EIGENVALUES AND ARITHMETIC FUNCTIONS ON $\mathbf{PSL_2}(\mathbb{Z})$

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Abstract

Over the past decade, various properties of the irrational factor function $I(n) = \prod_{p^\nu||n} p^{1/\nu}$ and strong restrictive factor function $R(n) = \prod_{p^\nu||n} p^{\nu-1}$ have been investigated by several authors. This study led to a generalization to a class of arithmetic functions associated to elements of $\mathrm{PSL}_2(\mathbb{Z})$. In the present paper, we study the possible influence of the eigenvalues of an element A of $\mathrm{PSL}_2(\mathbb{Z})$ on the behavior of the associated arithmetic function $f_A(n) = \prod_{p^\nu\|n} p^{A(\nu)}$, where $A(z) = (az+b)/(cz+d)$ is the linear fractional transformation induced by the matrix A. In particular, we obtain results on the local density of eigenvalues through their natural connection to a particular surface.

1. Introduction and Statement of Results

There has been recent interest in examining the behavior of the arithmetic functions $f_A(n)$ defined on natural numbers n in terms of the action of a matrix A in $\mathrm{PSL}_2(\mathbb{Z})$. Given an element

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

of $\mathrm{PSL}_2(\mathbb{Z})$, one may consider the linear fractional transformation induced by A,

$$A(z) = \frac{az+b}{cz+d},$$

and define the arithmetic function given for each positive integer n by

$$f_A(n) = \prod_{p^\nu \| n} p^{A(\nu)}.$$

These functions generalize the two arithmetic functions

$$I(n) = \prod_{p^\nu || n} p^{1/\nu}$$

and

$$R(n) = \prod_{p^\nu || n} p^{\nu-1},$$

which were introduced by Atanassov in [2] and [3]. These multiplicative functions satisfy the inequality

$$I(n)R(n)^2 \geq n,$$

for each $n \geq 1$, with equality if and only if n is square-free. If $S(n)$ denotes the square-free part of n and if n is k-power free, then $S(n)$ satisfies the inequalities

$$S(n) \geq n^{1/(k-1)}$$

and

$$I(n) \geq S(n)^{1/(k-1)} \geq n^{1/(k-1)^2}.$$

On the other hand, if n is k-power full, then $S(n)$ satisfies the inequality

$$I(n) \leq S(n)^{1/k}.$$

In this fashion, $I(n)$ roughly measures how far a given integer n is away from being either k-power free or k-power full.

In [11], two of the authors more fully develop this measure by studying weighted combinations $I(n)^\alpha R(n)^\beta$ for real-valued α and β. In [10], Panaitopol showed that

$$\sum_{n=1}^{\infty} \frac{1}{I(n)R(n)\varphi(n)} < e^2.$$

He further proved that the arithmetic function

$$G(n) = \prod_{\nu=1}^{n} I(\nu)^{1/n}$$

satisfies the inequalities

$$\frac{n}{e^7} < G(n) < n,$$

for each $n \geq 1$. Alkan and two of the authors [1] established an asymptotic formula for $G(n)$ and proved that the sequence $\{G(n)/n\}_{n\geq 1}$ is convergent. They further obtained results that show that $I(n)$ is very regular on average. Further improvements have recently been obtained by Koninck and Kátai [7]. Asymptotic formulas for certain weighted real moments of $R(n)$ were obtained in [9].

In the above more general setting, one realizes $I(n)$ and $R(n)$ as $f_{A_1}(n)$ and $f_{A_2}(n)$, respectively, with

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$A_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Results on averages of $f_A(n)$ have recently been established in [12]. That work generalizes $I(n)$ and $R(n)$ to a class of elements of $\mathrm{PSL}_2(\mathbb{Z})$ and explores some of the properties of these maps.

For each given matrix A and a positive real number x, we define the weighted average

$$M_A(x) = \sum_{1\leq n\leq x} \left(1 - \frac{n}{x}\right) f_A(n).$$

We also consider λ_A^+ and λ_A^-, the positive and negative real eigenvalues of A, respectively. Thus, λ_A^+ and λ_A^- are solutions of the quadratic equation

$$\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0,$$

with

$$\lambda_A^+ = \frac{a+d+\sqrt{(a+d)^2+4}}{2} \tag{1}$$

and

$$\lambda_A^- = \frac{a+d-\sqrt{(a+d)^2+4}}{2}. \tag{2}$$

Furthermore, λ_A^+ and λ_A^- satisfy the inequalities $\lambda_A^- < 0 < \lambda_A^+$ and the identity $\lambda_A^+\lambda_A^- = -1$.

In the present paper, for a large Q and a much larger x, we consider the following subset of $\mathrm{PSL}_2(\mathbb{Z})$:

$$\mathcal{A}(Q,x) = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : 1 \leq a,b,c,d \leq Q, ad-bc = -1, \right.$$
$$\left. \left(\frac{\lambda_A^+}{Q}, Q\lambda_A^-, \frac{\log M_A(x)}{\log x} \right) \in \mathcal{S} \right\},$$

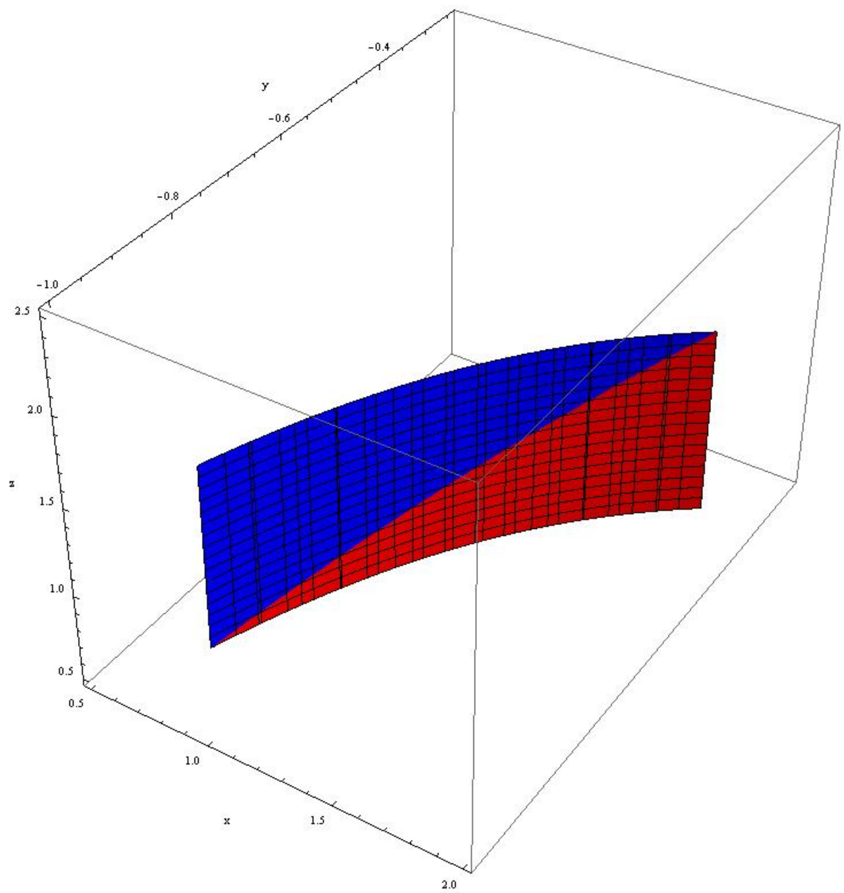

Figure 1: The surface $\mathcal{S}$.

where the surface $\mathcal{S}$ is given by

$$\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 \colon 1 < x, z < 2, xy = -1\}.$$

(See Figure 1.)

The map

$$\Psi_{Q,x} \colon \mathcal{A}(Q, x) \longrightarrow \mathcal{S},$$

defined by

$$\Psi_{Q,x}(A) = \left(\frac{\lambda_A^+}{Q}, Q\lambda_A^-, \frac{\log M_A(x)}{\log x} \right),$$

associates to each matrix $A \in \mathcal{A}(Q, x)$ a unique point on $\mathcal{S}$. In the first and second coordinates of such a point on $\mathcal{S}$, the eigenvalues λ_A^+ and λ_A^- of A are normalized, as λ_A^+ is divided by Q and λ_A^- is multiplied by Q. Furthermore, λ_A^+ is close to $a+d$, which can be $2Q$ at most. It follows that $\lambda_A^+/Q < 2$, with very few exceptions.

For the sake of simplicity, we restrict our attention to the case when λ_A^+/Q is in the interval $(1,2)$ and leave to the reader to make the adaptation to the case when λ_A^+/Q is in the interval $(0,1)$, as the two cases are similar.

In the third coordinate of such a point on $\mathcal{S}$, we observe that for any A with positive entries, $f_A(n) \geq 1$ for all n. It follows that $M_A(x) > x/2$. Hence,

$$\frac{\log M_A(x)}{\log x} > 1 - \frac{\log 2}{\log x}.$$

Finally, for simplicity's sake, we consider only the case when z is in the interval $(1,2)$. In like manner, one can study the case when z is in the interval $(2,\infty)$.

In the present paper, our purpose is to investigate the possible influence of the eigenvalues λ_A^+ and λ_A^- of A on the behavior of the associated arithmetic function $f_A(n)$. We seek to understand the joint distribution of λ_A^+, λ_A^-, and $(\log M_A(x))/\log x$, that is to say, the image of $\Psi_{Q,x}$ on $\mathcal{S}$. More precisely, for a given point $(\alpha,-1/\alpha,\beta)$ on $\mathcal{S}$ we consider, for each small $\delta > 0$, the neighborhood $\mathcal{V}_{\alpha,\beta,\delta}$ of $(\alpha,-1/\alpha,\beta)$ in $\mathcal{S}$ given by

$$\mathcal{V}_{\alpha,\beta,\delta} = \{(x,y,z) \in \mathcal{S} \colon |x-\alpha| < \delta, |z-\beta| < \delta\}.$$

We would like to estimate the number of matrices A in $\mathcal{A}(Q,x)$ for which $\Psi_{Q,x}(A)$ lies in $\mathcal{V}_{\alpha,\beta,\delta}$. We expect the number of such matrices to grow like a constant times $\delta^2 Q^2$ as Q and x tend to infinity, with x much larger than Q, while $\delta > 0$ is kept fixed. This leads us to consider the limit of the ratio

$$\frac{\#\{\Psi_{Q,x}^{-1}(\mathcal{V}_{\alpha,\beta,\delta})\}}{\delta^2 Q^2} = \frac{\#\{A \in \mathcal{A}(Q,x) \colon \Psi_{Q,x}(A) \in \mathcal{V}_{\alpha,\beta,\delta}\}}{\delta^2 Q^2},$$

as x approaches infinity and then Q approaches infinity. Lastly, we take the limit of this expression as $\delta \to 0^+$.

Our main result can be summarized as follows.

Theorem. *Fix a point $(\alpha,-1/\alpha,\beta) \in \mathcal{S}$, where α and β are real numbers such that $1 < \alpha, \beta < 2$. Then we have*

$$\lim_{\delta\to 0}\lim_{Q\to\infty}\lim_{x\to\infty} \frac{\#\{A \in \mathcal{A}(Q,x) \colon \Psi_{Q,x}(A) \in \mathcal{V}_{\alpha,\beta,\delta}\}}{\delta^2 Q^2} = \begin{cases} \dfrac{24}{\pi^2}\left(\dfrac{\beta-\alpha}{\beta-1}\right), & \text{if } \beta \geq \alpha; \\ 0, & \text{if } \beta < \alpha. \end{cases}$$

Thus, the images via $\Psi_{Q,x}$ of almost all matrices A lie on the part of the surface $\mathcal{S}$ where $z \geq x$, depicted in blue in Figure 1. If we fix two points $P_1 = (\alpha_1,-1/\alpha_1,\beta_1)$ and $P_2 = (\alpha_2,-1/\alpha_2,\beta_2)$ on that part of the surface $\mathcal{S}$ and compare the local densities of the points in $\Psi_{Q,x}(\mathcal{A}(Q,x))$ around P_1 and respectively P_2, as a direct consequence of our theorem we deduce the following corollary.

Corollary. *Let α_j and β_j be real numbers such that $1 < \alpha_j < \beta_j < 2$ for $j \in \{1, 2\}$. Then we have*

$$\lim_{\delta\to 0}\lim_{Q\to\infty}\lim_{x\to\infty}\frac{\#\{A \in \mathcal{A}(Q,x)\colon \Psi_{Q,x}(A) \in \mathcal{V}_{\alpha_1,\beta_1,\delta}\}}{\#\{A \in \mathcal{A}(Q,x)\colon \Psi_{Q,x}(A) \in \mathcal{V}_{\alpha_2,\beta_2,\delta}\}} = \frac{(\beta_1-\alpha_1)(\beta_2-1)}{(\beta_2-\alpha_2)(\beta_1-1)}.$$

2. Proof of the Theorem

We begin the proof by fixing an α and β in the interval $(1, 2)$ and a $\delta > 0$ small enough so that α and β belong to the interval $(1 + \delta, 2 - \delta)$. We also consider the set of matrices

$$\begin{aligned}\mathcal{D}_{\alpha,\beta,\delta,Q,x} = \Bigg\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{A}(Q,x)\colon 1 \le a,b,c \le d \le Q,\ ad-bc=-1,\\ (\alpha-\delta)Q \le a+d \le (\alpha+\delta)Q,\\ (\beta-1-\delta)d < b < (\beta-1+\delta)d \Bigg\}.\end{aligned}$$

The cardinality of $\mathcal{D}_{\alpha,\beta,\delta,Q,x}$ is given by

$$\begin{aligned}\#\mathcal{D}_{\alpha,\beta,\delta,Q,x} &= \sum_{1\le d\le Q}\ \sum_{\substack{1\le c\le d\\ \gcd(c,d)=1}} \#\{(a,b)\colon 1 \le a,b \le d, ad-bc=-1,\\ &\qquad (\alpha-\delta)Q \le a+d \le (\alpha+\delta)Q,\\ &\qquad (\beta-1-\delta)d < b < (\beta-1+\delta)d\} \\ &= \sum_{1\le d\le Q}\ \sum_{\substack{1\le c\le d\\ \gcd(c,d)=1\\ (\alpha-\delta)Q\le d+(c\bar{c}-1)/d\le(\alpha+\delta)Q\\ (\beta-1-\delta)d<\bar{c}<(\beta-1+\delta)d}} 1.\end{aligned} \tag{3}$$

Here, $\bar{c}$ is used to denote the unique multiplicative inverse of c modulo d in the interval $[1, d]$. The second step in (3) follows from the fact that the conditions $1 \le b \le d$ and $ad - bc = -1$ force b to equal $\bar{c}$. Hence, a is uniquely determined and given by $a = (bc - 1)/d$. Furthermore, the contribution of the terms in (3) for which $d < (\alpha - \delta)Q/2$ is zero. Indeed, since $a \le d$, we see that if $d < (\alpha - \delta)Q/2$, then $a + d < (\alpha - \delta)Q$.

Hence, setting $q = d$, $x = c$ and $y = \bar{c}$, we obtain $\#\mathcal{D}_{\alpha,\delta,Q}$ in the form

$$\#\mathcal{D}_{\alpha,\beta,\delta,Q,x} = \sum_{(\alpha-\delta)Q/2\le q\le Q} \#\{(x,y) \in \Omega_{\alpha,\beta,\delta,Q,q} \cap \mathbb{Z}^2\colon xy \equiv 1 \pmod q\}, \tag{4}$$

where

$$\begin{aligned}\Omega_{\alpha,\beta,\delta,Q,q} = \{(u,v) \in \mathbb{R}^2\colon 1 \le u,v \le q,\ (\alpha-\delta)qQ - q^2 \le uv \le (\alpha+\delta)qQ - q^2,\\ (\beta-1-\delta)q \le v \le (\beta-1+\delta)q\}.\end{aligned} \tag{5}$$

We estimate the summand in (4) by using a lemma due to Boca and Gologan [5].

Lemma 1 (Lemma 2.3 from [5]). *Assume that $q \geq 1$ and h are two integers, that $\mathcal{I}$ and $\mathcal{J}$ are intervals of length less than q, and that $f\colon \mathcal{I} \times \mathcal{J} \to \mathbb{R}$ is a C^1 function. Then for any integer $T > 1$ and any $\epsilon > 0$, we have*

$$\sum_{\substack{a \in \mathcal{I}, b \in \mathcal{J} \\ ab \equiv h \pmod q \\ \gcd(b,q)=1}} f(a,b) = \frac{\phi(q)}{q^2} \iint\limits_{\mathcal{I} \times \mathcal{J}} f(x,y)\, dx\, dy + \mathcal{E},$$

with

$$\mathcal{E} = O_\epsilon\left(T^2 \|f\|_\infty q^{1/2+\epsilon} \gcd(h,q)^{1/2} + T \|\nabla f\|_\infty q^{3/2+\epsilon} \gcd(h,q)^{1/2} + \frac{\|\nabla f\|_\infty |\mathcal{I}||\mathcal{J}|}{T} \right),$$

where $\phi(q)$ is the Euler totient function, $\|f\|_\infty$ and $\|\nabla f\|_\infty$ denote the sup-norm of f and $|\partial f/\partial x| + |\partial f/\partial y|$ on the region $\mathcal{I} \times \mathcal{J}$, respectively.

We break the region $\Omega_{\alpha,\beta,\delta,Q,q}$ into squares of side length $L = [Q^\eta]$ for some $0 < \eta < 1$, and denote by I_j those squares lying entirely within $\Omega_{\alpha,\beta,\delta,Q,q}$ and B_i those squares which intersect both $\Omega_{\alpha,\beta,\delta,Q,q}$ and its complement in $\mathbb{R}^2$, where $1 \leq j \leq n$ and $1 \leq i \leq m$ for some natural numbers n and m. We have

$$\begin{aligned} \#\{(u,v) \in \Omega_{\alpha,\beta,\delta,Q,q} \colon ab \equiv 1 \pmod q\} = &\sum_{1 \leq j \leq n} \#\{(u,v) \in I_j \colon ab \equiv 1 \pmod q\} \\ &+ \sum_{1 \leq i \leq m} \#\{(u,v) \in B_i \cap \Omega_{\alpha,\beta,\delta,Q,q} \colon \\ &\qquad ab \equiv 1 \pmod q\}. \end{aligned}$$

By Lemma 1, each of the summands on the right-hand side above is equal to

$$\frac{\phi(q)}{q^2} L^2 + O_\epsilon(q^{1/2+\epsilon}).$$

If we take Ω' to be the subset of $\Omega_{\alpha,\beta,\delta,Q,q}$ formed by removing from $\Omega_{\alpha,\beta,\delta,Q,q}$ an $L\sqrt{2}$-width neighborhood of the boundary of $\Omega_{\alpha,\beta,\delta,Q,q}$, then we find that $\Omega' \subset \bigcup I_j \subset \Omega_{\alpha,\beta,\delta,Q,q}$ and

$$\text{Area}(\Omega_{\alpha,\beta,\delta,Q,q}) - \text{Area}(\Omega') = O(qL).$$

Hence,

$$\text{Area}\left(\bigcup I_j\right) = \text{Area}(\Omega_{\alpha,\beta,\delta,Q,q}) + O(QL).$$

Since

$$\begin{aligned} \text{Area}\left(\bigcup I_j\right) &= \sum_{1 \leq j \leq n} \#\{(u,v) \in I_j \colon ab \equiv 1 \pmod q\} \\ &= n \frac{\phi(q)}{q^2} L^2 + O_\epsilon(n q^{1/2+\epsilon}), \end{aligned}$$

we have

$$nL^2 = \text{Area}(\Omega_{\alpha,\beta,\delta,Q,q}) + O(QL),$$

and in particular

$$n = O\left(\frac{Q^2}{L^2}\right).$$

Thus,

$$\begin{aligned}\sum_{1\le j\le n} \#\{(u,v)\in I_j \colon ab\equiv 1 \pmod q\} &= n\frac{\phi(q)}{q^2}L^2 + O_\epsilon(nq^{1/2+\epsilon})\\ &= \frac{\phi(q)}{q^2}(\text{Area}(\Omega_{\alpha,\beta,\delta,Q,q}) + O(QL))\\ &\quad + O_\epsilon\left(\frac{Q^2}{L^2}q^{1/2+\epsilon}\right)\\ &= \frac{\phi(q)}{q^2}\text{Area}(\Omega_{\alpha,\beta,\delta,Q,q}) + O(L)\\ &\quad + O_\epsilon\left(\frac{Q^{5/2+\epsilon}}{L^2}\right).\end{aligned}$$

Similarly, we find that $m = O(Q/L)$ and

$$\begin{aligned}0 &\le \sum_{1\le i\le m} \#\{(u,v)\in B_i\cap\Omega_{\alpha,\beta,\delta,Q,q} \colon ab\equiv 1 \pmod q\}\\ &\le \sum_{1\le i\le m} \#\{(u,v)\in B_i \colon ab\equiv 1 \pmod q\}\\ &= m\frac{\phi(q)}{q^2}L^2 + O_\epsilon(mq^{1/2+\epsilon}) = O(L) + O_\epsilon\left(\frac{Q^{3/2+\epsilon}}{L}\right).\end{aligned}$$

Taking $\eta = 5/6$, we have

$$\#\{(u,v)\in\Omega_{\alpha,\beta,\delta,Q,q} \colon ab\equiv 1 \pmod q\} = \frac{\phi(q)}{q^2}\text{Area}(\Omega_{\alpha,\beta,\delta,Q,q}) + O_\epsilon(Q^{5/6+\epsilon}).$$

Thus,

$$\#\mathcal{D}_{\alpha,\beta,\delta,Q,x} = M + E, \tag{6}$$

where

$$M = \sum_{(\alpha-\delta)Q/2\le q\le Q} \frac{\phi(q)}{q^2}\text{Area}(\Omega_{\alpha,\beta,\delta,Q,q}), \tag{7}$$

and

$$E = \sum_{(\alpha-\delta)Q/2\le q\le Q} E_{\alpha,\beta,\delta,Q,q} = O_\epsilon(Q^{11/6+\epsilon}). \tag{8}$$

To examine the main term M in (7), we recall from the definition of the set $\Omega_{\alpha,\beta,\delta,Q,q}$ in (5) that

$$(\alpha-\delta)qQ-q^2 \le uv \le (\alpha+\delta)qQ-q^2.$$

We first note that when $\alpha > \beta$ and δ is small enough, all the areas $\mathrm{Area}(\Omega_{\alpha,\beta,\delta,Q,q})$ are zero for all values of q. Indeed, if $\alpha > \beta$ and $(u,v) \in \mathrm{Area}(\Omega_{\alpha,\beta,\delta,Q,q})$, then

$$(\alpha-1-\delta)q^2 \le (\alpha-\delta)qQ-q^2 \le uv \le qv \le (\beta-1+\delta)q^2.$$

This shows that for $\delta > 0$ small enough, all of the sets $\mathrm{Area}(\Omega_{\alpha,\beta,\delta,Q,q})$ are empty. In what follows we will restrict to the case $\alpha < \beta$. From the position of the hyperbolas $uv = (\alpha-\delta)qQ-q^2$ and $uv = (\alpha+\delta)qQ-q^2$, the horizontal lines $v = (p-1-\delta)q$ and $v = (p-1+\delta)q$, and their points of intersection with the boundary of the square $[1,q]\times[1,q]$, we find that

$$\Omega_{\alpha,\beta,\delta,Q,q} = \mathcal{L} \cap ([1,q]\times[1,q]),$$

where $\mathcal{L}$ is the "parallelogram shaped" region that lies between the hyperbolas and horizontal lines.

It is easy to see that if $q < (\alpha-\delta)Q/(\beta+\delta)$, then $\mathcal{L}$ lies completely outside the square $[1,q]\times[1,q]$. Furthermore, one can verify that if $(\alpha-\delta)Q/(\alpha+\delta) \le q \le (\alpha+\delta)Q/(\beta-\delta)$, then $\mathcal{L}$ intersects the square $[1,q]\times[1,q]$ but does not lie entirely inside it. This forces $\mathcal{L}$ to lie close enough to the boundary of the square $[1,q]\times[1,q]$, so that the total contribution of these values of q to the main term M is negligible. Hence, we are left with the sum

$$\sum_{(\alpha+\delta)Q/(\beta-\delta)\le q\le Q} \frac{\phi(q)}{q^2}\mathrm{Area}(\mathcal{L}). \tag{9}$$

Here, $\mathrm{Area}(\mathcal{L})$ is asymptotic to the area of the parallelogram. That is, if δ is small enough, then we have

$$\begin{aligned}\mathrm{Area}(\mathcal{L}) \sim 2\delta q\left[\frac{(\alpha+\delta)qQ-q^2}{(\beta-1)q} - \frac{(\alpha-\delta)qQ-q^2}{(\beta-1)q}\right] &= 2\delta q\left(\frac{2\delta Q}{\beta-1}\right) \\ &= \frac{4\delta^2 qQ}{\beta-1},\end{aligned} \tag{10}$$

as $Q\to\infty$. Inserting (10) into (9), we obtain

$$M \sim \frac{4\delta^2 Q}{\beta-1}\sum_{(\alpha+\delta)Q/(\beta-\delta)\le q\le Q}\frac{\phi(q)}{q}. \tag{11}$$

We estimate the summation in (11) by employing the following result from [4].

Lemma 2 (Lemma 2.3 from [4]). *Suppose that a and b are two real numbers such that $0 < a < b$, $q \in \mathbb{N}^*$ and f is a piecewise C^1 function defined on $[a, b]$. Then we have*

$$\sum_{a<q\le b} \frac{\phi(q)}{q} f(q) = \frac{1}{\zeta(2)} \int_a^b f(x)\,dx + O\left(\log b \left(\|f\|_\infty + \int_a^b |f'(x)|\,dx\right)\right).$$

Applying Lemma 2, we get

$$\sum_{(\alpha+\delta)Q/(\beta-\delta)\le q\le Q} \frac{\phi(q)}{q} = \frac{1}{\zeta(2)} \int_{(\alpha+\delta)Q/(\beta-\delta)}^{Q} dt + O(\log Q). \tag{12}$$

Then inserting (12) into (11), we find that

$$\frac{M}{\delta^2 Q^2} \to \frac{4}{(\beta-1)\zeta(2)} \left(1 - \frac{\alpha}{\beta}\right), \tag{13}$$

as $Q \to \infty$ first and then followed by $\delta \to 0$.

Next, we consider the set of matrices

$$\begin{aligned} \mathcal{C}_{\alpha,\beta,\delta,Q,x} = \Bigg\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{A}(Q,x)\colon\, & 1 \le a, b, d \le c \le Q,\ ad - bc = -1, \\ & (\alpha-\delta)Q \le a + d \le (\alpha+\delta)Q, \\ & (\beta-1-\delta)c \le a \le (\beta-1+\delta)c \Bigg\}. \end{aligned}$$

Estimating the cardinality of $\mathcal{C}_{\alpha,\delta,\delta,Q,x}$ in a similar fashion to that in (3), we write

$$\#\mathcal{C}_{\alpha,\beta,\delta,Q,x} = \sum_{1\le c\le Q} \sum_{\substack{1\le d\le c \\ \gcd(c,d)=1 \\ (\alpha-\delta)Q\le c-\bar{d}+d\le(\alpha+\delta)Q \\ (\beta-\delta)c\le c-\bar{d}\le(\beta-1+\delta)c}} 1. \tag{14}$$

The equality in (14) follows by noticing that the conditions $1 \le a \le c$ and $ad - bc = -1$ force a to equal $c - \bar{d}$, where $\bar{d}$ is the multiplicative inverse of d modulo c in the interval $[1, c]$. Furthermore, let us note in (14) that the terms for which $c < (\alpha-\delta)Q/2$ have no contribution to the sum. Indeed, the inequality $(\alpha-\delta)Q \le c - \bar{d} + d$ implies $(\alpha-\delta)Q < 2q$. Hence, setting $q = c$, $x = d$ and $y = \bar{d}$, we obtain $\#\mathcal{C}_{\alpha,\beta,\delta,Q,x}$ in the form

$$\#\mathcal{C}_{\alpha,\beta,\delta,Q,x} = \sum_{(\alpha-\delta)Q/2\le q\le Q} \#\{(x,y) \in \Gamma_{\alpha,\beta,\delta,Q,q} \cap \mathbb{Z}^2\colon\, xy \equiv 1 \pmod{q}\}, \tag{15}$$

where

$$\Gamma_{\alpha,\beta,\delta,Q,q} = \{(u,v) \in \mathbb{R}^2 \colon 1 \le u, v \le q, \\ (\alpha-\delta)Q - q \le u - v \le (\alpha+\delta)Q - q, \\ (2-\beta-\delta)q \le v \le (2-\beta+\delta)q\}. \tag{16}$$

Applying Lemma 1 as before, we obtain

$$\#\{(x,y) \in \Gamma_{\alpha,\beta,\delta,Q,q} \cap \mathbb{Z}^2 \colon xy \equiv 1 \pmod q\} = \frac{\phi(q)}{q^2}\mathrm{Area}(\Gamma_{\alpha,\beta,\delta,Q,q}) \\ + E'_{\alpha,\beta,\delta,Q,q}, \tag{17}$$

where

$$E'_{\alpha,\beta,\delta,Q,q} = O_\epsilon(Q^{5/6+\epsilon}). \tag{18}$$

Then inserting (17) and (18) into (15), we get

$$\#\mathcal{C}_{\alpha,\beta,\delta,Q,x} = M' + E', \tag{19}$$

where

$$M' = \sum_{(\alpha-\delta)Q/2 \le q \le Q} \frac{\phi(q)}{q^2}\mathrm{Area}(\Gamma_{\alpha,\beta,\delta,Q,q}) \tag{20}$$

and

$$E' = \sum_{(\alpha-\delta)Q/2 \le q \le Q} E'_{\alpha,\beta,\delta,Q,q} = O_\epsilon(Q^{11/6+\epsilon}). \tag{21}$$

From the definition of the set $\Gamma_{\alpha,\beta,\delta,Q,q}$ in (16), we see that

$$\Gamma_{\alpha,\beta,\delta,Q,q} = \mathcal{M} \cap ([1,q] \times [1,q]),$$

where $\mathcal{M}$ is the parallelogram that lies between the slant lines $v = u + q - (\alpha+\delta)Q$ and $v = u + q - (\alpha-\delta)Q$ and the horizontal lines $v = (2-\beta-\delta)q$ and $v = (2-\beta+\delta)q$. First, we observe that if $\alpha > \beta$, then for δ small enough all parallelograms $\mathcal{M}$ lie outside the square $[1,q] \times [1,q]$. In this situation, the sets $\Gamma_{\alpha,\beta,\delta,Q,q}$ are empty. Hence, the main term M' is zero.

In what follows, we consider the case when $\alpha < \beta$. If $q < (\alpha-\delta)Q/(\beta+\delta)$, then the parallelograms $\mathcal{M}$ still lie outside the square $[1,q] \times [1,q]$. Hence, we may restrict to the interval $[(\alpha-\delta)Q/(\beta+\delta), Q]$.

Next, if q belongs to the interval $[(\alpha-\delta)Q/(\beta+\delta), (\alpha+\delta)Q/(\beta-\delta)]$, then $\mathcal{M}$ intersects the square $[1,q] \times [1,q]$ but is not entirely contained in it. This forces $\mathcal{M}$ to lie close to the boundary of the square $[1,q] \times [1,q]$, so that all those values of q satisfying this property have negligible contribution to the main term M'.

Hence, we may restrict the summation over q to the interval $[(\alpha+\delta)Q/(\beta-\delta), Q]$. For all such values of q, we see that $\mathcal{M}$ is entirely contained in the square $[1,q] \times [1,q]$

and its area is equal to exactly $4\delta^2 qQ$. Hence, the main term in (20) is given by

$$M' = \sum_{(\alpha+\delta)Q/(\beta-\delta)\le q\le Q} \frac{\phi(q)}{q^2}\mathrm{Area}(\Gamma_{\alpha,\beta,\delta,Q,q}) = 4\delta^2 Q \sum_{(\alpha+\delta)Q/(\beta-\delta)\le q\le Q} \frac{\phi(q)}{q}. \tag{22}$$

Using Lemma 2, we find that

$$\sum_{(\alpha+\delta)Q/(\beta-\delta)\le q\le Q} \frac{\phi(q)}{q} = \frac{Q}{\zeta(2)}\left(1-\frac{\alpha+\delta}{\beta-\delta}\right) + O(\log q). \tag{23}$$

Then inserting (23) into (22), we see that

$$\frac{M'}{\delta^2 Q^2} \to \frac{4}{\zeta(2)}\left(1-\frac{\alpha+\delta}{\beta-\delta}\right), \tag{24}$$

as $Q\to\infty$ first and then followed by $\delta\to 0$.

On combining the above estimates for $\#\mathcal{D}_{\alpha,\beta,\delta,Q,x}$ and $\#\mathcal{C}_{\alpha,\beta,\delta,Q,x}$ when β is larger than α and recalling that both quantities are zero when β is less than α, we deduce that

$$\begin{aligned}\lim_{\delta\to 0}\lim_{Q\to\infty}\lim_{x\to\infty} \frac{\#\mathcal{D}_{\alpha,\beta,\delta,Q,x}+\#\mathcal{C}_{\alpha,\beta,\delta,Q,x}}{\delta^2 Q^2} &= \begin{cases} \dfrac{4\left(1-\dfrac{\alpha}{\beta}\right)}{(\beta-1)\zeta(2)} + \dfrac{4\left(1-\dfrac{\alpha}{\beta}\right)}{\zeta(2)}, & \text{if } \alpha\le\beta; \\ 0, & \text{if } \alpha<\beta; \end{cases} \\ &= \begin{cases} \dfrac{4}{\zeta(2)}\left(\dfrac{\beta-\alpha}{\beta-1}\right), & \text{if } \alpha\le\beta; \\ 0, & \text{if } \alpha>\beta. \end{cases}\end{aligned} \tag{25}$$

We have the following result, which is essentially Theorem 1.1 from [12].

Lemma 3. *Given a matrix*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

of determinant -1 *with* $a,b,c,d\ge 1$*, there are positive real-valued constants* K_A *and* c' *such that*

$$M_A(x) = K_A x^{1+(a+b)/(c+d)} + O_A(x^{1/2+(a+b)/(c+d)}\exp\{-c'(\log x)^{3/5}(\log\log x)^{-1/5}\}).$$

For the sake of completeness, we outline a sketch of the proof of Lemma 3. Consider the Dirichlet series

$$F_A(s) = \sum_{n=1}^{\infty}\frac{f_A(n)}{n^s}.$$

One can show that $F_A(s)$ converges in the half plane $\Re s = \sigma > 1 + (a+b)/(c+d)$ and has an Euler product in that region. Write

$$F_A(s) = \frac{\zeta(s - (a+b)/(c+d))}{\zeta(2s - 2(a+b)/(c+d))} T_A(s).$$

Furthermore, one can show that $\zeta(2s - 2(a+b)/(c+d))^{-1} T_A(s)$ is analytic on a larger half-plane $\sigma > \sigma_0$. Hence, $F_A(s)$ is meromorphic there with a simple pole at $s = 1 + (a+b)/(c+d)$.

Next, we utilize a variant of Perron's formula and write

$$\sum_{n \le x} \left(1 - \frac{n}{x}\right) f_A(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\zeta(s - (a+b)/(c+d))}{\zeta(2s - 2(a+b)/(c+d))} T_A(s) \frac{x^s}{s(s+1)}\, ds,$$

where $1 + (a+b)/(c+d) < c \le 5/4 + (a+b)/(c+d)$. We need to apply the zero-free region for $\zeta(s)$ due to Korobov [8] and Vinogradov [14] in the region

$$\sigma \ge 1 - c_0 (\log t)^{-2/3} (\log\log t)^{-1/3}$$

for $t \ge t_0$, in which

$$\frac{1}{|\zeta(s)|} = O((\log t)^{2/3} (\log\log t)^{1/3}).$$

(See the end-of-chapter notes for Chapter 6 in Titchmarsh's classical book [13]; see, also, Chapters 2 and 5 in Walfisz's book [15].) We then fix $0 < U < T \le x$, let $\nu = 1/2 + (a+b)/(c+d)$ and

$$\eta = \nu - c_0 (\log U)^{-2/3} (\log\log U)^{-1/3},$$

and deform the path of integration into the union of the line segments

$$\begin{cases} \gamma_1, \gamma_9 \colon s = c + it, & \text{if } |t| \ge T; \\ \gamma_2, \gamma_8 \colon s = \sigma \pm iT, & \text{if } \nu \le \sigma \le c; \\ \gamma_3, \gamma_7 \colon s = \nu + it, & \text{if } U \le |t| \le T; \\ \gamma_4, \gamma_6 \colon s = \sigma \pm iU, & \text{if } \eta \le \sigma \le \nu; \\ \gamma_5 \colon s = \eta + it, & \text{if } |t| \le U. \end{cases}$$

Here, we note that the integrand is analytic on and within this modified contour. Hence, by the residue theorem

$$M_A(x) = \frac{1}{(1 + (a+b)/(c+d))(2 + (a+b)/(c+d))\zeta(2)} T_A\left(1 + \frac{a+b}{c+d}\right) \times x^{1 + (a+b)/(c+d)} + \sum_{k=1}^{9} J_k,$$

with the main term coming from the residue at the simple pole at $s = 1 + (a + b)/(c + d)$. Note that we will take

$$K_A = \frac{1}{(1 + (a + b)/(c + d))(2 + (a + b)/(c + d))\zeta(2)} T_A \left(1 + \frac{a + b}{c + d}\right)$$

in the statement of the lemma.

We estimate the integral along our modified contour and make use of the well-known bounds

$$|\zeta(\sigma + it)| = \begin{cases} O(t^{(1-\sigma)/2}), & \text{if } 0 \leq \sigma \leq 1 \text{ and } |t| \geq 1; \\ O(\log t), & \text{if } 1 \leq \sigma \leq 2; \\ O(1), & \text{if } \sigma \geq 2. \end{cases}$$

(See Theorem 1.9 in Ivić's classical book [6].) Upon collecting all estimates, we have the statement of the lemma.

Lemma 3 shows us that

$$\frac{\log M_A(x)}{\log x} \sim 1 + \frac{a + b}{c + d},$$

as $x \to \infty$. Since

$$\frac{a + b}{c + d} = \frac{a}{c} - \frac{\det(A)}{c(c + d)} = \frac{b}{d} + \frac{\det(A)}{d(c + d)},$$

when $d > c$ we see that

$$\left| \frac{\log M_A(x)}{\log x} - \frac{b}{d} \right| = O\left(\frac{1}{d^2}\right),$$

as $x \to \infty$. When $c > d$, we have

$$\left| \frac{\log M_A(x)}{\log x} - \frac{a}{c} \right| = O\left(\frac{1}{c^2}\right),$$

as $x \to \infty$.

We partition $\mathcal{A}(Q, x)$ into two subsets, according to whether $1 \leq \max(c, d) \leq \sqrt{Q}$ or $\max(c, d) > \sqrt{Q}$. There are at most $O(Q^{3/2})$ matrices of the first type, and for the second type we have $O(1/d^2) = O(1/Q)$ and $O(1/c^2) = O(1/Q)$ when $d > c$ and $c > d$, respectively, as $Q \to \infty$.

We note that the δ in our definitions of $\mathcal{D}_{\alpha,\beta,\delta,Q,x}$ and $\mathcal{C}_{\alpha,\beta,\delta,Q,x}$ should be replaced by an expression of the form $\delta + \delta_E(Q)$, where the function $\delta_E(Q) = O(1/Q)$, but in what follows we let Q tend to infinity before letting δ tend to zero, so in our case we may replace one by the other.

Since $1 + (a + b)/(c + d) < \beta + \delta < 2$, we find that $a < c$, and similarly $b \leq d$. So the conditions $a, b \leq d$ and $a, b \leq c$ in $\mathcal{D}_{\alpha,\beta,\delta,Q,x}$ and $\mathcal{C}_{\alpha,\beta,\delta,Q,x}$ are satisfied. Thus,

$$\lim_{x\to\infty}\left|\frac{\#\mathcal{D}_{\alpha,\beta,\delta,Q,x}+\#\mathcal{C}_{\alpha,\beta,\delta,Q,x}}{\delta^2Q^2}-\frac{\#\{A\in\mathcal{A}(Q,x)\colon \Psi_{Q,x}(A)\in\mathcal{V}_{\alpha,\beta,\delta}\}}{\delta^2Q^2}\right|=O\left(\frac{1}{\delta^2\sqrt{Q}}\right)$$

as $Q\to\infty$. Upon combining this with (25), the theorem is proved.

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