag{1}$$

$$4P_{n+1}(x) = 4(x+n)P_n(x) + \\ 2(2x+n+1)P_n(x+1) + (4x+n)l_{n-1}(x), \text{ if } n \text{ is even}, \tag{2}$$

where

$$l_n(x) = (x + \frac{n-1}{2})(x + \frac{n-3}{2}) \cdots (x+1).$$

The first few polynomials are ([8], sequence A174531):

$$P_1 = 1,$$

$$P_2 = 1,$$

$$P_3 = 3x + 4,$$

$$P_4 = 2x + 4,$$

$$P_5 = 5x^2 + 25x + 32,$$

$$P_6 = 3x^2 + 19x + 32,$$

$$P_7 = 7x^3 + 77x^2 + 294x + 384,$$

$$P_8 = 4x^3 + 52x^2 + 240x + 384,$$

$$P_9 = 9x^4 + 174x^3 + 1323x^2 + 4614x + 6144,$$

$$P_{10} = 5x^4 + 110x^3 + 967x^2 + 3934x + 6144,$$

$$P_{11} = 11x^5 + 330x^4 + 4169x^3 + 27258x^2 + 90992x + 122880,$$

$$P_{12} = 6x^5 + 200x^4 + 2842x^3 + 21040x^2 + 79832x + 122880.$$

According to our observations, the following conjectures are natural.

1) The coefficients of all the polynomials are integers. Moreover, the greatest common divisor of all coefficients is $n/rad(n)$, where $rad(n) = \prod_{p|n} p$;

2) $P_n(0) = 4^{\lfloor \frac{n-1}{2} \rfloor} \lfloor \frac{n-1}{2} \rfloor !$;

3) For even n, $P_n(1) = (2^n - 1)(\frac{n}{2})!/(n+1)$, and for odd n, $P_n(1) = (2^n - 1)(\frac{n-1}{2})!$;

4) $P_n(x)$ has a real rational root if and only if either $n = 3$ or $n \equiv 0 \pmod 4$. In the latter case, such a unique root is $-\frac{n}{2}$;

5) Coefficients of x^k increase when k decreases;
6) If n is even, then the coefficients of P_n do not exceed the corresponding coefficients of P_{n-1} and the equality holds only for the last ones; moreover, the ratios of coefficients of x^k of polynomials P_{n-1} and P_n monotonically decrease to 1 when k decreases;
7) All coefficients of P_n, except of the last one, are multiples of n if and only if n is prime.
The main results of our paper consist of the following two theorems.

Theorem 1. (Explicit formula for $P_n(k)$) *For an integer k we have*

$$P_n(k) = \begin{cases} \left(\binom{(n-1)/2+k-1}{k-1}/\binom{n+2k-2}{k-1}\right)(\frac{n-1}{2})!\ T_n(k), & \text{if } n \geq 1 \text{ is odd} \\ \left(\binom{n/2+k-1}{k}/\binom{n+2k-1}{k}\right)(\frac{n}{2}-1)!\ T_n(k), & \text{if } n \geq 2 \text{ is even}, \end{cases} \tag{3}$$

$$= 2^{-(\lfloor \frac{n}{2} \rfloor + k - 1)} \frac{(n+k-1)!}{(2\lfloor \frac{n}{2} \rfloor + 2k-1)!!} T_n(k), \tag{4}$$

where

$$T_n(k) = \sum_{i=1}^{n} 2^{i-1} \binom{n+2k-i-1}{k-1}. \tag{5}$$

Using Theorem 1, we prove Conjectures (2), (3) and the following main result.

Theorem 2. *For $n \geq 1$, $P_n(x)$ is a polynomial of degree $\lfloor \frac{n-1}{2} \rfloor$ with integer coefficients.*

Nevertheless, the subtle second part of Conjecture (1) remains open.

2. Representation of $P_n(k)$ Via a Polynomial in n of Degree $k-1$ with Integer Coefficients

Theorem 3. *For integers $k \geq 1$, $n \geq 1$, the following recursion holds*

$$P_n(k) = c_n(k)\left(2^{n+k-1} - \frac{R_k(n)}{(2k-2)!!}\right), \tag{6}$$

where $R_k(n)$ is a polynomial in n of degree $k-1$ with integer coefficients and

$$c_n(k) = \begin{cases} (\frac{n-1}{2})! \prod_{i=1}^{k-1} \frac{n+i}{n+2i}, & \text{if } n \text{ is odd}, \\ \frac{1}{2}(\frac{n}{2}-1)! \prod_{i=0}^{k-1} \frac{n+i}{n+2i+1}, & \text{if } n \text{ is even}. \end{cases} \tag{7}$$

Proof. Write (3) and (4) in the form

$$P_n(k+1) = -\frac{2f}{g}P_n(k) + 4P_{n+1}(k) - \frac{h}{g}(\frac{n-1}{2})!\binom{\frac{g-1}{2}}{k}, \ \textit{if}\ n \equiv 1 \ (\mathrm{mod}\ 2); \quad (8)$$

$$P_n(k+1) = -\frac{2f}{g+1}P_n(k) + \frac{2}{g+1}P_{n+1}(k) - \frac{h}{2(g+1)}(\frac{n}{2}-1)!\binom{\frac{g}{2}-1}{k} \quad (9)$$

$$\textit{if}\ n \equiv 0 \ (\mathrm{mod}\ 2),$$

where $f = n+k, \ g = n+2k, \ h = n+4k$.

Let n be odd. We use induction over k. For $k = 1$, (6) gives

$$R_1(n) = 2^n - \frac{P_n(1)}{c_n(1)} = Const(k). \quad (10)$$

Thus the base of induction is valid. Suppose the theorem is true for some value of k. Then, using this supposition and (6) to (9), we have

$$P_n(k+1) =$$

$$-\frac{2f}{g}(\frac{n-1}{2})!\Big(2^{n+k-1} - \frac{R_k(n)}{(2k-2)!!}\Big)\prod_{i=1}^{k-1}\frac{n+i}{n+2i}+$$

$$2(\frac{n-1}{2})!\Big(2^{n+k} - \frac{R_k(n+1)}{(2k-2)!!}\Big)\prod_{i=0}^{k-1}\frac{n+i+1}{n+2i+2}-$$

$$\frac{h}{g}(\frac{n-1}{2})!\frac{\frac{g-1}{2}\frac{g-3}{2}\cdots\frac{n+1}{2}}{k!}.$$

Note that

$$\frac{f}{g}\prod_{i=0}^{k-1}\frac{n+i}{n+2i} = \prod_{j=1}^{k}\frac{n+j}{n+2j} = \prod_{i=0}^{k-1}\frac{n+i+1}{n+2i+2}.$$

Therefore,

$$P_n(k+1) = (\frac{n-1}{2})!\Big(-2^{n+k} + \frac{2R_k(n)}{(2k-2)!!} + 2^{n+k+1} -$$

$$\frac{2R_k(n+1)}{(2k-2)!!} - \frac{h}{g}\frac{\frac{g-1}{2}\frac{g-3}{2}\cdots\frac{n+1}{2}}{k!}\prod_{j=1}^{k}\frac{n+2j}{n+j}\Big)\prod_{j=1}^{k}\frac{n+j}{n+2j}.$$

Here we note that

$$(g-1)(g-3)\cdots(n+1)\prod_{j=1}^{k}\frac{n+2j}{n+j} = (n+2k)_k,$$

where $(x)_k$ is a falling factorial. Hence

$$P_n(k+1) = c_n(k+1)\Big(2^{n+k} - 2\frac{R_k(n+1) - R_k(n)}{(2k-2)!!} -$$

$$\frac{4k+n}{(2k)!!}(n+2k-1)_{k-1}\Big) = c_n(k+1)\Big(2^{n+k} - \frac{R_{k+1}(n)}{(2k)!!}\Big),$$

where

$$R_{k+1}(n) = 4k(R_k(n+1) - R_k(n)) + (4k+n)(n+2k-1)_{k-1}. \tag{11}$$

Since, by the inductive supposition, $R_k(n)$ is a polynomial of degree $k-1$ with integer coefficients, then, by (11), $R_{k+1}(n)$ is a polynomial of degree k with integer coefficients. Note that the case of even n is considered quite analogously, obtaining the *same* formula (11). □

In (6) and (7), put $n = 1$. Then, for $k \geq 1$ we have

$$\Big(2^k - \frac{R_k(1)}{(2k-2)!!}\Big)\frac{k!}{(2k-1)!!} = 1,$$

from which

$$R_k(1) = (k-1)!\Big(2^{2k-1} - \binom{2k-1}{k}\Big). \tag{12}$$

In particular, $R_1(1) = 1$ and, since $R_1(n)$ is of degree 0, $R_1(n) = 1$. Further, we find polynomials $R_k(n)$ using the recursion (11). The first polynomials $R_k(n)$ are

$$R_1(n) = 1,$$

$$R_2(n) = n + 4,$$
$$R_3(n) = n^2 + 11n + 32,$$
$$R_4(n) = n^3 + 21n^2 + 152n + 384,$$
$$R_5(n) = n^4 + 34n^3 + 443n^2 + 2642n + 6144,$$
$$R_6(n) = n^5 + 50n^4 + 1015n^3 + 10510n^2 + 55864n + 122880.$$

3. Proof of Conjectures (2) and (3)

We start with the proof of Conjecture (3) for $P_n(1)$.

Proof. Note that, since $R_1(n) = 1$, from (10) we find

$$P_n(1) = c_n(1)(2^n - 1). \tag{13}$$

Besides, by (7), we have

$$c_n(1) = \begin{cases} (\frac{n-1}{2})!, \quad if \;\; n \;\; is \;\; odd, \\ \\ \frac{1}{2}(\frac{n}{2}-1)!\frac{n}{n+1} = (\frac{n}{2})!/(n+1), \quad if \;\; n \;\; is \;\; even, \end{cases} \tag{14}$$

and Conjecture (3) follows. □

Let us now prove Conjecture (2).

Proof. Note that (8) and (9), as in (1) and (2), are valid for every nonnegative k. For $k = 0$ and odd $n \geq 1$, (8) gives

$$P_n(1) = -2P_n(0) + 4P_{n+1}(0) - \big(\frac{n-1}{2}\big)!,$$

or, using (13) and (14), we have

$$4P_{n+1}(0) - 2P_n(0) = 2^n\big(\frac{n-1}{2}\big)!$$

Analogously, for $k = 0$ and even $n \geq 1$, from (9), (13) and (14) we find

$$P_{n+1}(0) - nP_n(0) = 2^{n-1}\big(\frac{n}{2}\big)!$$

Thus

$$P_{n+1}(0) = \begin{cases} \frac{1}{2}P_n(0) + 2^{n-2}(\frac{n-1}{2})!, \text{ if } n \text{ is odd,} \\ \\ nP_n(0) + 2^{n-1}(\frac{n}{2})!, \text{ if } n \text{ is even} \end{cases}$$

with $P_1(0) = 1, \;\; P_2(0) = 1$. Since the difference equation

$$y(n+1) = \begin{cases} \frac{1}{2}y(n) + 2^{n-2}(\frac{n-1}{2})!, \text{ if } n \text{ is odd,} \\ \\ ny(n) + 2^{n-1}(\frac{n}{2})!, \text{ if } n \text{ is even} \end{cases}$$

with the initial values $y(1) = 1, \;\; y(2) = 1$ has an unique solution, it is sufficient to verify that $y(n) = P_n(0) = 4^{\lfloor \frac{n-1}{2} \rfloor}\lfloor \frac{n-1}{2} \rfloor!$ is a solution. □

4. Explicit Formula for $R_k(n)$

Since from (11)

$$4kR_k(n+1) = 4kR_k(n)+ \\ R_{k+1}(n) - (4k+n)(n+2k-1)_{k-1}, \tag{15}$$

we have a recursion in n for $R_k(n)$ given by (12) and (15).
Our aim in this section is to find a generalization of (12) for an arbitrary integer $n \geq 1$. Note that we can write (12) in the form

$$R_k(1) = 2(k-1)!4^{k-1} - \frac{(2k-1)!}{k!}.$$

Using (15) and (12), after some transformations, we find

$$R_k(2) = 2^2(k-1)!4^{k-1} - 2\frac{(2k-1)!}{k!} - \frac{(2k)!}{(k+1)!}.$$

The regularity is fixed in the following theorem.

Theorem 4. *For integer $k \geq 1$, $n \geq 1$, we have*

$$R_k(n) = 2^n(k-1)!4^{k-1} - \sum_{i=1}^{n} 2^{n-i}\frac{(2k+i-2)!}{(k+i-1)!}. \tag{16}$$

Proof. Taking into account that $\frac{(2k+i-2)!}{(k+i-1)!} = \binom{2k+i-2}{k-1}(k-1)!$, we prove (16) in the following equivalent form:

$$R_k(n) = 2^n(k-1)!\Big(4^{k-1} - \sum_{i=1}^{n} 2^{-i}\binom{2k+i-2}{k-1}\Big). \tag{17}$$

We use induction over n. Suppose that (17) is valid for some value of n and an arbitrary integer $k \geq 1$. By (15), we have

$$R_k(n+1) = 2^n(k-1)!\Big(4^{k-1} - \sum_{i=1}^{n} 2^{-i}\binom{2k+i-2}{k-1}\Big)+$$

$$2^{n-2}(k-1)!\Big(4^k - \sum_{i=1}^{n} 2^{-i}\binom{2k+i}{k}\Big) - \frac{4k+n}{4k}(n+2k-1)_{k-1} =$$

$$2^n(k-1)!\Big(4^{k-1} - \sum_{i=1}^{n} 2^{-i}\binom{2k+i-2}{k-1}\Big) + 2^n(k-1)!\Big(4^{k-1} - \sum_{i=1}^{n} 2^{-i-2}\binom{2k+i}{k}\Big) -$$

$$\frac{(n+2k-1)!}{(n+k)!} - \frac{n}{4k}\frac{(n+2k-1)!}{(n+k)!}.$$

Thus we should prove the identity

$$2^{n+1}(k-1)!4^{k-1} - 2^n(k-1)!\sum_{i=1}^{n} 2^{-i}\binom{2k+i-2}{k-1} -$$

$$2^{n-2}(k-1)!\sum_{i=1}^{n} 2^{-i}\binom{2k+i}{k} - \frac{n+4k}{4k}\frac{(n+2k-1)!}{(n+k)!} =$$

$$2^{n+1}(k-1)!\Big(4^{k-1} - \sum_{i=1}^{n+1} 2^{-i}\binom{2k+i-2}{k-1}\Big),$$

which is easily reduced to the identity

$$4\sum_{i=1}^{n} 2^{-i}\binom{2k+i-2}{k-1} - \sum_{i=1}^{n} 2^{-i}\binom{2k+i}{k} =$$

$$2^{-n}\frac{n+4k}{4k}\frac{(n+2k-1)!}{(n+k)!} - 4\cdot 2^{-n}\binom{2k+n-1}{k-1}.$$

Note that, the right hand part is $\frac{n}{k2^n}\binom{2k+n-1}{k-1}$. Therefore, it is left to prove the identity

$$4\sum_{i=1}^{n} 2^{-i}\binom{2k+i-2}{k-1} - \sum_{i=1}^{n} 2^{-i}\binom{2k+i}{k} = \frac{n}{2^n k}\binom{2k+n-1}{k-1}.$$

Since this is trivially satisfied for $n = 0$, it is sufficient to verify the equality of the first differences of the left and the right hand parts, which is reduced to the identity

$$2(n+2k-1)\binom{2k+n-2}{k-1} = n\binom{2k+n-1}{k-1} + k\binom{2k+n}{k},$$

which is verified directly. □

5. Proof of Theorem 1

Now we are able to prove Theorem 1.

Proof. According to (5), we have

$$T_n(k) = \sum_{i=1}^{n} 2^{i-1}\binom{n+2k-i-1}{k-1} = \sum_{j=1}^{n} 2^{n-j}\binom{2k+j-2}{k-1}. \tag{18}$$

Hence, by (17), we find

$$R_k(n) = 2^n(k-1)!(4^{k-1} - 2^{-n}T_n(k)) =$$

$$(k-1)!(2^{n+2k-2}-T_n(k)). \tag{19}$$

Now from (6) and (19) we have

$$P_n(k)=2^{-(k-1)}c_n(k)T_n(k). \tag{20}$$

Let n be odd. Note that, by (7),

$$2^{-(k-1)}c_n(k)=$$

$$2^{-(k-1)}(\frac{n-1}{2})!\frac{(n+k-1)(n+k-2)\cdots(n+1)}{(n+2k-2)(n+2k-4)\cdots(n+2)}=$$

$$2^{-(k-1)}(\frac{n-1}{2})!\frac{(n+k-1)!n!!}{n!(n+2k-2)!!}. \tag{21}$$

Taking into account that

$$n!!=\frac{n!}{(n-1)!!}=\frac{n!}{2^{\frac{n-1}{2}}(\frac{n-1}{2})!}, \tag{22}$$

we find from (21)

$$2^{-(k-1)}c_n(k)=\frac{(n+k-1)!(\frac{n-1}{2}+k-1)!}{(n+2k-2)!}=$$

$$\frac{(\frac{n-1}{2}+k-1)!}{(k-1)!\binom{n+2k-2}{k-1}}=\frac{\binom{\frac{n-1}{2}+k-1}{k-1}}{\binom{n+2k-2}{k-1}}(\frac{n-1}{2})!$$

and (3) follows from (20). Furthermore, since by (22) $\frac{n!!(\frac{n-1}{2})!}{n!}=2^{-\frac{n-1}{2}}$, from (20) and (21) we find

$$P_n(k)=2^{-(\frac{n-1}{2}+k-1)}\frac{(n+k-1)!}{(n+2k-2)!!}T_n(k)$$

corresponds to (4) in the case of odd n. The case of even n is considered quite analogously. □

6. Bisection of Sequence $\{P_n(x)\}$

Note that $T_n(k)$, (5), has rather a simple structure, which allows us to find different relations for it. Using (3) and (4), we are able to find recursion relations for $P_n(x)$ which are simpler than the basis recursion (1) and (2). We start with the following simple recursions for $T_n(k)$.

Lemma 5.

$$T_n(k) - 2T_{n-1}(k) = \binom{n+2k-2}{k-1}, \quad k \geq 1; \tag{23}$$

$$T_n(k) - 4T_{n-2}(k) = \binom{n+2k-2}{k-1} + 2\binom{n+2k-3}{k-1}, \quad k \geq 2. \tag{24}$$

Proof. By (5), we have

$$T_n(k) - 2T_{n-1}(k) =$$

$$\sum_{i=1}^{n} 2^{i-1}\binom{n+2k-i-1}{k-1} - \sum_{j=1}^{n-1} 2^{j}\binom{n+2k-j-2}{k-1} =$$

$$\sum_{i=1}^{n} 2^{i-1}\binom{n+2k-i-1}{k-1} - \sum_{i=2}^{n} 2^{i-1}\binom{n+2k-i-1}{k-1}$$

and (23) follows; (24) is a simple corollary of (23). □

Theorem 6. (Bisection) *If $n \geq 3$ is odd, then*

$$(2x+n-2)P_n(x) = 2(x+n-1)(x+n-2)P_{n-2}(x)+$$

$$(4x+3n-4)(x+\frac{n-1}{2}-1)(x+\frac{n-1}{2}-2)\cdots x; \tag{25}$$

if $n \geq 4$ is even, then

$$(2x+n-1)P_n(x) = 2(x+n-1)(x+n-2)P_{n-2}(x)+$$

$$\frac{1}{2}(4x+3n-4)(x+\frac{n-2}{2}-1)(x+\frac{n-2}{2}-2)\cdots x. \tag{26}$$

Proof. According to (3), we have

$$T_n(k) = \begin{cases} \binom{n+2k-2}{k-1}/(\binom{(n-1)/2+k-1}{k-1}(\frac{n-1}{2})!) \;\; P_n(k), \text{ if } n \text{ is odd,} \\ \\ \binom{n+2k-1}{k}/(\binom{n/2+k-1}{k}(\frac{n}{2}-1)!) \;\; P_n(k), \text{ if } n \text{ is even.} \end{cases} \tag{27}$$

Substituting this to (24), after simple transformations, we obtain (25) and (26), where k is replaced by arbitrary x. □

Note that from (25) and (26), using a simple induction, we conclude that, for even $n \geq 4$, $P_n(x)$ is a polynomial of degree $\frac{n-2}{2}$, while, for odd $n \geq 3$, $P_n(x)$ is a polynomial of degree $\frac{n-1}{2}$. However, the structure of formulas (25) and (26) does not allow us to prove that all coefficients of $P_n(x)$ are integer. This will be done in the following section by the discovery of the special relationships with the required structure.

7. Proof of Theorem 2

Lemma 7. *For $n \geq 1$, we have*

$$T_n(k) - T_{n-2}(k+1) = \binom{n+2k-1}{k}. \tag{28}$$

Proof. By (18), we should prove that

$$\binom{2k+n-1}{k} = T_n(k) - T_{n-2}(k+1) =$$

$$\sum_{j=1}^{n} 2^{n-j}\binom{2k+j-2}{k-1} - \sum_{j=1}^{n-2} 2^{n-j-2}\binom{2k+j}{k} =$$

$$\sum_{j=1}^{n} 2^{n-j}\binom{2k+j-2}{k-1} - \sum_{i=1}^{n} 2^{n-i}\binom{2k+i-2}{k} +$$

$$2^{n-1}\binom{2k-1}{k} + 2^{n-2}\binom{2k}{k},$$

or

$$\sum_{j=1}^{n} 2^{-j}\left(\binom{2k+j-2}{k-1} - \binom{2k+j-2}{k}\right) =$$

$$2^{-n}\binom{2k+n-1}{k} - \frac{1}{2}\binom{2k-1}{k} - \frac{1}{4}\binom{2k}{k}. \tag{29}$$

It is verified directly that (29) is valid for $n = 1$. Therefore, it is sufficient to verify that the first differences over n of the left hand side and the right hand side coincide. The corresponding identity

$$2^{-n}\left(\binom{2k+n-2}{k-1} - \binom{2k+n-2}{k}\right) =$$

$$2^{-n}\binom{2k+n-1}{k} - 2^{-n+1}\binom{2k+n-2}{k}$$

reduces to the equality $\binom{2k+n-2}{k-1} + \binom{2k+n-2}{k} = \binom{2k+n-1}{k}$.

Now we are able to complete proof of Theorem 2. Considering even $n \geq 4$, by (27), we obtain the following relation for $P_n(k)$ corresponding to (28):

$$P_n(x) = (n+x-1)P_{n-2}(x+1)+$$

$$(x+\frac{n}{2}-1)(x+\frac{n}{2}-2)\cdots(x+1). \tag{30}$$

On the other hand, using (23), for odd $n \geq 3$, we obtain the following relation

$$P_n(x) = 2(x+n-1)P_{n-1}(x)+$$

$$(x+\frac{n-1}{2}-1)(x+\frac{n-1}{2}-2)\cdots x. \tag{31}$$

From (30), by simple induction, we see that, for even $n \geq 4$, $P_n(x)$ is a polynomial with integer coefficients. Then from (31) we find that $P_n(x)$, for odd n, is also a polynomial with integer coefficients. □

8. Other Relations

Together with (25), (26), (30) and (31) there exist many other relations for $P_n(x)$. All of them are corollaries of the corresponding relations for $T_n(k)$. Below we give a few pairs of some such relations.

As we saw, for odd $n \geq 3$, (31) follows from (23). Let us consider even $n \geq 4$. Then we obtain the second component of the following recursion

$$P_n(x) = \begin{cases} 2(x+n-1)P_{n-1}(x)+ \\ ((x+n-1)P_{n-1}(x)+ \end{cases}$$

$$\begin{cases} (x+\frac{n-1}{2}-1)(x+\frac{n-1}{2}-2)\cdots x, \text{ if } n \geq 3 \text{ is odd,} \\ (x+\frac{n}{2}-1)(x+\frac{n}{2}-2)\cdots x)/(2x+n-1), \text{ if } n \geq 4 \text{ is even.} \end{cases}$$

Lemma 8. *For $n \geq 1$, $k \geq 1$, we have*

$$T_n(k+1) = 4T_n(k) - \frac{n}{k}\binom{n+2k-1}{k-1}. \tag{32}$$

Proof. By (24) and (28), we have

$$T_n(k+1) = T_{n+2}(k) - \binom{n+2k+1}{k} =$$

$$4T_n(k) + \binom{n+2k}{k-1} + 2\binom{n+2k-1}{k-1} - \binom{n+2k+1}{k}.$$

It is left to note that

$$\binom{n+2k}{k-1} + 2\binom{n+2k-1}{k-1} - \binom{n+2k+1}{k} = -\frac{n}{k}\binom{n+2k-1}{k-1}.$$

□

From Lemma 8 and (27) we find the following recursion

$$\begin{cases} (2x+n)P_n(x+1) = 2(x+n)P_n(x) - \\ \\ (2x+n+1)P_n(x+1) = 2(x+n)P_n(x) - \end{cases}$$

$$\begin{cases} n(x+\frac{n-1}{2})(x+\frac{n-1}{2}-1)\cdots(x+1), \text{ if } n \geq 3 \text{ is odd,} \\ \\ \frac{n}{2}(x+\frac{n}{2}-1)(x+\frac{n}{2}-2)\cdots(x+1), \text{ if } n \geq 4 \text{ is even.} \end{cases}$$

Lemma 9. *For $n \geq 2, \;\; k \geq 1,$ we have*

$$(n+k-1)(T_n(k) - 4T_n(k-1)) = n(T_{n-1}(k) - 2T_n(k-1)). \tag{33}$$

Proof. By (32),

$$T_n(k) - 4T_n(k-1) = -\frac{n}{k-1}\binom{n+2k-3}{k-2}. \tag{34}$$

By (23),

$$T_n(k-1) = 2T_{n-1}(k-1) + \binom{n+2k-4}{k-2}.$$

Therefore,

$$T_{n-1}(k) - 2T_n(k-1) = T_{n-1}(k) - 4T_{n-1}(k-1) - 2\binom{n+2k-4}{k-2}.$$

Using again (32), we find

$$T_{n-1}(k) - 2T_n(k-1) = -(\frac{n-1}{k-1} + 2)\binom{n+2k-4}{k-2}. \tag{35}$$

Now the lemma follows from (34) and (35) since $(n+k-1)\binom{n+2k-3}{k-2} = (n+2k-3)\binom{n+2k-4}{k-2}$. □

Going from (33) to the corresponding formula for $P_n(x)$ in the case of odd $n \geq 3$ unexpectedly leads to a very simple homogeneous relation

$$P_n(x) = P_n(x-1) + nP_{n-1}(x) \tag{36}$$

which we use in Sections 9 and 12. The corresponding relation for even $n \geq 4$ is

$$(2x+n-1)P_n(x) = (2x+n-2)P_n(x-1) + \frac{n}{2}P_{n-1}(x). \tag{37}$$

Lemma 10. *For $n \geq 1$, $k \geq 2$, we have*

$$2T_n(k) - T_{n-1}(k+1) = \binom{n+2k-1}{k}.$$

Proof. By (23), we have

$$2T_n(k) - T_{n-1}(k+1) = 4T_{n-1}(k) + 2\binom{n+2k-2}{k-1} - T_{n-1}(k+1).$$

Furthermore, by (28),

$$T_{n-1}(k+1) = T_{n+1}(k) - \binom{n+2k}{k}.$$

Hence,

$$2T_n(k) - T_{n-1}(k+1) = 4T_{n-1}(k) - T_{n+1}(k) + 2\binom{n+2k-2}{k-1} + \binom{n+2k}{k}. \tag{38}$$

Finally, by (24),

$$T_{n+1}(k) - 4T_{n-1}(k) = \binom{n+2k-1}{k-1} + 2\binom{n+2k-2}{k-1}$$

and the lemma follows from (38). □

Using Lemma 10 and (27), for even $n \geq 4$, we find

$$2P_n(x) = P_{n-1}(x+1) + (x + \frac{n}{2} - 1)(x + \frac{n}{2} - 2) \cdots (x+1), \tag{39}$$

while, for odd $n \geq 3$,

$$P_n(x) = (2x+n)P_{n-1}(x+1) + (x + \frac{n-1}{2})(x + \frac{n-1}{2} - 1) \cdots (x+1).$$

Proposition 11. *For odd $n \geq 3$, we have $P_n(k) \equiv P_n(0) \pmod{n}$.*

Proof. From (36) we find that $\sum_{i=1}^{k} P_{n-1}(i) = (P_n(k) - P_n(0))/n$, and the proposition follows. □

9. On the Coefficients of $P_n(x)$

Using formulas (25) and (26), we give a recursion for the calculation of the coefficients of $P_n(x)$ with a fixed parity of n. Let

$$P_n(x) = a_0(n)x^m + a_1(n)x^{m-1} + \cdots + a_{m-1}(n)x + a_m(n),$$

where $m = \lfloor \frac{n-1}{2} \rfloor$. We prove the following.

Theorem 12. *For $n \geq 1$, we have*

$$a_0(n) = \begin{cases} n, & \text{if } n \text{ is odd} \\ \\ \frac{n}{2}, & \text{if } n \text{ is even;} \end{cases} \tag{40}$$

$$a_1(n) = \begin{cases} \frac{1}{24}(7n^3 - 12n^2 + 5n) \\ \\ \frac{1}{48}(7n^3 - 18n^2 + 8n) \end{cases} = \begin{cases} \frac{1}{24}n(n-1)(7n-5), & \text{if } n \text{ is odd,} \\ \\ \frac{1}{48}n(n-2)(7n-4), & \text{if } n \text{ is even.} \end{cases} \tag{41}$$

In general, for a fixed i, $a_i(n) = U_i(n)$, if n is odd, and $a_i(n) = V_i(n)$, if n is even, where U_i and V_i are polynomials in n of degree $2i+1$.

Proof. `Case 1).` Let n be even. Then, using (26), for integer x and $m = \frac{n-2}{2}$, we have

$$(2x+n-1)(a_0(n)x^m + a_1(n)x^{m-1} + \cdots) =$$
$$2(x+n-1)(x+n-2)(a_0(n-2)x^{m-1} + a_1(n-2)x^{m-2} + \cdots) +$$
$$\frac{1}{2}(\frac{n-2}{2})!(4x+3n-4)\binom{x-1+\frac{n-2}{2}}{\frac{n-2}{2}}. \tag{42}$$

Comparing the coefficient of x^{m+1} on both sides, we find

$$a_0(n) = a_0(n-2) + 1, \;\; n \geq 4, \;\; a_0(4) = 2.$$

Thus $a_0(6) = 3, a_0(8) = 4, \ldots, a_0(n) = n/2$.

Furthermore, comparing the coefficient of x^m on both sides of (42), we have

$$2a_1(n) + (n-1)a_0(n) = 2a_1(n-2) + 2(2n-3)a_0(n-2) +$$
$$Coef[x^m](\frac{1}{2}(4x+3n-4)(x+\frac{n-4}{2})(x+\frac{n-6}{2})\cdots(x+1)x). \tag{43}$$

Note that

$$Coef[x^m](\frac{1}{2}(4x+3n-4)(x+\frac{n-4}{2})(x+\frac{n-6}{2})\cdots(x+1)x) =$$
$$\frac{3n-4}{2} + 2(\frac{n-4}{2} + \frac{n-6}{2} + \cdots + 1) =$$
$$\frac{3n-4}{2} + \sum_{i=2}^{m}(n-2i) = \frac{n^2}{4}.$$

Therefore, by (43),

$$a_1(n) - a_1(n-2) = \frac{(2n-3)(n-2)}{2} - \frac{(n-1)n}{4} + \frac{n^2}{8} = \frac{7n^2 - 26n + 24}{8}.$$

Hence

$$a_1(n) = \sum_{i=4,6,\cdots,n} (a_1(i) - a_1(i-2)) = \frac{1}{8} \sum_{i=4,6,\ldots,n} (7i^2 - 26i + 24) =$$

$$\frac{1}{2} \sum_{j=2}^{n/2} (7j^2 - 13j + 6) = \frac{1}{48}(7n^3 - 18n^2 + 8n).$$

Finally, comparing the coefficient of x^{m-i} on both sides of (42), we find

$$2a_{i+1}(n) + (n-1)a_i(n) = 2a_{i+1}(n-2)+$$

$$2(2n-3)a_i(n-2) + 2(n-1)(n-2)a_{i-1}(n-2)+$$

$$\frac{1}{2} Coef[x^{m-i}]((4x + 3n - 4)(x + \frac{n-4}{2})(x + \frac{n-6}{2}) \cdots (x+1)(x)). \quad (44)$$

Note that, polynomial $(4x + 3n - 4)(x + \frac{n-4}{2})(x + \frac{n-6}{2}) \cdots (x+1)x$ has degree $m + 1$. Therefore, in order to calculate $Coef[x^{m-i}]$ in (44), we should choose, in all possible ways, in $m - i$ brackets (from $m + 1$ ones) $x's$, and for the other $i + 1$ brackets we choose linear forms of n. Thus $\frac{1}{2} Coef[x^{m-i}]$ in (44) is a polynomial $r_i(n)$ of degree $i + 1$. Further we use induction over i with the formulas (40) and (41) as the inductive base. Write (44) in the form

$$2(a_{i+1}(n) - a_{i+1}(n-2)) =$$

$$2(2n-3)a_i(n-2) - (n-1)a_i(n) + 2(n-1)(n-2)a_{i-1}(n-2) + r_i(n). \quad (45)$$

By the inductive supposition, $a_{i-1}(n)$ and $a_i(n)$ are polynomials of degree $2i - 1$ and $2i+1$ respectively. Thus $a_{i+1}(n) - a_{i+1}(n-2)$ is a polynomial of degree $2i + 2$. This means that $a_{i+1}(n)$ is a polynomial of degree $2i + 3$.

`Case 2).` Let n be odd. By (25), for integer x and $m = \frac{n-1}{2}$, we have

$$(2x + n - 2)(a_0(n)x^m + a_1(n)x^{m-1} + \cdots) =$$

$$2(x + n - 1)(x + n - 2)(a_0(n-2)x^{m-1} + a_1(n-2)x^{m-2} + \cdots)+$$

$$(\frac{n-1}{2})!(4x + 3n - 4)\binom{x + \frac{n-3}{2}}{\frac{n-1}{2}}. \quad (46)$$

Hence, comparing the coefficient of x^{m+1} on both sides, we find

$$a_0(n) = a_0(n-2) + 2, \;\; n \geq 3, \;\; a_0(1) = 1.$$

Thus $a_0(3) = 3, a_0(5) = 5, \dots, a_0(n) = n$.
Furthermore, comparing the coefficient of x^m on both sides of (46), using the same arguments as in 1), we have

$$a_1(n) = a_1(n-2) + \frac{7n^2 - 22n + 19}{4}, \quad n \geq 3, \quad a_1(1) = 0.$$

Since $a_1(n) = \sum_{i=3,5,\dots,n}(a_1(i) - a_1(i-2))$, we find

$$a_1(n) = \frac{1}{4} \sum_{i=3,5,\dots,n} (7i^2 - 22i + 19) = \frac{1}{24}(7n^3 - 12n^2 + 5n).$$

Finally, comparing the coefficient of x^{m-i} on both sides of (46), we find

$$\begin{aligned} &2(a_{i+1}(n) - a_{i+1}(n-2)) = \\ &2(2n-3)a_i(n-2) - (n-2)a_i(n) + \\ &2(n-1)(n-2)a_{i-1}(n-2) + s_i(n), \end{aligned} \tag{47}$$

where

$$s_i(n) = Coef[x^{m-i}]((4x + 3n - 4)(x + \frac{n-3}{2})(x + \frac{n-5}{2}) \cdots (x+1)x)$$

and, as in 1), the statement is proved by induction over i. □

A few such polynomials are the following:
For odd n :

$$U_0(n) = n,$$

$$U_1(n) = \frac{1}{24}(n-1)n(7n-5),$$

$$U_2(n) = \frac{1}{640}(n-3)(n-1)n(29n^2 - 44n + 7),$$

$$U_3(n) = \frac{1}{322560}(n-5)(n-3)(n-1)n(1581n^3 - 3775n^2 + 1587n + 223);$$

For even n :

$$V_0(n) = \frac{1}{2}n,$$

$$V_1(n) = \frac{1}{48}(n-2)n(7n-4),$$

$$V_2(n) = \frac{1}{3840}(n-4)(n-2)n(87n^2 - 98n + 16),$$

$$V_3(n) = \frac{1}{645120}(n-6)(n-4)(n-2)n(1581n^3 - 2686n^2 + 936n + 64).$$

Proposition 13.

$$a_i(n) \equiv \begin{cases} r_i(n), & \text{if } n \text{ is even,} \\ \\ s_i(n), & \text{if } n \text{ is odd.} \end{cases} \pmod 2$$

Proof. The proposition follows from (45), (47) and Theorem 2. □

Finally, note that, from (36) and (37) the following homogeneous recursions for the coefficients of $P_n(x)$ follow.

Theorem 14. *For odd $n \geq 3$ and $i \geq 0$,*

$$(m-i)a_i(n) = na_i(n-1) + \sum_{j=0}^{i-1}(-1)^{i-j+1}\binom{m-j}{m-i-1}a_j(n).$$

For even $n \geq 4$ and $i \geq 0$,

$$(n-2i-1)a_i(n) = \frac{n}{2}a_i(n-1)+$$

$$2\sum_{j=0}^{i-1}(-1)^{i-j+1}\left(m\binom{m-j}{m-i} - \binom{m-j}{m-i-1}\right)a_j(n).$$

10. Arithmetic Proof of the Integrality $P_n(x)$ in Integer Points

From Theorem 2 we conclude that the polynomial, $P_n(x)$, takes integer values for integer $x = k$. Here we give an independent arithmetic proof of this fact using the explicit expression (3). It is well known (cf. [5], Section 8, Problem 87) that, if a polynomial $P(x)$ of degree m takes integer values for $x = 0, 1, \ldots, m$, then it takes integer values for every integer x. Since, as we proved at the end of Section 6, $\deg P_n(k) = \lfloor \frac{n-1}{2} \rfloor$, we suppose that $0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$. Moreover, from the results of Section 3, $P_n(0)$ and $P_n(1)$ are integers. (In the case when $n+1$ is an odd prime, $P_n(1) = (2^n - 1)(\frac{n}{2})!/(n+1)$ is integer, since $2^n - 1 \equiv 0 \pmod{n+1}$, while in the case when $n+1$ is an odd composite number, no divisor exceeds $\frac{n+1}{3}$, therefore, $(\frac{n}{2})! \equiv 0 \pmod{n+1}$.) Thus we can suppose that

$$2 \leq k \leq \lfloor \frac{n-1}{2} \rfloor. \tag{48}$$

Suppose that n is *even* (the case of odd n is considered quite analogously). Let p be a prime. Denote the maximal power of p dividing n by $[n]_p$. We say that, for integer l, h, the fraction $\frac{l}{h}$ is *p-integer,* if $[l]_p - [h]_p \geq 0$.

A) Firstly, we show that, for $n \geq 4$, $P_n(k)$ is 2-integer. Indeed, $2k+n-1$ is odd, while $4k+3n-4$ is even. Therefore, by (26), using a trivial induction, we see that $P_n(k)$ is 2-integer.

Further we use the explicit formula (3) of Theorem 1.

B) Let p be an odd prime divisor of $\binom{n+2k-2}{k-1}$ which does not coincide with any factor of the product $(n+2k-1)(n+2k-2)\cdots(n+k)$. Thus p could divide one or several *composite* factors of this product. Therefore, the following condition holds

$$3 \leq p \leq \frac{n+2k-1}{3}. \tag{49}$$

Let us show that

$$a(n;k) = \frac{\binom{\frac{n}{2}+k-1}{k}}{\binom{n+2k-1}{k}}\left(\frac{n-2}{2}\right)! =$$
$$2^{-k}\frac{(n+2k-2)(n+2k-4)\cdots n}{(n+2k-1)(n+2k-2)\cdots(n+k)}\left(\frac{n-2}{2}\right)!$$

is p-integer and, consequently, $P_n(k)$ is p-integer.

Let $k \geq 3$ be even. Then, after a simplification, we have

$$2^k a(n;k) = \frac{(n+k-2)(n+k-4)\cdots n}{(n+2k-1)(n+2k-3)\cdots(n+k+1)}\left(\frac{n-2}{2}\right)!,$$

or

$$2^{\frac{k}{2}} a(n;k) = \frac{\left(\frac{n+k-2}{2}\right)!}{(n+2k-1)(n+2k-3)\cdots(n+k+1)} \tag{50}$$

We distinguish several cases.

`Case a).` For $t \geq 2$, let p^t divide at least one factor of the denominator. Then $p \leq (n+2k-1)^{\frac{1}{t}}$ Let us show that $p \leq \frac{n+k-2}{2t}$. We should show that $n+2k-1 \leq (\frac{n+k-2}{2t})^t$, or, since, by (48), $k \leq \frac{n-2}{2}$, it is sufficient to show that $\frac{3}{2}(n+k-2) \leq (\frac{n+k-2}{2t})^t$, or $(2t)^{\frac{t}{t-1}} \leq (\frac{2}{3})^{\frac{1}{t-1}}(n+k-2)$. Since $(\frac{2}{3})^{\frac{1}{t-1}} \geq \frac{2}{3}$, it is sufficient to prove that $(2t)^{\frac{t}{t-1}} \leq \frac{2}{3}(n+k-2)$. Note that $e^t < p^t \leq n+2k-2,\ t \leq \ln(n+2k-2)$. Therefore we find $(2t)^{\frac{t}{t-1}} \leq (2\ln(n+2k-2))^2$. Furthermore, note that, if $n \geq 152$, then $\ln^2 n < \frac{n}{6}$. Thus $(2t)^{\frac{t}{t-1}} \leq \frac{2}{3}(n+k-2)$. It is left to add that up to $n = 161$ we verified that the polynomials $P_n(k)$ have integer coefficients and, consequently, is integer-valued.

`Case b).` Let p divide only one factor of the denominator. Then, in view of (48) and (49), $p \leq \frac{n+2k-1}{3} \leq \frac{n+k-2}{2}$ and, by (50), $a(n;k)$ is p-integer.

`Case c).` Let p divide exactly l factors of the denominator. Then

$$p \leq \frac{(n+2k-1)-(n+k+1)}{l} = \frac{k-2}{l},$$

and, since, by (48), $n \geq 2k+2$, we conclude that $\frac{n+k-2}{2} \geq \frac{3k}{2} \geq k-2 \geq lp$. Hence, by (50), $a(n;k)$ is p-integer.
It is left to notice that the case of odd k is considered quite analogously.
C) Suppose that, as in B), $k \geq 2$ is even. Let p be an odd prime divisor of $\binom{n+2k-1}{k}$ which coincides with some factor of the product $(n+2k-1)(n+2k-3)\cdots(n+k+1)$. In this case the fraction (50) is not integer. Thus in order to prove that $P_n(k)$ is p-integer, we should prove that $T_n(k)$ (18) is p-integer. By the condition, p has form

$$p = n+2k-1-2r, \;\; 0 \leq r \leq \frac{k-2}{2}. \tag{51}$$

According to (18) and (51), we should prove that

$$\sum_{j=0}^{n-1} 2^j \binom{n+2k-j-2}{k-1} =$$

$$\sum_{j=0}^{n-1} 2^j \binom{p+2r-1-j}{k-1} \equiv 0 \pmod{p}, \tag{52}$$

or

$$A(n,r,k) :=$$

$$\sum_{j=0}^{n-1} 2^j (j-(2r-1))(j+1-(2r-1))\cdots(j+k-2-(2r-1)) \equiv 0 \pmod{p}.$$

Note that, since $n-2r = p-2k+1$, we have

$$\sum_{j=0}^{n-1} x^{j+k-2-(2r-1)} =$$

$$(x^{n+k-2r-1} - x^{k-2r-1})(x-1)^{-1} = (x^{p-k} - x^{k-2r-1})(x-1)^{-1}.$$

Therefore,

$$A(n,r,k) = 2^{2r} \sum_{j=0}^{n-1} (x^{j+k-2-(2r-1)})^{(k-1)} \mid_{x=2} =$$

$$2^{2r}((x^{p-k} - x^{k-2r-1})(x-1)^{-1})^{(k-1)} \mid_{x=2} .$$

Thus we should prove that

$$((x^{p-k} - x^{k-2r-1})(x-1)^{-1})^{(k-1)} \mid_{x=2} \equiv 0 \pmod{p},$$

or, using the Leibnitz formula,

$$\sum_{j=0}^{k-1} (-1)^{k-j-1} \binom{k-1}{j} (k-j-1)!(p-k)(p-k-1)\cdots(p-k-j+1)2^{p-k-j} \equiv$$

$$\sum_{j=0}^{k-1}(-1)^{k-j-1}\binom{k-1}{j}(k-j-1)!(k-2r-1)(k-2r-2)\cdots(k-2r-j)2^{k-2r-j-1} \pmod p.$$

Since $2^{p-1} \equiv 1 \pmod p$, we should prove the identity

$$\sum_{j=0}^{k-1}(-1)^{k-j-1}\binom{k-1}{k-j-1}(k-j-1)!(p-k)(p-k-1)\cdots(p-k-j+1)\mid_{p=0} 2^{-k-j+1} =$$

$$\sum_{j=0}^{k-1}(-1)^{k-j-1}\binom{k-1}{k-j-1}(k-j-1)!(k-2r-1)(k-2r-2)\cdots(k-2r-j)2^{k-2r-j-1},$$

or, after simple transformations, the identity

$$\sum_{j=0}^{k-1}\binom{k+j-1}{j}2^{-j} = 2^{2k-2r-2}\sum_{j=0}^{k-1}(-1)^j\binom{k-2r-1}{j}2^{-j}. \tag{53}$$

It is known (([7], Ch.1, problem 7), that

$$\sum_{i=0}^{n}\binom{2n-i}{n}2^{i-n} = 2^n.$$

Putting $n-i=j$, we have

$$\sum_{j=0}^{n}\binom{n+j}{n}2^{-j} = \sum_{j=0}^{n}\binom{n+j}{j}2^{-j} = 2^n.$$

Therefore, the left hand side of (53) is 2^{k-1} and it is left to prove that

$$\sum_{j=0}^{k-1}(-1)^j\binom{k-2r-1}{j}2^{k-j} = 2^{2r+1}.$$

We have

$$\sum_{j=0}^{k-1}(-1)^j\binom{k-2r-1}{j}2^{k-j} = \sum_{j=0}^{k-2r-1}(-1)^j\binom{k-2r-1}{j}2^{k-j} =$$

$$2^{2r+1}\sum_{j=0}^{k-2r-1}(-1)^j\binom{k-2r-1}{j}2^{k-2r-1-j} = 2^{2r+1}(2-1)^{k-2r-1} = 2^{2r+1}$$

and we are done. The case of odd $k \ge 3$ is considered quite analogously. So, formulas (50) and (51) take the form

$$2^{\frac{k-1}{2}}a(n;k) = \frac{(\frac{n+k-1}{2})!}{(n+2k-1)(n+2k-3)\cdots(n+k)},$$

$$p = n+2k-2r-1, \ \ 0 \le r \le \frac{k-1}{2},$$

and, for odd k, the proof reduces to the same congruence (52).

11. Representation of $P_n(x)$ in Basis $\{\binom{x}{i}\}$

The structure of the explicit formula (3) allows us to conjecture that the coefficients of $P_n(x)$ in basis $\{\binom{x}{i}\}$ have simpler properties. A process of expansion of a polynomial $P(x)$ in the binomial basis is indicated in [5] in a solution of Problem 85: "Functions $1,\ x,\ x^2, \dots, x^n$ one can consecutively express in the form of linear combinations with the constant coefficients of $1,\ \frac{x}{1},\ \frac{x(x-1)}{2}, \dots, \frac{x(x-1)\cdots(x-n+1)}{n!}$." Therefore,

$$P(x) = b_0\binom{x}{m} + b_1\binom{x}{m-1} + \cdots + b_{m-1}\binom{x}{1} + b_m,$$

where $b_0,\ b_1, \dots, b_m$ are defined from the equations

$$\begin{aligned} P(0) &= b_m, \\ P(1) &= b_m + \binom{1}{1}b_{m-1}, \\ P(2) &= b_m + \binom{2}{1}b_{m-1} + \binom{2}{2}b_{m-2}, \\ &\vdots \\ P(m) &= b_m + \binom{m}{1}b_{m-1} + \cdots + \binom{m}{m}b_0. \end{aligned}$$

This process can be simplified in the following way. In the identity

$$n^x = (1 + (n-1))^x =$$

$$1 + (n-1)\binom{x}{1} + (n-1)^2\binom{x}{2} + \cdots + (n-1)^x\binom{x}{x} =$$

$$n^0 + (n-n^0)\binom{x}{1} + (n-n^0)^2\binom{x}{2} + \cdots + (n-n^0)^x\binom{x}{x}$$

we can evidently replace powers $n^j,\ \ j = 0, \dots, x$, by the arbitrary numbers $a_j,\ \ j = 0, \dots, x$. Thus we have a general identity

$$a_x = a_0 + (a_1 - a_0)\binom{x}{1} + (a_2 - 2a_1 + a_0)\binom{x}{2} + (a_3 - 3a_2 + 3a_1 - a_0)\binom{x}{3} + \cdots +$$

$$(a_x - \binom{x}{1}a_{x-1} + \binom{x}{2}a_{x-2} - \cdots + (-1)^x\binom{x}{x}a_0)\binom{x}{x}.$$

Essentially, we quickly obtained a special case of the so-called "Newton's forward difference formula" (cf. [10]). Here, put $a_j = P(j),\ \ j = 0, \dots, m$, and, firstly, consider values $0 \le x \le m$. Since $\binom{x}{l} = 0$ for $l > m$, we obtain the required representation under the condition $0 \le x \le m$:

$$P(x) = P(0) + (P(1) - P(0))\binom{x}{1} +$$

$$(P(2) - 2P(1) + P(0))\binom{x}{2} + \cdots + (P(m) - \binom{m}{1}P(m-1) +$$

$$\binom{m}{2}P(m-2) - \cdots + (-1)^m\binom{m}{m}P(0))\binom{x}{m}. \tag{54}$$

It is left to note that, since a polynomial of degree m is fully defined by its values in $m+1$ points $0, 1, \ldots, m$, then (54) is the required representation for all x.

So, for the polynomials $\{P_n(x)\}$, we have

$$P_1 = 1,$$

$$P_2 = 1,$$

$$P_3 = 3\binom{x}{1} + 4,$$

$$P_4 = 2\binom{x}{1} + 4,$$

$$P_5 = 10\binom{x}{2} + 30\binom{x}{1} + 32,$$

$$P_6 = 6\binom{x}{2} + 22\binom{x}{1} + 32,$$

$$P_7 = 42\binom{x}{3} + 196\binom{x}{2} + 378\binom{x}{1} + 384,$$

$$P_8 = 24\binom{x}{3} + 128\binom{x}{2} + 296\binom{x}{1} + 384,$$

$$P_9 = 216\binom{x}{4} + 1368\binom{x}{3} + 3816\binom{x}{2} + 6120\binom{x}{1} + 6144,$$

$$P_{10} = 120\binom{x}{4} + 840\binom{x}{3} + 2664\binom{x}{2} + 5016\binom{x}{1} + 6144,$$

$$P_{11} = 1320\binom{x}{5} + 10560\binom{x}{4} + 38544\binom{x}{3} + 84480\binom{x}{2} + 122760\binom{x}{1} + 122880,$$

$$P_{12} = 760\binom{x}{5} + 6240\binom{x}{4} + 25152\binom{x}{3} + 62112\binom{x}{2} + 103920\binom{x}{1} + 122880.$$

12. On Coefficients of $P_n(x)$ in Basis $\{\binom{x}{i}\}$

Let

$$P_n(x) = b_0(n)\binom{x}{m} + b_1(n)\binom{x}{m-1} + \cdots + b_{m-1}(n)\binom{x}{1} + b_m(n),$$

where $m = \lfloor \frac{n-1}{2} \rfloor$.
Since, for integer k, we have the explicit formula for $P_n(k)$, (3), then, according to (54), we have the following explicit formula for $b_i(n), \ \ i = 0, \dots, m:$

$$b_i(n) = \sum_{k=0}^{m-i} (-1)^{m-i-k} \binom{m-i}{k} P_n(k). \tag{55}$$

Let

$$P_n(x) = \sum_{j=0}^{m} a_j(n) x^{m-j}.$$

Then

$$b_i(n) = \sum_{j=0}^{m} a_j(n) \sum_{k=0}^{m-i} (-1)^{m-i-k} k^{m-j} \binom{m-i}{k}. \tag{56}$$

Since the l-th difference of $f(x)$ is (cf. [1], formula 25.1.1)

$$\Delta^l f(x) = \sum_{k=0}^{l} (-1)^{l-k} \binom{l}{k} f(x+k),$$

one can write (56) in the form

$$b_i(n) = \sum_{j=0}^{m} a_j(n) \Delta^{m-i} x^{m-j} \mid_{x=0} .$$

Here the summands corresponding to $j > i$, evidently, equal 0. Therefore, we have

$$b_i(n) = \sum_{j=0}^{i} a_j(n) \Delta^{m-i} x^{m-j} \mid_{x=0} . \tag{57}$$

Theorem 15. *For $n \geq 1$, we have*

$$b_0(n) = \begin{cases} n(\frac{n-1}{2})!, & \text{if } n \text{ is odd}, \\ (\frac{n}{2})!, & \text{if } n \text{ is even}; \end{cases} \tag{58}$$

$$b_1(n) = \begin{cases} \frac{1}{6} n(5n-7)(\frac{n-1}{2})!, & \text{if } n \text{ is odd}, \\ \frac{1}{6}(5n-8)(\frac{n}{2})!, & \text{if } n \text{ is even}. \end{cases} \tag{59}$$

In general, for a fixed i, $b_i(n) = (m-i)!Y_i(n)$, if n is odd, and $b_i(n) = (m-i)!Z_i(n)$, if n is even, where Y_i and Z_i are polynomials in n of degree $2i+1$.

Proof. Note that the Stirling number of the second kind $S(n,m)$ is connected with the m-th difference of $\Delta^m x^n \mid_{x=0}$ in the following way (see [1], formulas 24.1.4):

$$S(n,m)m! = \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}k^n = \Delta^m x^n \mid_{x=0} . \tag{60}$$

In particular, since $S(m,m) = 1$, $S(m+1,m) = \binom{m+1}{2}$, we have

$$\Delta^m x^m \mid_{x=0} = m!$$

and

$$\Delta^m x^{m+1} \mid_{x=0} = \frac{m}{2}(m+1)! .$$

Therefore, by (57),

$$b_0(n) = m!a_0(n),$$

$$b_1(n) = \frac{m-1}{2}m!a_0(n) + (m-1)!a_1(n),$$

and, by (40) and (41) (where $m = \lfloor \frac{n-1}{2} \rfloor$), we find formulas (58) and (59).

Further, we need the following lemma.

Lemma 16. *$S(n+k,n)$ is a polynomial in n of degree $2k$.*

Proof. For $k \geq 1$, let

$$Q_k(n) = S(n+k,n).$$

Note that, since $S(n,n) = 1$, we have $Q_0(n) = 1$. Further, since $S(n,0) = \delta_{n,0}$, for $k \geq 1$, $Q_k(0) = 0$. From the main recursion for $S(n,m)$ which is $S(n,m) = mS(n-1,m) + S(n-1,m-1)$, we have

$$Q_k(n) - Q_k(n-1) = nQ_{k-1}(n).$$

Also, in view of $Q_k(0) = 0$, we find the recursion

$$Q_0(n) = 1, \;\; Q_k(n) = \sum_{i=1}^{n} iQ_{k-1}(i). \tag{61}$$

Using a simple induction, from (61) we obtain the lemma. □

Remark 17. The first few polynomials $\{Q_k(n)\}$ are:

$$Q_0 = 1,$$
$$Q_1 = \frac{1}{2}n(n+1),$$
$$Q_2 = \frac{1}{24}n(n+1)(n+2)(3n+1),$$
$$Q_3 = \frac{1}{48}n^2(n+1)^2(n+2)(n+3),$$
$$Q_4 = \frac{1}{5760}n(n+1)(n+2)(n+3)(n+4)(15n^3+30n^2+5n-2)\,.$$

It can be proven that the sequence of denominators coincides with $A053657$ [8], such that the denominator of $Q_k(n)$ is $\prod p^{\sum_{j\geq 0}\lfloor\frac{k}{(p-1)p^j}\rfloor}$, where the product is over all primes.

Note that from (57) and (60) we find

$$b_i(n) = (m-i)!\sum_{j=0}^{i} a_j(n)S(m-j, m-i), \;\; m = \lfloor\frac{n-1}{2}\rfloor.$$

Since, by Lemma 10, $S(m-j, m-i)$ is a polynomial in n of degree $2((m-j)-(m-i)) = 2(i-j)$, while, by Theorem 12, $a_j(n)$ is a polynomial of degree $2j+1$, it follows that $a_j(n)S(m-j, m-i)$ is a polynomial of degree $2i+1$. Thus $\sum_{j=0}^{i} a_j(n)S(m-j, m-i)$ is a polynomial of degree $2i+1$. This completes the proof. □

The first polynomials $Y_i(n)$, $Z_i(n)$ are

$$Y_0 = n,$$
$$Y_1 = \frac{1}{12}(n-1)n(5n-7),$$
$$Y_2 = \frac{1}{480}(n-3)(n-1)n(43n^2-168n+149),$$
$$Y_3 = \frac{1}{13440}(n-5)(n-3)(n-1)n(177n^3-1319n^2+3063n-2161);$$
$$Z_0 = \frac{n}{2},$$
$$Z_1 = \frac{1}{24}(n-2)n(5n-8),$$
$$Z_2 = \frac{1}{960}(n-4)(n-2)n(43n^2-182n+184),$$
$$Z_3 = \frac{1}{26880}(n-6)(n-4)(n-2)n(3n-8)(59n^2-306n+352)).$$

Finally, we prove the following attractive result.

Theorem 18. 1) *For odd* n, $b_j(n)/n, j = 0, \ldots, m-1$, *are integer. Moreover, for* $n \geq 3$,

$$b_i(n) = n(b_i(n-1) + b_{i-1}(n-1)), \ \ i = 1, \ldots, m-1.$$

2) *For even* $n \geq 4$,

$$2b_i(n) = b_i(n-1) + b_{i-1}(n-1) + m!\binom{m}{i}, \ \ i = 1, \ldots, m-1. \tag{62}$$

Proof. 1) According to (55), we should prove that for odd $n \geq 3$,

$$\sum_{k=0}^{m-i}(-1)^{m-i-k}\binom{m-i}{k}P_n(k) =$$

$$n\Big(\sum_{k=0}^{m-i}(-1)^{m-i-k}\binom{m-i}{k}P_{n-1}(k)+$$

$$\sum_{k=0}^{m-i-1}(-1)^{m-i-k-1}\binom{m-i-1}{k}P_{n-1}(k)\Big), \ \ i = 1, 2, \ldots, m-1,$$

or, putting $m - i = t$,

$$\sum_{k=0}^{t}(-1)^k\binom{t}{k}P_n(k) = n\Big(\sum_{k=0}^{t}(-1)^k\binom{t}{k}P_{n-1}(k)-$$

$$\sum_{k=0}^{t}(-1)^k\binom{t-1}{k}P_{n-1}(k)\Big), \ \ t = 1, 2, \ldots, m-1,$$

or, finally, for $t = 1, \ldots, \frac{n-3}{2}$,

$$\sum_{k=1}^{t}(-1)^{k-1}\Big(\binom{t}{k}P_n(k) - \binom{t-1}{k-1}nP_{n-1}(k)\Big) = P_n(0). \tag{63}$$

To prove (63), note that, by (36), $nP_{n-1}(k) = P_n(k) - P_n(k-1)$. Hence,

$$\binom{t}{k}P_n(k) - \binom{t-1}{k-1}nP_{n-1}(k) =$$

$$P_n(k)\Big(\binom{t}{k} - \binom{t-1}{k-1}\Big) + \binom{t-1}{k-1}P_n(k-1) =$$

$$\binom{t-1}{k}P_n(k) + \binom{t-1}{k-1}P_n(k-1).$$

Thus the summands of (63) are

$$(-1)^{k-1}\Big(\binom{t}{k}P_n(k) - \binom{t-1}{k-1}nP_{n-1}(k)\Big) =$$

$$(-1)^{k-1}\binom{t-1}{k}P_n(k)-(-1)^{k-2}\binom{t-1}{k-1}P_n(k-1),$$

and the summing gives

$$\sum_{k=1}^{t}(-1)^{k-1}\Big(\binom{t}{k}P_n(k)-\binom{t-1}{k-1}nP_{n-1}(k)\Big)=$$

$$(-1)^{k-1}\binom{t-1}{k}P_n(k)\mid_{k=t}-(-1)^{k-2}\binom{t-1}{k-1}P_n(k-1)\mid_{k=1}=P_n(0).$$

2) Analogously, the proof of (62) reduces to the proof of the following equality for $t=1,2,\dots,m-1$:

$$\sum_{k=0}^{t+1}(-1)^k\Big(2\binom{t}{k}P_n(k)+\binom{t}{k-1}P_{n-1}(k)\Big)=(-1)^t m!\binom{m}{t}. \tag{64}$$

Note that, by (39),

$$(-1)^k\Big(2\binom{t}{k}P_n(k)+\binom{t}{k-1}P_{n-1}(k)\Big)=$$

$$(-1)^k\binom{t}{k}P_{n-1}(k+1)-$$

$$(-1)^{k-1}\binom{t}{k-1}P_{n-1}(k)+(-1)^k\binom{t}{k}\binom{k+m}{m}m! \tag{65}$$

Since

$$\sum_{k=0}^{t+1}\Big((-1)^k\binom{t}{k}P_{n-1}(k+1)-(-1)^{k-1}\binom{t}{k-1}P_{n-1}(k)\Big)=$$

$$(-1)^k\binom{t}{k}P_{n-1}(k)\mid_{k=t+1}-(-1)^{k-1}\binom{t}{k-1}P_{n-1}(k)\mid_{k=0}=0,$$

by (64) and (65), the proof reduces to the known combinatorial identity

$$\sum_{k=0}^{t}(-1)^k\binom{t}{k}\binom{k+m}{m}=(-1)^t\binom{m}{t},\;\; t=1,\dots,m-1$$

(see [7], Ch.1, formula (8) with $p=0$ up to the notations). □

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