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SOME PARTITION THEOREMS FOR INFINITE AND FINITE MATRICES

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Abstract

Let A be a finite or infinite matrix with integer entries and only finitely many nonzero entries in each row. Then A is *image partition regular* (over $\mathbb{N}$) provided whenever $\mathbb{N}$ is finitely colored, there must exist $\vec{x}$ with entries from $\mathbb{N}$ (and the same number of entries as A has columns) such that the entries of $A\vec{x}$ are the same color.

Let $k \in \mathbb{N}$ and let $\vec{a} = \langle a_1, a_2, \ldots, a_k \rangle$ be a sequence in $\mathbb{Z} \setminus \{0\}$. Let $B(\vec{a})$ denote an infinite matrix consisting of all rows whose nonzero entries are $a_1, a_2, \ldots, a_k$ in order, each occurring once, and for $m \geq k$, $B_m(\vec{a})$ is a matrix with m columns and all rows whose nonzero entries are $a_1, a_2, \ldots, a_k$ in order, each occurring once. Also define $M(\vec{a}) = \begin{pmatrix} I \\ B(\vec{a}) \end{pmatrix}$ and $M_m(\vec{a}) = \begin{pmatrix} I_m \\ B_m(\vec{a}) \end{pmatrix}$, where I is the $\omega \times \omega$ identity matrix and I_m is the $m \times m$ identity matrix. We provide a simple characterization of those sequences $\vec{a}$ with the property that for sufficiently large m, $M_m(\vec{a})$ is image partition regular. We also provide a simple characterization of those sequences $\vec{a}$ such that $M(\vec{a})$ is image partition regular in the special case where there is a fixed $\beta \in \mathbb{N}$ such that each a_i is either a power of β or the negative of a power of β.

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1. Introduction

In this paper, $\mathbb{N} = \{1, 2, \ldots\}$ denotes the set of natural numbers. We have indicated the notion of image partition regularity in the abstract. A historically older notion is that of *kernel partition regularity* (which is often referred to as simply "partition regularity" in the literature).

Definition 1.1. Let A be a finite or infinite matrix with integer entries and only finitely many nonzero entries in each row. Then A is *kernel partition regular* (over $\mathbb{N}$) provided whenever $\mathbb{N}$ is finitely colored, there must exist a monochromatic $\vec{x}$ with the same number of entries as A has rows such that $A\vec{x} = \vec{0}$.

Kernel partition regularity for finite matrices was characterized by Rado in 1933 [8] in terms of a computable condition known as the *columns condition.*

Definition 1.2. A finite integer-valued $u \times v$-matrix A satisfies the *columns condition* if and only if there is some $m \in \mathbb{N}$ such that the set $\{1, 2, \ldots, v\}$ of column indices can be partitioned as $\{1, 2, \ldots, v\} = I_1 \cup \ldots \cup I_m$ such that

(i) the sum of all columns with indices in I_1 add up to the zero vector, and

(ii) for $j = 2, 3, \ldots, m$, the sum of all columns with indices in I_j is a rational linear combination of all columns with indices in $I_1 \cup \ldots \cup I_{j-1}$.

Theorem 1.3 (Rado's Theorem). *Let A be a finite integer-valued matrix. The matrix A is kernel partition regular if and only if it satisfies the columns condition.*

Proof. [8, Satz IV]. □

A motivation explicitly stated in [8] for the study of kernel partition regularity was van der Waerden's Theorem [12], which says that for any $\sigma \in \mathbb{N}$, if $\mathbb{N}$ is finitely colored, there must be a length σ monochromatic arithmetic progression. The fact that the following matrix is kernel partition regular establishes the length 4 version of van der Waerden's Theorem, with the added conclusion that the increment is the same color as the terms of the sequence.

$$\begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix}$$

This matrix satisfies the columns condition with $I_1 = \{1, 2, 3, 4\}$ and $I_2 = \{5\}$. (Any three of the first four columns are linearly independent and so span $\mathbb{Q}^3$.) One might think that the fact that $\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix}$ satisfies the columns condition (with $I_1 = \{1, 2, 3, 4\}$) would suffice for the length 4 version of van der Waerden's Theorem, but its kernel includes any constant sequence.

In 1973 Deuber [1] used the notion of (m, p, c)-sets to prove Rado's Conjecture [8], namely that if a subset of $\mathbb{N}$ contains a kernel of each kernel partition regular matrix, and it is partitioned into finitely many sets, then one of those sets would also contain a kernel of each kernel partition regular matrix. Deuber's (m, p, c)-sets are images of certain image partition regular matrices, a fact which he established in [1, Satz 3.1].

Definition 1.4. Let A be a finite or infinite matrix with integer entries and only finitely many nonzero entries in each row. Then A is *image partition regular* (over $\mathbb{N}$) provided whenever $\mathbb{N}$ is finitely colored, there must exist $\vec{x}$ with entries from $\mathbb{N}$ (and the same number of entries as A has columns) such that the entries of $A\vec{x}$ are monochromatic. (That is to say, all entries of $A\vec{x}$ are the same color.)

It takes a small amount of thought to come up with kernel partition regular matrices establishing the validity of van der Waerden's Theorem. By contrast, the length 4 version, and the strengthened length 4 version with the increment the same color, are given by the image partition regularity of the following two matrices, which require no thought to produce:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}.$$

In spite of the utility of image partition regular matrices and the ease of stating natural problems using them, they were not characterized until 1993 in [5]. Two of these characterizations were in terms of the kernel partition regularity of related matrices. In a special case with which we are concerned in this paper, namely determining the image partition regularity of $M_m(\vec{a})$, there is a very simply stated relationship between image and kernel partition regularity. Given $n \in \mathbb{N}$, we denote the $n \times n$ identity matrix by I_n.

Lemma 1.5. *Let $u, v \in \mathbb{N}$ and let M be a $u \times v$ matrix with entries from $\mathbb{Z}$. Then the $(u+v) \times v$ matrix $\begin{pmatrix} I_v \\ M \end{pmatrix}$ is image partition regular if and only if the $u \times (u+v)$ matrix $\begin{pmatrix} M & -I_u \end{pmatrix}$ is kernel partition regular.*

Proof. Let $\vec{x} \in \mathbb{N}^v$. Then

$$\begin{pmatrix} I_v \\ M \end{pmatrix} \vec{x} = \begin{pmatrix} \vec{x} \\ M\vec{x} \end{pmatrix} \text{ and } \begin{pmatrix} M & -I_u \end{pmatrix} \begin{pmatrix} \vec{x} \\ M\vec{x} \end{pmatrix} = \vec{0}\,.$$

□

The corresponding infinite version of Lemma 1.5 is also valid, but not of much interest since there are no known characterizations of either image or kernel partition regularity of infinite matrices.

Definition 1.6. Let $k \in \mathbb{N}$ and let $\vec{a} = \langle a_1, a_2, \ldots, a_k \rangle$ be a sequence in $\mathbb{Z} \setminus \{0\}$. Let $B(\vec{a})$ denote an infinite matrix consisting of all rows whose nonzero entries are $a_1, a_2, \ldots, a_k$ in order, each occurring once, and for $m \geq k$, $B_m(\vec{a})$ is a matrix with m columns and all rows whose nonzero entries are $a_1, a_2, \ldots, a_k$ in order, each occurring once. Also define $M(\vec{a}) = \begin{pmatrix} I \\ B(\vec{a}) \end{pmatrix}$ and $M_m(\vec{a}) = \begin{pmatrix} I_m \\ B_m(\vec{a}) \end{pmatrix}$, where I is the $\omega \times \omega$ identity matrix and I_m is the $m \times m$ identity matrix.

There are quite a few partial results about image partition regularity of infinite matrices. See the survey [4, Section 6] or [2] for a discussion of several of them. What is probably the most general known class of infinite image partition regular matrices are the *Milliken-Taylor* matrices.

Definition 1.7. (1) Let $n \in \mathbb{N}$ and let $\vec{b} = \langle b_1, b_2, \ldots, b_n \rangle$ be a sequence in $\mathbb{Z} \setminus \{0\}$. The *compressed form* $c(\vec{b})$ of $\vec{b}$ is the sequence $\vec{a}$ obtained from $\vec{b}$ by deleting all but one occurrence of adjacent repeated terms.

(2) Let $k \in \mathbb{N}$ and let $\vec{a} = \langle a_1, a_2, \ldots, a_k \rangle$ be a sequence in $\mathbb{Z} \setminus \{0\}$ with no adjacent repeated terms (so that $\vec{a} = c(\vec{a})$). Then $MT(\vec{a})$ is a matrix with all rows with finitely many nonzero entries such that, if $\vec{b}$ is in order the nonzero entries of a row, then $c(\vec{b}) = \vec{a}$.

For example, $c(\langle 1, 1, 1, 3, -2, -2, 1, 2, 2, 3, 3, -2 \rangle) = \langle 1, 3, -2, 1, 2, 3, -2 \rangle$. Of course, there are infinitely many matrices fitting the definition of $MT(\vec{a})$, since any permutation of the rows of such a matrix is another such a matrix. (A similar situation applies to the matrices $B(\vec{a})$ and $M(\vec{a})$ defined in Definition 1.6.)

Theorem 1.8 (Milliken-Taylor). *Let $k \in \mathbb{N}$ and let $\vec{a} = \langle a_1, a_2, \ldots, a_k \rangle$ be a sequence in $\mathbb{Z} \setminus \{0\}$ with no adjacent repeated terms and $a_k > 0$. Then $MT(\vec{a})$ is image partition regular.*

Proof. This follows easily from [7, Theorem 2.2] or [11, Lemma 2.2]. For the details see [6, Corollary 17.33]. (There the entries are assumed to come from $\mathbb{N}$ but that has no effect on the proof.) □

It is easy to see that the requirement that $a_k > 0$ in Theorem 1.8 is necessary. Indeed, if $a_k < 0$, there is no sequence $\vec{x} = \langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ with all entries of $MT(\vec{a})$ in $\mathbb{N}$. (Suppose one has such and let $m = \sum_{i=1}^{k-1} a_i x_i$. If $m \leq 0$ then $\sum_{i=1}^{k} a_i x_i$ is a negative entry of $MT(\vec{a})\vec{x}$. Otherwise $m + \sum_{i=k}^{k+m} a_k x_i$ is a negative entry of $MT(\vec{a})\vec{x}$.)

Moreover, it is an immediate consequence of Theorem 1.8 that if $k \in \mathbb{N}$ and $\vec{a} = \langle a_1, a_2, \ldots, a_k \rangle$ is a sequence in $\mathbb{Z} \setminus \{0\}$ with $a_k > 0$, then $B(\vec{a})$ is image partition regular. For example, if $\vec{a} = \langle -1, -1, 2, 3, -2, -2, 4, 4 \rangle$ then all of the rows of $B(\vec{a})$ are rows of $MT(\langle -1, 2, 3, -2, 4 \rangle)$.

In this paper we investigate the question of when we can add the requirement that whenever $\mathbb{N}$ is finitely colored, not only do we get some $\vec{x} \in \mathbb{N}^\omega$ with the entries of $B(\vec{a})\vec{x}$ monochromatic, but the entries of $\vec{x}$ are also the same color as the entries of $B(\vec{x})$. That is, we investigate when $M(\vec{a})$ is image partition regular. Note that for $\vec{a} = \langle 1 \rangle$ Theorem 1.8 is the Finite Sums Theorem [3], while for $\vec{a} = \langle a_1, a_2, \ldots, a_k \rangle = \langle 1, 1, \ldots, 1 \rangle$ with $M(\vec{a})\vec{x}$ we consider only k-term sums.

In Section 2, we obtain a complete answer to this question in the event that there is some $\beta \in \mathbb{N}$ such that each a_t is a power of β or the negative of a power of β (and we conjecture that the same answer holds in general).

In Section 3 we address the analogous question for the finite versions of the same question. We show that if $k \in \mathbb{N}$, $k \geq 2$, $\vec{a} = \langle a_1, a_2, \ldots, a_k \rangle$ is a sequence in $\mathbb{Z} \setminus \{0\}$, and $m \geq 2k - 2$, then $M_m(\vec{a})$ is image partition regular if and only if one of (1) $a_1 = 1$, (2) $a_k = 1$, (3) $a_1 + a_2 + \ldots + a_k = 1$, or (4) $a_1 + a_2 + \ldots + a_k = 0$. We also show that the bound $m \geq 2k - 2$ is best possible.

2. The Infinite Version

The Finite Sums Theorem [3] is the case $\vec{a} = \langle 1 \rangle$ of Theorem 1.8. As an immediate consequence of this theorem we have the following.

Theorem 2.1. *Let $k \in \mathbb{N}$ and let $\vec{a} = \langle 1, 1, \ldots, 1 \rangle$ be the all ones sequence of length k. Then $M(\vec{a})$ is image partition regular.*

By a theorem of Rado [9, Lemma page 932] we know also that for the sequence $\vec{a} = \langle -1, 1 \rangle$ the matrix $M(\vec{a})$ is image partition regular.

It is trivial that if $a_1 + a_2 + \ldots + a_k = 1$, then $M(\vec{a})$ is image partition regular since then any constant sequence $\vec{x} = \langle x_t \rangle_{t=1}^{\infty}$ satisfies $M(\vec{a})\vec{x}$ is monochromatic.

In this section we prove the following theorem, which extends these results.

Theorem 2.2. *Let $k \in \mathbb{N}$, let $\beta \in \mathbb{N} \setminus \{1\}$, and let $\vec{a} = \langle a_1, a_2, \ldots, a_k \rangle$ be a sequence such that for each $t \in \{1, 2, \ldots, k\}$, there is some $\gamma \in \omega$ such that $a_t = \beta^\gamma$ or $a_t = -\beta^\gamma$. Then $M(\vec{a})$ is image partition regular if and only if one of*

(1) $a_1 + a_2 + \ldots + a_k = 0$ *and* $a_k = 1$;

(2) $a_1 + a_2 + \ldots + a_k = 1$; *or*

(3) $a_1 = a_2 = \ldots = a_k = 1$.

We let $\omega = \{0, 1, 2, \ldots\} = \mathbb{N} \cup \{0\}$. For a set X, let $\mathcal{P}_f(X)$ be the set of all finite nonempty subsets of X.

Definition 2.3. Let $p \in \mathbb{N}\backslash\{1\}$. We define the *support function* $\text{supp}_p \colon \mathbb{N} \to \mathcal{P}_f(\omega)$ by, for $x \in \mathbb{N}$, $\text{supp}_p(x)$ is the set of locations of nonzero digits in the base p expansion of x. And we define the function $\eta_{p,x} \colon \text{supp}_p(x) \to \{1, 2, \ldots, p-1\}$ by $x = \sum_{i \in \text{supp}_p(x)} \eta_{p,x}(i) p^i$.

Lemma 2.4. *Let $k \in \mathbb{N}$, and let $\vec{a} = \langle a_1, a_2, \ldots, a_k \rangle$ be a sequence in $\mathbb{Z} \setminus \{0\}$ such that $a_1 + a_2 + \ldots + a_k \neq 1$. There is a 2-coloring φ of $\mathbb{N}$ such that there do not exist $b \in \mathbb{N}$ and $\vec{x} = \langle x_t \rangle_{t=1}^{\infty}$ such that $M(\vec{a})\vec{x}$ is monochromatic with respect to φ, and some k terms of $\vec{x}$ are equal to b.*

Proof. If $a_1 + a_2 + \ldots + a_k \leq 0$, then there is no $b \in \mathbb{N}$ such that $(a_1 + a_2 + \ldots + a_k)b \in \mathbb{N}$, so assume that $a_1 + a_2 + \ldots + a_k = S \geq 2$. Define $\varphi \colon \mathbb{N} \to \{0, 1\}$ by $\varphi(x) = 0$ if and only if there is some $i \in \omega$ such that $S^{2i} \leq x < S^{2i+1}$. Given $b \in \mathbb{N}$, pick $j \in \omega$ such that $S^j \leq b < S^{j+1}$. Then $S^{j+1} \leq (a_1 + a_2 + \ldots + a_k)b < S^{j+2}$. □

Lemma 2.5. *Let $k \in \mathbb{N}$, and let $\vec{a} = \langle a_1, a_2, \ldots, a_k \rangle$ be a sequence in $\mathbb{Z} \setminus \{0\}$ such that $a_1 + a_2 + \ldots + a_k \neq 1$. If $M(\vec{a})$ is image partition regular, then $a_k = 1$.*

Proof. Pick a 2-coloring φ of $\mathbb{N}$ as guaranteed by Lemma 2.4 and pick $p \in \mathbb{N}$ such that $p > \max\big\{|a_t| : t \in \{1, 2, \ldots, k\}\big\}$. Let ψ be a finite coloring of $\mathbb{N}$ with the property that for $x, y \in \mathbb{N}$, $\psi(x) = \psi(y)$ if and only if

(1) $\varphi(x) = \varphi(y)$;

(2) $\max \text{supp}_p(x) \equiv \max \text{supp}_p(y) \pmod 5$;

(3) $\eta_{p,x}\big(\max \text{supp}_p(x)\big) = \eta_{p,y}\big(\max \text{supp}_p(y)\big)$; and

(4) $\eta_{p,x}\big(\max \text{supp}_p(x) - 1\big) = \eta_{p,y}\big(\max \text{supp}_p(y) - 1\big)$.

Requirements (3) and (4) say that if $\psi(x) = \psi(y)$, then the two most significant digits in the base p expansions of x and y agree.

Pick $\vec{x} = \langle x_t \rangle_{t=1}^{\infty}$ such that the entries of $M(\vec{a})\vec{x}$ are monochromatic with respect to ψ. By Lemma 2.4, $\vec{x}$ does not have any k terms equal. So by thinning the sequence we can assume that for each $t \in \mathbb{N}$, $\max \text{supp}_p(t) < \max \text{supp}_p(t+1)$. For each $t \in \mathbb{N}$, let $m_t = \max \text{supp}_p(t)$. Then by requirement (2) of the coloring, we have for each t that $m_{t+1} \geq m_t + 5$. Note that, since the sequence $\langle x_t \rangle_{t=1}^{\infty}$ is increasing, we trivially have that $a_k > 0$.

Pick $u \in \{1, 2, \ldots, p-1\}$ and $v \in \{0, 1, \ldots, p-1\}$ such that for each $t \in \mathbb{N}$, $\eta_{p,x_t}(m_t) = u$ and $\eta_{p,x_t}(m_t - 1) = v$. Then for each $t \in \mathbb{N}$, $x_t = up^{m_t} + vp^{m_t - 1} + y_t$ for some $y_t \in \{0, 1, \ldots, p^{m_t - 1} - 1\}$.

Note that $\sum_{t=1}^{k-1}|a_t|x_t < \sum_{t=1}^{k-1} p^{m_t+2} < p^{m_k-2}$ because $m_{k-1}+2 < m_k - 2$. Therefore,

$$(*) \qquad -p^{m_k-2} < \sum_{t=1}^{k-1} a_t x_t < p^{m_k-2}$$

and in particular

$$p^{m_k} - p^{m_k-2} \leq a_k x_k - p^{m_k-2} < \sum_{t=1}^{k} a_t x_t < a_k x_k + p^{m_k-2} < p^{m_k+1} + p^{m_k-2}\,.$$

Consequently, $m_k - 1 \leq \max \operatorname{supp}_p(\sum_{t=1}^{k} a_t x_t) \leq m_k + 1$. Therefore, since $\psi(\sum_{t=1}^{k} a_t x_t) = \psi(x_k)$, we must have that $\max \operatorname{supp}_p(\sum_{t=1}^{k} a_t x_t) = m_k$.

Case 1. $a_k v p^{m_k-1} + a_k y_k + \sum_{t=1}^{k-1} a_t x_t \geq 0$. Then

$$\sum_{t=1}^{k} a_t x_t = a_k u p^{m_k} + a_k v p^{m_k-1} + a_k y_k + \sum_{t=1}^{k-1} a_t x_t$$

and, using $(*)$,

$$0 \leq a_k v p^{m_k-1} + a_k y_k + \sum_{t=1}^{k-1} a_t x_t < p^{m_k+1} + p^{m_k} + p^{m_k-2}$$

so there exist some z and w with $0 \leq z < p+1$ and $0 \leq w < p^{m_k}$ such that $a_k v p^{m_k-1} + a_k y_k + \sum_{t=1}^{k-1} a_t x_t = z p^{m_k} + w$.

Then $\sum_{t=1}^{k} a_t x_t = (a_k u + z)p^{m_k} + w$. And since $\max \operatorname{supp}_p(\sum_{t=1}^{k} a_t x_t) = m_k$, we have that $a_k u + z < p$. Consequently $a_k u + z = u$ so $(a_k - 1)u + z = 0$ so $a_k = 1$ as claimed.

Case 2. $a_k v p^{m_k-1} + a_k y_k + \sum_{t=1}^{k-1} a_t x_t < 0$. In this case, we must have $v = 0$, since otherwise $a_k v p^{m_k-1} + a_k y_k + \sum_{t=1}^{k-1} a_t x_t > p^{m_k-1} - p^{m_k-2} > 0$. Also, $a_k y_k \leq p^{m_k-2}$ since otherwise $a_k y_k + \sum_{t=1}^{k-1} a_t x_t > 0$. Now

$$\sum_{t=1}^{k} a_t x_t = a_k u p^{m_k} + a_k y_k + \sum_{t=1}^{k-1} a_t x_t = a_k u p^{m_k} - \left|a_k y_k + \sum_{t=1}^{k-1} a_t x_t\right|.$$

Since $|a_k y_k + \sum_{t=1}^{k-1} a_t x_t| < 2p^{m_k-2}$, the digit in position $m_k - 1$ of $\sum_{t=1}^{k} a_t x_t$ is $p - 1 \neq 0 = v$, a contradiction. □

Lemma 2.6. *Let $k \in \mathbb{N}$ and let $\vec{a} = \langle a_1, a_2, \ldots, a_k \rangle$ be a sequence in $\mathbb{Z} \setminus \{0\}$. If $M(\vec{a})$ is image partition regular, then*

(1) $a_1 + a_2 + \ldots + a_k = 0$ and $a_k = 1$;

(2) $a_1 + a_2 + \ldots + a_k = 1$; or

(3) $a_1 = a_k = 1$.

Proof. If $a_1 + a_2 + \ldots + a_k = 1$, we are done, so assume that $a_1 + a_2 + \ldots + a_k \neq 1$. Then by Lemma 2.5, we have $a_k = 1$. Pick a coloring φ of $\mathbb{N}$ as guaranteed by Lemma 2.4.

Pick a prime $p > \sum_{t=1}^{k} |a_t|$. Let ψ be a finite coloring of $\mathbb{N}$ with the property that for $x, y \in \mathbb{N}$, $\psi(x) = \psi(y)$ if and only if

(1) $\varphi(x) = \varphi(y)$ and

(2) $\eta_{p,x}\big(\min \operatorname{supp}_p(x)\big) = \eta_{p,y}\big(\min \operatorname{supp}_p(y)\big)$.

Pick $\vec{x} = \langle x_t\rangle_{t=1}^{\infty}$ such that the entries of $M(\vec{a})\vec{x}$ are monochromatic with respect to ψ. By Lemma 2.4 we have that $\vec{x}$ does not have any k terms equal. So by thinning the sequence we may presume that either

(a) there is some $m \in \omega$ such that for each $t \in \mathbb{N}$, $\min \operatorname{supp}_p(x_t) = m$ or

(b) for each $t \in \mathbb{N}$, $\min \operatorname{supp}_p(x_t) < \min \operatorname{supp}_p(x_{t+1})$.

Let $v = \eta_{p,x_1}\big(\min \operatorname{supp}_p(x_1)\big)$, the constant value of the least significant digit in the base p expansion of x_t.

In case (a), the digit in position m of $\sum_{t=1}^{k} a_t x_t$ is congruent to $v\sum_{t=1}^{k} a_t$ (mod p). If $\sum_{t=1}^{k} a_t \neq 0$, this says that $v\sum_{t=1}^{k} a_t \equiv v$ (mod p), so that $\sum_{t=1}^{k} a_t = 1$, which we have forbidden. Therefore $\sum_{t=1}^{k} a_t \neq 0$ and conclusion (1) holds.

In case (b), the rightmost nonzero digit in the base p expansion of $\sum_{t=1}^{k} a_t x_t$ is congruent to $a_1 v$ (mod p) so $a_1 v \equiv v$ (mod p) and thus conclusion (3) holds. □

Lemma 2.7. *Let $k \in \mathbb{N}$, and let $\vec{a} = \langle a_1, a_2, \ldots, a_k\rangle$ be a sequence in $\mathbb{Z} \setminus \{0\}$. If one of the following holds, then $M(\vec{a})$ is image partition regular*

(1) $a_1 + a_2 + \ldots + a_k = 0$ and $a_k = 1$;

(2) $a_1 + a_2 + \ldots + a_k = 1$; or

(3) $a_1 = a_2 = \ldots = a_k = 1$.

Proof. We only have to consider case (1), as case (2) is trivial, and case (3) follows by Theorem 2.1. Let $r \in \mathbb{N}$ be fixed and let $\varphi \colon \mathbb{N} \to \{1, 2, \ldots, r\}$ be an arbitrary coloring. We induce another coloring $\varphi' \colon [\mathbb{N}]^k \to \{1, 2, \ldots, r+1\}$ of k-tuples by

$$\varphi'(\{z_1, z_2, \ldots, z_k\}_<) = \varphi(\textstyle\sum_{i=1}^{k} a_i z_i),$$

whenever $\sum_{i=1}^{k} a_i z_i > 0$. If $\sum_{i=1}^{k} a_i z_i < 0$, let $\varphi'(\{z_1, z_2, \ldots, z_k\}_<) = r+1$.

By Ramsey's Theorem [10] there exists an infinite set $Y \subseteq \mathbb{N}$ such that $[Y]^k$ is colored monochromatically by φ', say in color g. With $a_k = 1$ we have $g < r+1$. Let $\langle y_t\rangle_{t=1}^{\infty}$ enumerate Y in increasing order. For $j = 1, 2, \ldots$, set

$$x_j = \textstyle\sum_{i=1}^{k-1} a_i y_i + a_k y_{k+j-1}.$$

For $j = 1, 2, \ldots$, by choice of the coloring φ' we infer

$$\varphi(x_j) = \varphi'(\{y_1, \ldots, y_{k-1}, y_{k+j-1}\}_<) = g.$$

Fix any positive integers $j_1 < j_2 < \ldots < j_k$. With $a_k = 1$ and $\sum_{i=1}^k a_i = 0$ we obtain

$$\sum_{i=1}^k a_i x_{j_i} = \sum_{i=1}^k a_i (\sum_{\sigma=1}^{k-1} a_\sigma y_\sigma) + \sum_{i=1}^k a_i a_k y_{k+j_i-1} = \sum_{i=1}^k a_i y_{k+j_i-1}.$$

As $\varphi'(\{y_{k+j_1-1}, y_{k+j_2-1}, \ldots, y_{k+j_k-1}\}_<) = \varphi(\sum_{i=1}^k a_i y_{k+j_i-1}) = g$, we infer that $\varphi(\sum_{i=1}^k a_i x_{j_i}) = \varphi(x_{j_1}) = \varphi(x_{j_2}) = \ldots = \varphi(x_{j_k})$, i.e., $M(\vec{a})$ is image partition regular. $\square$

Lemma 2.8. *Let $\beta \in \mathbb{N} \setminus \{1\}$. Let $k \in \mathbb{N}$, let $\vec{a} = \langle a_1, a_2, \ldots, a_k \rangle$ be a sequence in $\mathbb{Z} \setminus \{0\}$, and assume that $a_1 + a_2 + \ldots + a_k \notin \{0, 1\}$. Let $M = M(\vec{a})$. There is a finite coloring of $\mathbb{N}$ such that there is no $\vec{x} = \langle x_t \rangle_{t=1}^\infty$ such that for all $t, s \in \mathbb{N}$, $\min \mathrm{supp}_\beta(x_t) = \min \mathrm{supp}_\beta(x_s)$ and $M\vec{x}$ is monochromatic with respect to this coloring.*

Proof. Pick $\tau \in \mathbb{N}$, distinct primes $p_1, p_2, \ldots, p_\tau$, and $\alpha(1), \alpha(2), \ldots, \alpha(\tau) \in \mathbb{N}$ such that $\beta = \prod_{i=1}^\tau p_i^{\alpha(i)}$. Let $a = \sum_{t=1}^k a_t$, let

$$\begin{aligned} \sigma = 1 + \quad & \max(\{t \in \mathbb{N} : (\exists i \in \{1, 2, \ldots, \tau\})\ (p_i^t \text{ divides } a)\} \cup \\ & \cup \{t \in \mathbb{N} : (\exists i \in \{1, 2, \ldots, \tau\})\ (p_i^t \text{ divides } a - 1)\}) \end{aligned}$$

and let $m = \sigma + \max\big\{\alpha(i) : i \in \{1, 2, \ldots, \tau\}\big\}$.

For $x \in \mathbb{N}$, let $q(x) = \min \mathrm{supp}_\beta(x)$. Let ν be a finite coloring of $\mathbb{N}$ so that for $x, y \in \mathbb{N}$, $\nu(x) = \nu(y)$ if and only if $q(x) \equiv q(y) \pmod{m}$ and $x/\beta^{q(x)} \equiv y/\beta^{q(y)} \pmod{\beta^m}$. Thus ν has $m(\beta^m - 1)$ color classes. Suppose that we have $\vec{x} = \langle x_t \rangle_{t=1}^\infty$ such that for all $t, s \in \mathbb{N}$, $q(x_t) = q(x_s)$ and $M\vec{x}$ is monochromatic with respect to ν. Let q be the constant value of $q(x_t)$.

Pick $b \in \{0, 1, \ldots, \beta^m - 1\}$ and for each $t \in \mathbb{N}$ pick $c_t \in \omega$ such that $x_t/\beta^q = c_t \beta^m + b$. Then each $x_t = c_t \beta^{m+q} + b\beta^q$, and since $q(x_t) = q$, we have that β does not divide b. Pick $i \in \{1, 2, \ldots, \tau\}$ and $\delta \in \{0, 1, \ldots, \alpha(i) - 1\}$ such that p_i^δ divides b and $p_i^{\delta+1}$ does not divide b.

Pick $z \in \mathbb{Z}$ and $d \in \{0, 1, \ldots, \beta^m - 1\}$ such that $a = z\beta^m + d$. Then $\sum_{t=1}^k a_t x_t = (\sum_{t=1}^k a_t c_t + zb)\beta^{m+q} + db\beta^q$. Since $m > \sigma$, $d \neq 0$.

Case 1. β divides db. Let r be the largest integer such that β^r divides db and let s be the largest integer such that p_i^s divides d. Then $s < \sigma$ since otherwise we would have p_i^σ divides a. Also $p_i^{s+\delta}$ is the largest power of p_i which divides bd so $r \leq s + \delta < m$. Thus $q(\sum_{i=1}^k a_t x_t) = r + q \not\equiv q \pmod{m}$, a contradiction.

Case 2. β does not divide db. Then $q(\sum_{t=1}^k a_t x_t) = q$ and

$$(\sum_{t=1}^k a_t x_t)/\beta^q = (\sum_{t=1}^k a_t c_t + zb)\beta^m + db\,.$$

Therefore $db \equiv b \pmod{\beta^m}$ so β^m divides $(d-1)b$. Let s be the largest integer such that p_i^s divides $d - 1$. Then $s < \sigma$ since otherwise we would have p_i^σ divides $a - 1$. Also $p_i^{s+\delta}$ is the largest power of p_i which divides $(d-1)b$ so $m \leq s + \delta$, a contradiction. $\square$

Convention 2.9. For the rest of this section we will assume that we have a fixed $\beta \in \mathbb{N} \setminus \{1\}$, a fixed $k \in \mathbb{N}$, and a fixed sequence $\vec{a} = \langle a_1, a_2, \ldots, a_k \rangle$ in $\mathbb{Z} \setminus \{0\}$ such that for each t, either a_t or $-a_t$ is a non-negative power of β.

Definition 2.10. (1) For each $t \in \{1, 2, \ldots, k\}$, let $\gamma(t) \in \omega$ such that either $a_t = \beta^{\gamma(t)}$ or $a_t = -\beta^{\gamma(t)}$.

(2) Let $\mu = \max\big(\big\{2\gamma(t) + 1 : t \in \{1, 2, \ldots, k\}\big\} \cup \{\beta\}\big)$.

(3) Let $x, y \in \mathbb{N}$. Then $x \gg y$ if and only if $\max \operatorname{supp}_\beta(y) + \mu + 2 < \min \operatorname{supp}_\beta(x)$.

Definition 2.11. (1) A *digit block* in the base β expansion of an integer x is a maximal set of consecutive occurrences of the same digit located between $\min \operatorname{supp}_\beta(x)$ and $\max \operatorname{supp}_\beta(x)$.

(2) For each $j \in \{1, 2, \ldots, \mu\}$, define $g_j : \mathbb{N} \to \omega$ by $g_j(x)$ the number of digit blocks in the base β expansion of x of length congruent to $j \pmod{\mu}$.

For example, if $\beta = 5$ and $x = 3222000042311003320000000$, then there are five digit blocks of length 1, three of length 2, and one each of length 3 and 4. (We do not count 000000.)

Lemma 2.12. *Assume that $x \gg y$.*

(1) If the length of the rightmost digit blocks of x and y are both 1, the least significant nonzero digit of both x and y is $v \in \{1, 2, \ldots, \beta - 1\}$ and the digit of the next to rightmost digit blocks in both x and y is $w \neq v - 1$, then for all $j \in \{1, 2, \ldots, \mu\}$, $g_j(x - y) = g_j(x + y)$.

(2) If the length of the rightmost digit blocks of x and y are both 1, the least significant nonzero digit of both x and y is $v \in \{1, 2, \ldots, \beta - 1\}$, the digit of the next to rightmost digit blocks in both x and y is $v - 1$, and the lengths of the next to rightmost digit blocks of x and y are congruent to $\sigma \in \{1, 2, \ldots, \mu\} \pmod{\mu}$, then

(a) if $\sigma = 1$, then $g_1(x - y) = g_1(x + y) - 4$, $g_2(x - y) = g_2(x + y) + 2$, and for all $j \in \{3, 4, \ldots, \mu\}$, $g_j(x - y) = g_j(x + y)$;

(b) if $\sigma = \mu$, then $g_\mu(x - y) = g_\mu(x + y) - 2$ and for all $j \in \{1, 2, \ldots, \mu - 1\}$, $g_j(x - y) = g_j(x + y)$; and

(c) if $\sigma \notin \{1, \mu\}$, then $g_1(x - y) = g_1(x + y) - 2$, $g_\sigma(x - y) = g_\sigma(x + y) - 2$, $g_{\sigma+1}(x - y) = g_{\sigma+1}(x + y) + 2$ and for all $j \in \{1, 2, \ldots, \mu\} \setminus \{1, \sigma, \sigma + 1\}$, $g_j(x - y) = g_j(x + y)$.

(3) If the lengths of the rightmost digit blocks of x and y are both congruent to $2 \pmod{\mu}$, then $g_1(x - y) = g_1(x + y) + 4$, $g_2(x - y) = g_2(x + y) - 2$, and for all $j \in \{3, 4, \ldots, \mu\}$, $g_j(x - y) = g_j(x + y)$.

(4) *If the lengths of the rightmost digit blocks of x and y are both bigger than 1 but congruent to* $1 \pmod{\mu}$*, then* $g_\mu(x-y) = g_\mu(x+y)+2$ *and for all* $j \in \{1,2,\ldots,\mu-1\}$*,* $g_j(x-y) = g_j(x+y)$*.*

(5) *If* $\sigma \in \{3,4,\ldots,\mu\}$ *and the lengths of the rightmost digit blocks of x and y are both congruent to* $\sigma \pmod{\mu}$*, then* $g_1(x-y) = g_1(x+y)+2$*,* $g_\sigma(x-y) = g_\sigma(x+y)-2$*,* $g_{\sigma-1}(x-y) = g_{\sigma-1}(x+y)+2$*, and for all* $j \in \{1,2,\ldots,\mu\}\setminus\{1,\sigma,\sigma+1\}$*,* $g_j(x-y) = g_j(x+y)$*.*

Proof. The block of 0's between $\max \operatorname{supp}_\beta(y)$ and $\min \operatorname{supp}_\beta(x)$ in $x+y$ becomes a block of $\beta-1$'s in the same location and of the same length in $x-y$. And any block of v's in y above the rightmost two blocks becomes a block of $\beta-1-v$'s in $x-y$. The adjustments in each case come from observing what happens to the rightmost one or two digit blocks of x and the rightmost one or two digit blocks of y when y is subtracted from x. □

Lemma 2.13. *Assume that* $a_1 = a_k = 1$ *and it is not the case that* $a_1 = a_2 = \ldots = a_k = 1$*. Let* $M = M(\vec{a})$*. Let* φ *be a coloring of* $\mathbb{N}$ *(with* $(k-1)^\mu$ *color classes) such that for x and y in* $\mathbb{N}$*,* $\varphi(x) = \varphi(y)$ *if and only if* $g_j(x) \equiv g_j(y) \pmod{k-1}$ *for each* $j \in \{1,2,\ldots,\mu\}$*. There does not exist* $\vec{x} = \langle x_t\rangle_{t=1}^\infty$ *such that for each* $t \in \mathbb{N}$*,* $x_{t+1} \gg x_t$ *and* $M\vec{x}$ *is monochromatic with respect to* φ*.*

Proof. Suppose we have such an $\vec{x}$. By thinning the sequence we may suppose that we have ρ and λ in $\{1,2,\ldots,\mu\}$ such that for all $t \in \mathbb{N}$, $\max \operatorname{supp}_\beta(x_t) \equiv \lambda \pmod{\mu}$ and $\min \operatorname{supp}_\beta(x_t) \equiv \rho \pmod{\mu}$.

Define for $j \in \{1,2,\ldots,\mu\}$,

$$y_j = |\{t \in \{1,2,\ldots,k-1\} : \rho + \gamma(t+1) - \lambda - \gamma(t) - 1 \equiv j \pmod{\mu}\}|.$$

Then y_j is the number of blocks of 0's of length congruent to $j \pmod{\mu}$ that are introduced between $\beta^{\gamma(t+1)}x_{t+1}$ and $\beta^{\gamma(t)}x_t$ for $t \in \{1,2,\ldots,k\}$.

We now show

$$(**) \quad \text{it is not the case that } y_j \equiv 0 \pmod{k-1} \text{ for each } j \in \{1,2,\ldots,\mu\}.$$

To see this, suppose that $y_j \equiv 0 \pmod{k-1}$ for each $j \in \{1,2,\ldots,\mu\}$. Pick $j \in \{1,2,\ldots,\mu\}$ such that $j \equiv \rho + \gamma(2) - \lambda - \gamma(1) - 1 \pmod{\mu}$. Then $y_j > 0$ so $y_j = k-1$. Therefore for $t \in \{2,3,\ldots,k-1\}$, $\gamma(t+1)-\gamma(t) \equiv \gamma(2)-\gamma(1) \pmod{\mu}$. We claim that each $\gamma(t) = 0$, so that $a_1 = a_2 = \ldots = a_k = 1$, which we have forbidden.

Suppose then that t is the largest in $\{1,2,\ldots,k\}$ such that $\gamma(t) \neq 0$ and note that $t \notin \{1,k\}$. Then $0 - \gamma(t) \equiv \gamma(2) - 0 \pmod{\mu}$ so $\gamma(t) + \gamma(2) \equiv 0 \pmod{\mu}$ while $0 < \gamma(t) + \gamma(2) < \mu$, a contradiction. Thus $(**)$ has been established.

We start with the case where $a_1, a_2, \ldots, a_k > 0$. Then for each $j \in \{1, 2, \ldots, \mu\}$,

$$g_j(\sum_{t=1}^k a_t x_t) = \sum_{t=1}^k g_j(x_t) + y_j \text{ and } g_j(\sum_{t=1}^k a_t x_t) \equiv g_j(x_k) \pmod{k-1}$$

so

$$\sum_{t=1}^{k-1} g_j(x_t) + y_j \equiv 0 \pmod{k-1}.$$

Since also $g_j(x_t) \equiv g_j(x_s) \pmod{k-1}$ for $t, s \in \{1, 2, \ldots, k\}$, we have each $y_j \equiv 0 \pmod{k-1}$, contradicting $(**)$.

Next assume that at least one of $a_1, a_2, \ldots, a_k$ is negative. By thinning we may further suppose that for all $t, s \in \mathbb{N}$,

(1) the length of the rightmost digit blocks of x_t and x_s are congruent $\pmod{\mu}$;

(2) the length of the next to rightmost digit blocks of x_t and x_s are congruent $\pmod{\mu}$;

(3) the length of the rightmost digit block of x_t is 1 if and only if the length of the rightmost digit block of x_s is 1;

(4) the least significant digits of x_t and x_s are equal; and

(5) the digits of the next to rightmost digit blocks of x_t and x_s are equal.

Pick $m \in \omega$ and $0 = \delta(0) < \alpha(1) < \delta(1) < \ldots < \alpha(m) < \delta(m) < \alpha(m+1) = k$ such that for $i \in \{0, 1, \ldots, m\}$, if $\delta(i) < t \leq \alpha(i+1)$, then $a_t = \beta^{\gamma(t)}$, and for $i \in \{1, 2, \ldots, m\}$, if $\alpha(i) < t \leq \delta(i)$, then $a_t = -\beta^{\gamma(t)}$. Then $1 \leq 2m+1 \leq k$. As at least one a_i is negative, we must have $m \geq 1$.

Now

$$\begin{aligned} \sum_{t=1}^k a_t x_t = \ & \sum_{i=1}^m \big((\sum_{t=\delta(i)+1}^{\alpha(i+1)} \beta^{\gamma(t)} x_t) - (\sum_{t=\alpha(i)+1}^{\delta(i)} \beta^{\gamma(t)} x_t) \big) \\ & + \sum_{t=1}^{\alpha(1)} \beta^{\gamma(t)} x_t \,. \end{aligned}$$

For each $i \in \{1, 2, \ldots, m\}$ and each $j \in \{1, 2, \ldots, \mu\}$ let

$$\begin{aligned} w_{i,j} \ & = g_j\big((\sum_{t=\delta(i)+1}^{\alpha(i+1)} \beta^{\gamma(t)} x_t) - (\sum_{t=\alpha(i)+1}^{\delta(i)} \beta^{\gamma(t)} x_t) \big) \\ & \quad - g_j\big((\sum_{t=\delta(i)+1}^{\alpha(i+1)} \beta^{\gamma(t)} x_t) + (\sum_{t=\alpha(i)+1}^{\delta(i)} \beta^{\gamma(t)} x_t) \big) \end{aligned}$$

and for $j \in \{1, 2, \ldots, \mu\}$, let $z_j = \sum_{i=1}^m w_{i,j}$. Note that z_j is the total change in g_j going from $\sum_{t=1}^k \beta^{\gamma(t)} x_t$ (wherein the minus signs are ignored) to $\sum_{t=1}^k a_t x_t$.

For $j \in \{1, 2, \ldots, \mu\}$, let

$$v_j = |\{t \in \{1, 2, \ldots, m-1\} : \rho + \gamma(\alpha(i)+1) - \lambda - \gamma\big(\alpha(i)\big) - 1 \equiv j \pmod{\mu}\}|\,.$$

Note that v_j is the number of blocks of 0's of length congruent to $j \pmod{\mu}$ between terms of the form $\beta^{\gamma(\alpha(i))} x_{\alpha(i)}$ and $\beta^{\gamma(\alpha(i)+1)} x_{\alpha(i)+1}$. Here $\beta^{\gamma(\alpha(i))} x_{\alpha(i)}$ is the largest term in $(\sum_{t=\delta(i-1)+1}^{\alpha(i)} \beta^{\gamma(t)} x_t) + (\sum_{t=\alpha(i-1)+1}^{\delta(i-1)} \beta^{\gamma(t)} x_t)$ if $i > 1$ and is the largest term in $\sum_{t=1}^{\alpha(1)} \beta^{\gamma(t)} x_t$ if $i = 1$. Moreover $\beta^{\gamma(\alpha(i)+1)} x_{\alpha(i)+1}$ is the smallest term in $(\sum_{t=\delta(i)+1}^{\alpha(i+1)} \beta^{\gamma(t)} x_t) + (\sum_{t=\alpha(i)+1}^{\delta(i)} \beta^{\gamma(t)} x_t)$.

Then for each $j \in \{1, 2, \ldots, \mu\}$,

$$\begin{aligned}
\textstyle\sum_{t=1}^{k} g_j(x_t) + y_j \quad &= g_j(\textstyle\sum_{t=1}^{k} \beta^{\gamma(t)} x_t) \\
&= \textstyle\sum_{i=1}^{m} g_j\big((\sum_{t=\delta(i)+1}^{\alpha(i+1)} \beta^{\gamma(t)} x_t) + (\sum_{t=\alpha(i)+1}^{\delta(i)} \beta^{\gamma(t)} x_t)\big) + \\
&\quad\; g_j(\textstyle\sum_{t=1}^{\alpha(1)} \beta^{\gamma(t)} x_t) + v_j\,.
\end{aligned}$$

So for each $j \in \{1, 2, \ldots, \mu\}$,

$$\begin{aligned}
g_j(\textstyle\sum_{t=1}^{k} a_t x_t) \quad &= \textstyle\sum_{i=1}^{m} g_j\big((\sum_{t=\delta(i)+1}^{\alpha(i+1)} \beta^{\gamma(t)} x_t) - (\sum_{t=\alpha(i)+1}^{\delta(i)} \beta^{\gamma(t)} x_t)\big) + \\
&\quad\; g_j(\textstyle\sum_{t=1}^{\alpha(1)} \beta^{\gamma(t)} x_t) + v_j \\
&= \textstyle\sum_{i=1}^{m} w_{i,j} + \\
&\quad\; \textstyle\sum_{i=1}^{m} g_j\big((\sum_{t=\delta(i)+1}^{\alpha(i+1)} \beta^{\gamma(t)} x_t) + (\sum_{t=\alpha(i)+1}^{\delta(i)} \beta^{\gamma(t)} x_t)\big) + \\
&\quad\; g_j(\textstyle\sum_{t=1}^{\alpha(1)} \beta^{\gamma(t)} x_t) + v_j \\
&= z_j + \textstyle\sum_{t=1}^{k} g_j(x_t) + y_j\,.
\end{aligned}$$

Since $g_j(\sum_{t=1}^{k} a_t x_t) \equiv g_j(x_k) \pmod{k-1}$ we have that $0 \equiv z_j + \sum_{t=1}^{k-1} g_j(x_t) + y_j \pmod{k-1}$ and since $g_j(x_t) \equiv g_j(x_s) \pmod{k-1}$ when $t, s \in \{1, 2, \ldots, k-1\}$, we have for each $j \in \{1, 2, \ldots, \mu\}$ that $y_j + z_j \equiv 0 \pmod{k-1}$.

Notice also that for each $j \in \{1, 2, \ldots, \mu\}$ and each $i, i' \in \{1, 2, \ldots, m\}$, $w_{i,j} = w_{i',j}$. (This is because the lengths and digits of the two rightmost digit blocks of x are not affected when x is multiplied by a power of β and so the relevant case of Lemma 2.12 is the same for both.)

To complete the proof we will show that for each $j \in \{1, 2, \ldots, \mu\}$,

$$z_j \equiv 0 \pmod{k-1}\,,$$

and therefore for each $j \in \{1, 2, \ldots, \mu\}$, $y_j \equiv 0 \pmod{k-1}$, contradicting $(**)$.

Now for $t, s \in \{1, 2, \ldots, k\}$, x_t and x_s satisfy the same one of the cases described in Lemma 2.12. If that is case (1), i.e., if the length of the rightmost digit block of x_t is 1, the least significant nonzero digit of x_t is $v \in \{1, 2, \ldots, \beta - 1\}$, and the digit of the next to rightmost digit block in x_t is $w \neq v - 1$, then we have that $w_{i,j} = 0$ for each $i \in \{1, 2, \ldots, m\}$ and each $j \in \{1, 2, \ldots, \mu\}$ so that $z_j = 0$ for each $j \in \{1, 2, \ldots, \mu\}$.

If that case is any of the other cases described in Lemma 2.12, then we have that for each $i \in \{1, 2, \ldots, m\}$, $\sum_{j=1}^{\mu} w_{i,j} = -2$ or for each $i \in \{1, 2, \ldots, m\}$, $\sum_{j=1}^{\mu} w_{i,j} = 2$. Thus

$$\textstyle\sum_{j=1}^{\mu} z_j = \sum_{j=1}^{\mu} \sum_{i=1}^{m} w_{i,j} = \sum_{i=1}^{m} \sum_{j=1}^{\mu} w_{i,j} = -2m \text{ or } \sum_{j=1}^{\mu} z_j = 2m\,.$$

Therefore $\sum_{j=1}^{\mu} y_j + \sum_{j=1}^{\mu} z_j = k-1+\sum_{j=1}^{\mu} z_j \equiv 2m \pmod{k-1}$ or $\sum_{j=1}^{\mu} y_j + \sum_{j=1}^{\mu} z_j \equiv -2m \pmod{k-1}$. Since also $\sum_{j=1}^{\mu} y_j + \sum_{j=1}^{\mu} z_j = \sum_{j=1}^{\mu}(y_j+z_j) \equiv 0 \pmod{k-1}$ we conclude that $2m = k-1$. In each of cases (2), (3), (4), or (5) of Lemma 2.12 and each $j \in \{1,2,\ldots,\mu\}$ we have that z_j is an even multiple of m, and thus $z_j \equiv 0 \pmod{k-1}$ as claimed. □

For the statement of the main theorem we restate our standing hypotheses.

Theorem 2.2. *Let $k \in \mathbb{N}$, let $\beta \in \mathbb{N}\setminus\{1\}$, and let $\vec{a} = \langle a_1, a_2, \ldots, a_k\rangle$ be a sequence such that for each $t \in \{1,2,\ldots,k\}$, there is some $\gamma \in \omega$ such that $a_t = \beta^\gamma$ or $a_t = -\beta^\gamma$. Then $M(\vec{a})$ is image partition regular if and only if one of*

(1) $a_1 + a_2 + \ldots + a_k = 0$ *and* $a_k = 1$*;*

(2) $a_1 + a_2 + \ldots + a_k = 1$*; or*

(3) $a_1 = a_2 = \ldots = a_k = 1$.

Proof. The sufficiency is Lemma 2.7. For the necessity, by Lemma 2.6 we have one of

(1) $a_1 + a_2 + \ldots + a_k = 0$ and $a_k = 1$;

(2) $a_1 + a_2 + \ldots + a_k = 1$; or

(3) $a_1 = a_k = 1$.

If (1) or (2) holds, we are done. So assume that (3) holds and $a_1 + a_2 + \ldots + a_k \notin \{0,1\}$. Let $M = M(\vec{a})$. Let φ be the coloring of $\mathbb{N}$ in Lemma 2.13 and let ν be a coloring of $\mathbb{N}$ as guaranteed by Lemma 2.8. Define a coloring ψ of $\mathbb{N}$ such that $\psi(x) = \psi(y)$ if and only if $\varphi(x) = \varphi(y)$ and $\nu(x) = \nu(y)$. Suppose we have a sequence $\vec{x} = \langle x_t\rangle_{t=1}^{\infty}$ such that $M\vec{x}$ is monochromatic with respect to ψ. By thinning we may suppose that either for all $t, s \in \mathbb{N}$, $\min \operatorname{supp}_\beta(x_t) = \min \operatorname{supp}_\beta(x_s)$, or for all $t \in \mathbb{N}$, $x_{t+1} \gg x_t$. The first assumption contradicts Lemma 2.8. By Lemma 2.13, we must have that $a_1 = a_2 = \ldots = a_k = 1$. □

We state now the natural conjecture.

Conjecture 2.14. *Let $k \in \mathbb{N}$ and let $\vec{a} = \langle a_1, a_2, \ldots, a_k\rangle$ be a sequence in $\mathbb{Z}\setminus\{0\}$. Then $M(\vec{a})$ is image partition regular if and only if one of*

(1) $a_1 + a_2 + \ldots + a_k = 0$ *and* $a_k = 1$*;*

(2) $a_1 + a_2 + \ldots + a_k = 1$*; or*

(3) $a_1 = a_2 = \ldots = a_k = 1$.

3. The Finite Version

Recall that if $k, m \in \mathbb{N}$ with $m \geq k$ and $\vec{a} = \langle a_1, a_2, \ldots, a_k \rangle$ is a sequence in $\mathbb{Z} \setminus \{0\}$, then $M_m(\vec{a})$ is a $\left(\begin{pmatrix} m \\ k \end{pmatrix} + m \right) \times m$ matrix with all rows of 0's and 1's with a single 1 as well as all rows whose nonzero entries are $a_1, a_2, \ldots, a_k$ in order.

Of course, as with the infinite version, there are many choices (obtained by permuting rows) for what one calls $M_m(\vec{a})$. The following is a reasonable choice for $M_4(\langle -2, 3 \rangle)$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ -2 & 0 & 3 & 0 \\ -2 & 0 & 0 & 3 \\ 0 & -2 & 3 & 0 \\ 0 & -2 & 0 & 3 \\ 0 & 0 & -2 & 3 \end{pmatrix}$$

Lemma 1.5 provides a computable way of verifying when $M_m(\vec{a})$ is image partition regular. Recall that I_n is the $n \times n$ identity matrix. Since $M_m(\vec{a})$ includes all of the rows of I_m, it is trivial that $\begin{pmatrix} I_m \\ M_m(\vec{a}) \end{pmatrix}$ is image partition regular if and only if $M_m(\vec{a})$ is image partition regular. Consequently, we have that $M_m(\vec{a})$ is image partition regular if and only if $\begin{pmatrix} M_m(\vec{a}) & -I_n \end{pmatrix}$ is kernel partition regular where $n = \begin{pmatrix} m \\ k \end{pmatrix} + m$.

Lemma 3.1. *Let $k, m \in \mathbb{N}$ with $m \geq k$ and let $\vec{a} = \langle a_1, a_2, \ldots, a_k \rangle$ be a sequence in $\mathbb{Z} \setminus \{0\}$. Then $M_m(\vec{a})$ is image partition regular if one of*

(1) $a_1 = 1$*;*

(2) $a_k = 1$*;*

(3) $a_1 + a_2 + \ldots + a_k = 1$*; or*

(4) $a_1 + a_2 + \ldots + a_k = 0$.

Proof. In cases (3) and (4) it is immediate that $\begin{pmatrix} M_m(\vec{a}) & -I_n \end{pmatrix}$ satisfies the columns condition where $n = \begin{pmatrix} m \\ k \end{pmatrix} + m$. (In case (3) take $I_1 = \{1, 2, \ldots, m+n\}$. In case (4), take $I_1 = \{1, 2, \ldots, m\}$.)

It is not hard to show that $\begin{pmatrix} M_m(\vec{a}) & -I_n \end{pmatrix}$ satisfies the columns condition in cases (1) and (2), but it is much easier to note that if $p = \max\{|a_1|, |a_2|, \ldots, |a_k|\}$, then, for any $(m,p,1)$-set X in $\mathbb{N}$, there is an $\vec{x}$ such that all entries of $M_m(\vec{a})\vec{x}$ are contained in X, so $M_m(\vec{a})$ is image partition regular by [1, Satz 3.1].

Alternatively, this follows easily from any of several characterizations in [6, Theorem 15.24]. □

We shall show that for sufficiently large m (and not all that large as such things go) the conditions of Lemma 3.1 do characterize the image partition regularity of $M_m(\vec{a})$. If $k = 1$ it is trivial that for all $m \geq 1$, $M_m(\vec{a})$ is image partition regular if and only if $a_1 = 1$. And if $k = 2$ it is at least routine to verify that for all $m \geq 2$, $M_m(\vec{a})$ is image partition regular if and only if $a_1 = 1$, $a_2 = 1$, $a_1 + a_2 = 1$, or $a_1 + a_2 = 0$.

Theorem 3.2. *Let $m, k \in \mathbb{N}$ with $k \geq 3$ and let $\langle a_1, a_2, \ldots, a_k \rangle$ be a sequence in $\mathbb{Z} \setminus \{0\}$. Assume that $m \geq 2k - 2$. Then $M_m(\vec{a})$ is image partition regular if and only if one of*

(1) $a_1 = 1$;

(2) $a_k = 1$;

(3) $a_1 + a_2 + \ldots + a_k = 1$; *or*

(4) $a_1 + a_2 + \ldots + a_k = 0$.

Proof. The sufficiency is Lemma 3.1.

For the necessity, let $B = B_m(\vec{a})$ and denote the columns of B by $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_m$. Then

$$M_m(\vec{a}) = \begin{pmatrix} I_m \\ B \end{pmatrix}$$

and $M_m(\vec{a})$ is image partition regular so by Lemma 1.5 $\begin{pmatrix} B & -I_n \end{pmatrix}$ is kernel partition regular and therefore satisfies the columns condition, where $n = \begin{pmatrix} m \\ k \end{pmatrix}$. In particular, there is a set J_1 of columns in $\begin{pmatrix} B & -I_n \end{pmatrix}$ summing to $\vec{0}$.

Let $J = J_1 \cap \{1, 2, \ldots, m\}$ and note that $\sum_{t \in J} \vec{c}_t$ is a vector all of whose entries are 0 or 1. Let $r = |J|$ and let $J = \{t_1, t_2, \ldots, t_r\}_<$. Let $D = \begin{pmatrix} \vec{c}_{t_1} & \vec{c}_{t_2} & \ldots & \vec{c}_{t_r} \end{pmatrix}$. Thus all rows of D sum to 0 or 1. Let $g = t_1$ and $h = t_r$.

`Case 1.` $r \geq k$. Then $\begin{pmatrix} a_1 & a_2 & \ldots & a_k & 0 & 0 & \ldots & 0 \end{pmatrix}$ is a row of D. (Or, if $r = k$, $\begin{pmatrix} a_1 & a_2 & \ldots & a_k \end{pmatrix}$ is a row of D.) So (3) or (4) holds.

`Case 2.` $r \leq k - 1$.

`Case 2(a).` $g \le m-r-k+2$. Then $|\{g+1, g+2, \ldots, m\} \setminus J| = m-g-(r-1) \ge k-1$ so $(\ a_1 \quad 0 \quad 0 \quad \ldots \quad 0\ )$ is a row of D (because there is room in the omitted columns to put $a_2, a_3, \ldots, a_k$) and thus $a_1 = 1$.

`Case 2(b).` $h \ge r+k-1$. Then $|\{1, 2, \ldots, h-1\} \setminus J| = h-1-(r-1) \ge k-1$ so $(\ 0 \quad 0 \quad \ldots \quad 0 \quad a_k\ )$ is a row of D so $a_k = 1$.

`Case 2(c).` $g \ge m-r-k+3$ and $h \le r+k-2$. Note that $g \ge m-r-k+3 \ge (2k-2)-(k-1)-k+3 = 2$ and $h \le r+k-2 \le 2k-3 \le m-1$. We also note that we cannot have $r = 1$. For if so, we would have $g = h$ and thus $m-1-k+3 \le g = h \le 1+k-2$ so $m \le 2k-3$, a contradiction.

`Case 2(c)(i).` $h > g+r-1$. We claim that in this case $(\ a_g \quad 0 \quad 0 \quad \ldots \quad 0\ )$, $(\ 0 \quad 0 \quad \ldots \quad 0 \quad a_{h-r+1}\ )$, and $(\ a_g \quad 0 \quad \ldots \quad 0 \quad a_{h-r+1}\ )$ are all rows of D. This will be a contradiction since it forces $a_g = a_{h-r+1} = 1$ and $a_g + a_{h-r+1} = 0$.

Now $m \ge 2k-2 \ge k+r-1$ so $(m-g)-(r-1) \ge k-g$. Consequently $|\{g+1, g+2, \ldots, m\} \setminus J| \ge k-g$, so there is room in the omitted columns above g to put $a_{g+1}, a_{g+2}, \ldots, a_k$, and thus $(\ a_g \quad 0 \quad 0 \quad \ldots \quad 0\ )$ is a row of D.

Next $|\{1, 2, \ldots, h-1\} \setminus J| = h-r$, so there is room in the omitted columns below h to put $a_1, a_2, \ldots, a_{h-r}$, and thus $(\ 0 \quad 0 \quad \ldots \quad 0 \quad a_{h-r+1}\ )$ is a row of D.

Finally (for this case) $|\{g+1, g+2, \ldots, h-1\} \setminus J| = (h-1-g)-(r-2) > h-r-g$, so there is room in the omitted columns between g and h to put $a_{g+1}, a_{g+2}, \ldots, a_{h-r}$. Also, $m \ge k+r-2$ so $|\{h+1, h+2, \ldots, m\}| = m-h \ge k-h+r-1$, so there is room in the omitted columns above h to put $a_{h-r+2}, a_{h-r+3}, \ldots, a_k$. Thus $(\ a_g \quad 0 \quad \ldots \quad 0 \quad a_{h-r+1}\ )$ is a row of D.

`Case 2(c)(ii).` $h = g+r-1$. Then $J = \{g, g+1, \ldots, g+r-1\}$. Assume first that $r \ge 3$. We have that $m \ge k+r-1 = k+h-g$ so

$$|\{h+1, h+2, \ldots, m\}| \ge k-g$$

so there is room above h to put $a_{g+1}, a_{g+2}, \ldots, a_k$ and thus $(\ a_g \quad 0 \quad \ldots \quad 0\ )$, $(\ a_{g-1} \quad a_g \quad 0 \quad \ldots \quad 0\ )$, and $(\ a_g \quad a_{g+1} \quad 0 \quad \ldots \quad 0\ )$ are all rows of D. Thus $a_g = 1$ and $a_{g-1} = a_{g+1} = -1$. If $r \ge 3$, then $(\ a_{g-1} \quad a_g \quad a_{g+1} \quad 0 \quad \ldots \quad 0\ )$ is a row of D while $a_{g-1} + a_g + a_{g+1} = -1$, a contradiction.

Therefore, $r = 2$. Now we will assume that $m \ge k+2$. (Since $m \ge 2k-2$, this automatically holds unless $k = 3$ and $m = 4$.) Then

$$|\{h+1, h+2, \ldots, m\}| = m-h \ge k+2-h = k-g+1,$$

so there is room above h for $a_g, a_{g+1}, \ldots, a_k$ and thus $(\ a_g \quad 0\ )$, $(\ a_{g-1} \quad 0\ )$, and $(\ a_{g-1} \quad a_g\ )$ are all rows of D, which is impossible.

Finally (for the whole proof) assume $k = 3$ and $m = 4$. We have

$$B = B_4(\vec{a}) = \begin{pmatrix} a_1 & a_2 & a_3 & 0 \\ a_1 & a_2 & 0 & a_3 \\ a_1 & 0 & a_2 & a_3 \\ 0 & a_1 & a_2 & a_3 \end{pmatrix}.$$

Since, as we saw at the start of case 2(c), $g \geq 2$ and $h \leq m-1$, we must have that $J = \{2,3\}$. We then have that each of $a_2 + a_3$, a_2, and $a_1 + a_2$ are in $\{0,1\}$ and $a_2 \neq 0$ so we must have that $a_2 = 1$ and $a_1 = a_3 = -1$ and thus

$$\begin{pmatrix} B & -I_4 \end{pmatrix} = \begin{pmatrix} -1 & 1 & -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

The only set of columns of this matrix that sum to $\vec{0}$ is $J_1 = \{2,3,6,7\}$. The span of these four columns is

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : x_4 = -x_1 \right\}$$

and as columns 1, 4, 5, and 8 have no positive entries, no sum of these remaining columns is in W. $\square$

From the characterizations of Theorems 2.2 and 3.2 we see that it is easy to produce examples of sequences $\vec{a}$ such that $M_m(\vec{a})$ is image partition regular for all sufficiently large m but $M(\vec{a})$ is not image partition regular.

We conclude by showing that the bound of Theorem 3.2 is best possible.

Theorem 3.3. *Let $k \in \mathbb{N}$ with $k \geq 3$, let $m = 2k-3$, let a_1 and a_k be any elements of $\mathbb{Z} \setminus \{0\}$ and let $a_2 = a_3 = \ldots = a_{k-1} = 1$. Then $M_m(\vec{a})$ is image partition regular.*

Proof. Let $B = B_m(\vec{a})$ and let $n = \begin{pmatrix} m \\ k \end{pmatrix}$. Then $M_m(\vec{a}) = \begin{pmatrix} I_m \\ B \end{pmatrix}$ so by Lemma 1.5 it suffices to show that $E = \begin{pmatrix} B & -I_n \end{pmatrix}$ satisfies the columns condition. Denote the columns of E by $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_{m+n}$.

For $j \in \{1,2,\ldots,k-2\}$, let $f(j) = k-2+j$. Let

$$I_1 = \{f(1)\} \cup \{m+t : t \in \{1,2,\ldots,n\} \text{ and } b_{t,f(1)} \neq 0\}.$$

Given $j \in \{2,3,\ldots,k-2\}$ let

$$I_j = \{f(j)\} \cup \{m+t : \begin{array}{l} t \in \{1,2,\ldots,n\},\, b_{t,f(j)} \neq 0\,, \\ \text{and } (\forall i \in \{1,2,\ldots,j-1\})(b_{t,f(i)} = 0)\}. \end{array}$$

Let $I_{k-1} = \{1,2,\ldots,m+n\} \setminus \bigcup_{j=1}^{k-2} I_j$. We shall show that $\{I_1, I_2, \ldots, I_{k-1}\}$ is as required to show that E satisfies the columns condition.

First, since there are only $k-2$ columns in front of column $f(1) = k-1$, a_k does not occur in column $f(1)$. And, since there are only $k-2$ columns after column

$f(1)$, a_1 does not occur in column $f(1)$. Therefore, $\sum_{i\in I_1}\vec{c}_i=\vec{0}$. Also, since each row of B has $k-3$ zeroes we have that

$$\{m+t : t\in\{1,2,\ldots,n\}\}\subseteq \bigcup_{j=1}^{k-2} I_j$$

so $\{\vec{c}_i : i\in\bigcup_{j=1}^{k-2} I_j\}$ spans $\mathbb{Q}^n$ and in particular $\sum_{i\in I_{k-1}}\vec{c}_i$ is in this span.

Thus it remains only to show that for $j\in\{2,3,\ldots,k-2\}$, $\sum_{i\in I_j}\vec{c}_i$ is in the span of $\{\vec{c}_i : i\in\bigcup_{s=1}^{j-1} I_s\}$. To this end, let $j\in\{2,3,\ldots,k-2\}$ and let $\vec{d}_j=\sum_{i\in I_j}\vec{c}_i$. It suffices to show that $\vec{d}_j$ is in the span of

$$\{\vec{c}_{m+t} : t\in\{1,2,\ldots,n\} \text{ and } (\exists i\in\{1,2,\ldots,j-1\})(b_{t,f(i)}\neq 0\}\,.$$

For this, we need to show that if $t\in\{1,2,\ldots,n\}$ and $d_j(t)\neq 0$, then $(\exists i\in\{1,2,\ldots,j-1\})(b_{t,f(i)}\neq 0)$. Then, letting

$$L=\{t\in\{1,2,\ldots,n\} : d_j(t)\neq 0\}\,,$$

we have $\vec{d}_j=\sum_{t\in L} d_j(t)\vec{c}_{m+t}$ and $\{m+t : t\in L\}\subseteq\bigcup_{s=1}^{j-1} I_s$. So let $t\in\{1,2,\ldots,n\}$, assume that $d_j(t)\neq 0$, and suppose that

$$(\forall i\in\{1,2,\ldots,j-1\})(b_{t,f(i)}=0)\,.$$

Then $m+t\in I_j$ so it must be that $b_{t,f(j)}\neq 1$. And since $f(j)\geq k$, this says that $b_{t,f(j)}=a_k$. Since $b_{t,k-1}=b_{t,k}=\ldots=b_{t,f(j-1)}=0$, there are at most $k-2$ nonzero entries in row t before column $f(j)$ so $b_{t,f(j)}\neq a_k$. This contradiction completes the proof. □

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