LONG MINIMAL ZERO-SUM SEQUENCES IN THE GROUPS $C_2^{r-1} \oplus C_{2k}$

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Abstract

The article discusses sufficiently long minimal zero-sum sequences over groups of the form $C_2^{r-1} \oplus C_{2k}$, with rank $r \geq 3$. Their structure is clarified by general results in the first part. The conclusions are applied to the Davenport problems, direct and inverse, for the rank-5 group $C_2^4 \oplus C_{2k}$. We determine its Davenport constant for $k \geq 70$ and describe the longest minimal zero-sum sequences in the more interesting case where k is odd.

1. Introduction

A non-empty sequence in an additively written abelian group G is a *minimal zero-sum sequence* if its sum is the zero element of G and none of its proper subsequences has sum zero. The classic *direct Davenport problem* asks for the maximum length $\mathsf{D}(G)$ of a minimal zero-sum sequence over G. This length is called the *Davenport constant* of G; finding it is an unsolved problem for most groups.

The associated *inverse Davenport problem* asks for a description of the minimal zero-sum sequences with length $\mathsf{D}(G)$. It proves to be not any easier than the direct one. A notable achievement concerns groups of rank 2. Gao and Geroldinger [2] conjectured the answer to the inverse problem for groups of the form C_n^2. Assuming their conjecture true, Schmid [8] solved the inverse problem conditionally for all abelian groups with rank 2. After a number of partial results Gao, Geroldinger and Grynkiewicz [3] reduced the proof of the conjecture for C_n^2 to the case of a prime n. Finally the crucial prime case was settled by Reiher [5]. Let us also mention that the inverse problem for the rank-3 group $C_2^2 \oplus C_{2k}$ was solved by Schmid [9].

A more general inverse zero-sum problem is to describe all sufficiently long minimal zero-sum sequences over G. It is solved for cyclic groups [6], [11] and for the rank-2 group $C_2 \oplus C_{2k}$ [7], by characterizing the "long" sequences in naturally emerging length ranges. The ranges are optimal with respect to the form of the obtained characterization.

In this article we deal with the general inverse problem for higher-rank groups of the form $C_2^{r-1} \oplus C_{2k}$, with rank $r \geq 3$. Here the "long" minimal zero-sum sequences are too diverse to be described satisfactorily by a single characterization theorem. The central result of the first part is a more modest-looking statement, Proposition 4.1. Without being an explicit characterization, it is a starting point of further investigations. Examples are the statements in Section 4 which establish non-obvious structural properties of the long minimal zero-sum sequences over $C_2^{r-1} \oplus C_{2k}$. In turn these properties form a basis for a systematic study of the *longest* minimal zero-sum sequences. In Section 5 we solve the Davenport problems for the rank-5 group $C_2^4 \oplus C_{2k}$ in a sense to be explained below.

The approach to our inverse problem for the group $G = C_2^{r-1} \oplus C_{2k}$ is the one developed for the rank-2 group $C_2 \oplus C_{2k}$ in [7]. It rests on the characterization theorem for long minimal zero-sum sequences in cyclic groups (Theorem 2.1) and related properties presented in Section 2. The proof of the main structural statement in Section 3 (Proposition 3.8) is reduced to considerations in the cyclic subgroup $2G = \{2x \mid x \in G\}$. We partition a given minimal zero-sum sequence over G into blocks with sums in $2G$ and study the obtained minimal zero-sum sequence over $2G \cong C_k$. For the scheme to function it is essential that the factor group $G/2G$ is isomorphic to the elementary 2-group C_2^r.

Our condition defining a "long" sequence is quite restrictive—strong enough to keep simple the standard idea just described. We are interested mostly in longest minimal zero-sum sequences, not so much in the optimal range where the general results hold true. Let us emphasize that, whatever the optimal length constraint, the approach is feasible only if the exponent of G is large with respect to its rank.

The ultimate goal of Sections 4 and 5 are the Davenport problems for $C_2^4 \oplus C_{2k}$. The statements for arbitrary rank in them are treated accordingly. A number of these can be refined or generalized, but we usually include versions sufficient for the immediate task.

It is a basic fact that, for a general abelian group $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 | \cdots | n_r$, the Davenport constant $\mathsf{D}(G)$ is at least $\sum_{j=1}^{r}(n_j - 1) + 1 = \mathsf{D}^*(G)$. For $G = C_2^{r-1} \oplus C_{2k}$ with $r \in \{2, 3, 4\}$ it is known that $\mathsf{D}(G) = \mathsf{D}^*(G)$. The first example showing that there exist abelian groups G with $\mathsf{D}(G) > \mathsf{D}^*(G)$ is due to Baayen (see [10]). It applies to the groups $C_2^{2k-2} \oplus C_{2k}$ with odd $k \geq 3$, hence to the rank-5 group $C_2^4 \oplus C_6$ in particular. Geroldinger and Schneider [4] found such an example for all groups of the form $C_2^4 \oplus C_{2k}$, with odd $k \geq 3$ again. We solve the direct Davenport problem in $C_2^4 \oplus C_{2k}$ for $k \geq 70$. The outcome is (Theorem 5.8):

$$\mathsf{D}(C_2^4 \oplus C_{2k}) = \begin{cases} 2k+4 = \mathsf{D}^*(C_2^4 \oplus C_{2k}) & \text{if } k \text{ is even;} \\ 2k+5 = \mathsf{D}^*(C_2^4 \oplus C_{2k}) + 1 & \text{if } k \text{ is odd.} \end{cases}$$

About the inverse problem, for odd $k > 70$ there is a unique longest minimal zero-sum sequence in $C_2^4 \oplus C_{2k}$ (Theorem 5.9). In contrast, there are many such sequences for even $k \geq 70$. We do not need new ideas to exhibit them; however the development indicates that doing so would not be particularly illuminating. A similar situation occurs already with the rank-3 group $C_2^2 \oplus C_{2k}$, as observed in [9].

The length, the sum and the sumset of a sequence α are denoted by $|\alpha|$, $S(\alpha)$ and $\Sigma(\alpha)$ respectively. For a subsequence β of α we say that α is *divisible* by β or β *divides* α, and write $\beta|\alpha$. The complementary subsequence of β is denoted by $\alpha\beta^{-1}$. Term multiplicities are indicated by exponents, e.g., $(e+a)a^3e^2$ is the sequence with terms $e+a, a, a, a, e, e$. The union of disjoint sequences is called their *product*. Let a sequence α be the product of its disjoint subsequences $\alpha_1, \ldots, \alpha_m$. We say that the α_i's form a *decomposition* of α with *factors* $\alpha_1, \ldots, \alpha_m$ and write $\alpha = \prod_{i=1}^m \alpha_i$. Quite often we study the sequence with terms $S(\alpha_1), \ldots, S(\alpha_m)$. For convenience of speech it is also said to be a decomposition of α with factors $\alpha_1, \ldots, \alpha_m$; sometimes we call terms $\alpha_1, \ldots, \alpha_m$ themselves.

2. Basis of a Sequence in a Cyclic Group

The foundation of all the work is the next theorem for cyclic groups.

Theorem 2.1 ([6],[11]). *Each minimal zero-sum sequence β of length $\ell \geq \lfloor k/2 \rfloor + 2$ in the cyclic group C_k, $k \geq 3$, has the form $\beta = \prod_{j=1}^{\ell}(x_j g)$, where g is a term of β that generates C_k and $x_1, \ldots, x_\ell$ are positive integers with sum k.*

The notions in this section and some of the next statements were introduced in [7], where the analogue of our inverse problem for the group $C_2 \oplus C_{2k}$ is considered. We omit the simple proof of the first lemma.

Lemma 2.2. *Let γ be a sequence with positive integer terms, sum $k \geq 3$ and length $|\gamma| \geq \lfloor k/2 \rfloor + 2$. If μ is the multiplicity of the term 1 in γ and $t \in \gamma$ is an arbitrary term then $\mu \geq 2|\gamma| - k$, $t \leq \mu - (2|\gamma| - k - 2)$ and $t \leq k - |\gamma| + 1$.*

Definition 2.3. *Let g be a generator of the cyclic group C_k. The g-*coordinate *of an element $a \in C_k$ is the unique integer $x_g(a) \in [1, k]$ such that $a = x_g(a)g$. The singleton $\{g\}$ is a* basis *of a sequence β in C_k if $\sum_{t \in \beta} x_g(t) = k$.*

Sequences with a basis are minimal zero-sum sequences. The converse is false in general; and if a basis exists, the sequence may not contain the basis element as a term. However for length $\geq \lfloor k/2 \rfloor + 2$ the notions of a minimal zero-sum sequence and a sequence with a basis are equivalent. Moreover the basis of a long minimal zero-sum sequence is unique, and the sequence has terms equal to its basis element.

Lemma 2.4. *Each minimal zero-sum sequence β of length $|\beta| \geq \lfloor k/2 \rfloor + 2$ in C_k, $k \geq 3$, has a uniquely determined basis $\{g\}$. The multiplicity μ of the basis element g in β and the g-coordinate $x_g(t)$ of each term t satisfy the inequalities $\mu \geq 2|\beta| - k$, $x_g(t) \leq \mu - (2|\beta| - k - 2)$ and $x_g(t) \leq k - |\beta| + 1$.*

Proof. A basis exists by Theorem 2.1. Its uniqueness is proven in [7, Lemma 5]. The three inequalities follow from their analogues in Lemma 2.2. □

Certain local changes do not affect the basis of a long minimal zero-sum sequence.

Lemma 2.5. *Let β be a minimal zero-sum sequence in C_k, $k \geq 3$, that satisfies the condition $2|\beta| - k \geq 4$. Let $w \leq 2|\beta| - k - 2$ be a positive integer. Suppose that w terms of β are replaced by at least w group elements so that the obtained sequence β' is a minimal zero-sum sequence. Then β and β' have the same basis.*

Proof. We use repeatedly Lemma 2.4. Both β and β' have uniquely determined bases $\{g\}$ and $\{h\}$ as $2|\beta| - k \geq 4$ and $|\beta'| \geq |\beta|$. Since g and h generate C_k, there is an integer $s \in (0, k)$ coprime to k such that $g = sh$. Then the h-coordinate of g is $x_h(g) = s$. Let μ and μ' be respectively the multiplicities of g and h in β and β'. At least $\mu - w$ terms g are not replaced, hence present in β' (note that $\mu - w > 0$). Since $\sum_{t \in \beta'} x_h(t) = k$ and $x_h(g) = s$, we obtain $s(\mu - w) \leq k$. If the removed w terms of β are replaced by $p \geq w$ group elements then $|\beta'| = |\beta| + p - w$ and so $\mu' \geq 2|\beta'| - k = (2|\beta| - k) + 2p - 2w$. Hence $\mu' > p$ by $2|\beta| - k > w$ and $p \geq w$. Thus β has a term h that is not replaced. Its g-coordinate satisfies $x_g(h) \leq \mu - (2|\beta| - k - 2) \leq \mu - w$, hence $sx_g(h) \leq s(\mu - w) \leq k$ by the above. Since $h = x_g(h)g = (sx_g(h))h$, we have $1 = x_h(h) = sx_g(h)$, implying $h = g$. □

3. Augmentations

Let G be an abelian group and H a subgroup of G. We call an *H-block* each sequence in G with sum in H. An *H-decomposition* of a sequence is a decomposition whose factors are H-blocks; clearly the sequence is an H-block itself. An H-block is *minimal* if its projection onto the factor group G/H under the natural homomorphism is a minimal zero-sum sequence. An H-decomposition whose factors are minimal H-blocks is an *H-factorization*. In accordance with our general terminology, an H-decomposition (H-factorization) is viewed in two different ways. Depending on the occasion, we regard it either as a partition of a sequence into

H-blocks (minimal H-blocks), or as a sequence over H, with terms the sums of its factors. In the latter case we sometimes call terms the factors themselves.

It is plain that H-factorizations exist for every H-block. Observe also that the minimal zero-sum sequences in G are H-blocks for every subgroup H of G. Moreover their H-decompositions and H-factorizations are minimal zero-sum sequences in H.

We are concerned with the group $G = C_2^{r-1} \oplus C_{2k}$ where $r \geq 2$ and, first of all, with its subgroup $2G = \{2x \colon x \in G\}$ which is cyclic of order k. In this section we consider certain $2G$-factorizations. The factor group $G/2G$ is isomorphic to C_2^r, the elementary 2-group of rank r. The minimal zero-sum sequences in C_2^r have the form $u_1 \cdots u_m(u_1 + \cdots + u_m)$ where $m \leq r$ and $u_1, \ldots, u_m$ are independent elements. Such a sequence generates a subgroup of rank m, with basis $\{u_1, \ldots, u_m\}$. So the blocks in a $2G$-factorization may have lengths $1, 2, \ldots, r+1$. Call *pairs* the blocks of length 2 and *long blocks* the ones of length ≥ 3. The blocks with length 1 are the terms of the original sequence in $2G$. They are terms of every $2G$-factorization.

For the rest of the section let $G = C_2^{r-1} \oplus C_{2k}$ with $r \geq 2$, and let α denote a minimal zero-sum sequence over G that satisfies

$$|\alpha| \geq k + \left\lceil \tfrac{3r-1}{r+1}(2^r - 1) \right\rceil + 1. \tag{3.1}$$

The length condition (3.1) imposes a restrictive constraint on k and r:

$$k \geq \left\lceil \tfrac{3r-1}{r+1}(2^r - 1) \right\rceil - 2^{r-1} + 2. \tag{3.2}$$

To justify (3.2) we use a general upper bound for the Davenport constant due to Bhowmik and Schlage-Puchta (see [1, Theorem 1.1]):

Let G be a finite abelian group with exponent $exp(G)$. If $exp(G) \geq \sqrt{|G|}$ then $\mathsf{D}(G) \leq exp(G) + |G|/exp(G) - 1$; if $exp(G) < \sqrt{|G|}$ then $\mathsf{D}(G) \leq 2\sqrt{|G|} - 1$.

The uniform (and less precise) estimate $\mathsf{D}(G) \leq exp(G) + |G|/exp(G) - 1$ follows because $exp(G) + |G|/exp(G) \geq 2\sqrt{|G|}$ always holds.

In the case $G = C_2^{r-1} \oplus C_{2k}$ the above yields $D(C_2^{r-1} \oplus C_{2k}) \leq 2k + 2^{r-1} - 1$. Then $|\alpha| \leq 2k + 2^{r-1} - 1$ for each minimal zero-sum sequence α over $C_2^{r-1} \oplus C_{2k}$. Now (3.2) follows in view of (3.1).

In summary, the length condition (3.1) is satisfied only if k is exponentially large with respect to r. We use only a very weak consequence of the implied relation (3.2) in Section 4, but a constraint stronger than (3.2) is needed in Section 5.

Definition 3.1. *An* augmentation *of α is a $2G$-factorization such that terms of α from the same proper $2G$-coset participate in at most 2 long blocks.*

All augmentations are sufficiently long, so that the conclusions of Section 2 apply.

Lemma 3.2. *The following statements hold true for every augmentation β of α:*
a) β has a uniquely determined basis, with basis element a generator of $2G$;
b) β has at least 2^r terms equal to the basis element that are not terms of α;
c) $2|\beta| - k \geq 4$ and the number of long blocks in β does not exceed $2|\beta| - k - 2$.

Proof. Let β have w long blocks with total length W. If α has y terms in $2G$ then $w + y$ terms of β are not sums of pairs. Hence $|\beta| = w + y + \frac{1}{2}(|\alpha| - y - W)$, yielding $2|\beta| = |\alpha| + 2w - W + y$. Observe that $W - 2w \leq \frac{r-1}{r+1}(2^{r+1} - 2)$ because $W \leq 2^{r+1} - 2$ (each of the $2^r - 1$ proper $2G$-cosets is involved in at most two long blocks) and $W \leq (r+1)w$ (the length of a block is $\leq r+1$). So (3.1) yields

$$2|\beta| - k = (|\alpha| - k) - (W - 2w) + y \geq \left\lceil \frac{3r-1}{r+1}(2^r - 1) \right\rceil + 1 - \frac{r-1}{r+1}(2^{r+1} - 2) + y \geq 2^r + y.$$

Since $r \geq 2$ and $y \geq 0$, we have $2|\beta| - k \geq 4$ and so Lemma 2.4 applies. Hence β has a unique basis $\{g\}$ where g generates $2G$; part (a) follows.

For part (b) Lemma 2.4 ensures at least $2|\beta| - k \geq 2^r + y$ terms of β equal to g. At most y of them are terms of α (from $2G$). The remaining ones are sums of pairs or long blocks, and there are at least 2^r of them.

The inequality $2|\beta| - k \geq 4$ from part (c) was already established. For the second claim in (c) it is enough to check $w + 2 \leq 2^r$ because $2|\beta| - k \geq 2^r + y \geq 2^r$. The length of a long block is at least 3, so $w \leq \frac{1}{3}W \leq \frac{1}{3}(2^{r+1} - 2)$. The inequality $\frac{1}{3}(2^{r+1} - 2) + 2 \leq 2^r$ is true for $r \geq 2$ and implies the desired $w + 2 \leq 2^r$. $\square$

Let $B|\alpha$ be a $2G$-block. A *standard partition* of B is a $2G$-factorization of B such that each proper $2G$-coset is represented at most once in its long blocks. The terms *pairs* and *long blocks* have the same meaning like above.

We say that a proper $2G$-coset is even or odd according as it contains an even or odd number of terms of α; a nonzero number of terms is assumed in the even case.

It is immediate that in a standard partition all terms of B from an even coset are partitioned into pairs. All terms except one from an odd coset are partitioned into pairs; the exceptional term is in a long block. It is also clear that if $u', u'' \in B$ are terms from the same proper $2G$-coset then there is a standard partition of B containing the pair $u'u''$.

Now form a standard partition of the complementary subsequence αB^{-1} of B (which is also a $2G$-block). The two partitions together form an augmentation of α since each proper $2G$-coset is involved in at most 2 long blocks. We justified the following simple observation.

Lemma 3.3. *Let $B|\alpha$ be a $2G$-block and $B = B_1 \cdots B_m$ a standard partition of B. Then the sums of $B_1, \ldots, B_m$ are terms of an augmentation of α.*

In particular the standard partitions of the entire α are augmentations which we call *standard*. They play a rôle in the next considerations.

We need an almost evident general statement. Let $\{a, b\}$ and $\{c, d\}$ be disjoint pairs in a multiset. A *swap* means replacing them by the pairs $\{a, c\}$ and $\{b, d\}$.

Lemma 3.4. *Every partition of a multiset with cardinality* $2m$ *into* m *pairs can be obtained from every other such partition by a sequence of swaps.*

Proof. We use induction on m. The base $m = 1$ is obvious. For the inductive step $m - 1 \mapsto m$, $m \geq 2$, consider a multiset M with $|M| = 2m$ and two different partitions $\mathcal{P}_1, \mathcal{P}_2$ of M into m pairs. Take a pair $\{x_1, x_2\}$ such that $\{x_1, x_2\} \in \mathcal{P}_2$, $\{x_1, x_2\} \notin \mathcal{P}_1$. Then x_1 and x_2 are in different pairs $\{x_1, x_1'\}, \{x_2, x_2'\} \in \mathcal{P}_1$. Apply the swap $\{x_1, x_1'\}, \{x_2, x_2'\} \mapsto \{x_1, x_2\}, \{x_1', x_2'\}$ in $\mathcal{P}_1$ to obtain a partition $\mathcal{P}_1'$ of M with m pairs; $\mathcal{P}_1'$ shares the pair $\{x_1, x_2\}$ with $\mathcal{P}_2$.

Now remove $\{x_1, x_2\}$ from M, $\mathcal{P}_1'$ and $\mathcal{P}_2$. This yields a multiset M' with cardinality $2(m-1)$ and two partitions of M' into $m - 1$ pairs. It remains to apply the induction hypothesis. □

Although we need the next statement only in this section, it adds an interesting detail to the general picture.

Lemma 3.5. *All augmentations of* α *have the same basis.*

Proof. We prove first that every augmentation β has the same basis as a standard one. Suppose that two long blocks B', B'' of β have terms u', u'' from the same proper $2G$-coset. Form a standard partition of the $2G$-block $B'B''$ that contains the pair $u'u''$: $B'B'' = B_1 \cdots B_m$, $m \geq 2$, with $B_1 = u'u''$. Replacing B', B'' by $B_1, \ldots, B_m$ yields a $2G$-factorization β' of α, with more pairs than β. No proper $2G$-coset is involved in more than two long blocks of β', so β' is an augmentation. In particular β' is a minimal zero-sum sequence in $2G$, obtained by removing two terms from β and adding $m \geq 2$ new ones. Now Lemma 2.5 can be applied with $w = 2$ since $2 \leq 2|\beta| - k - 2$ (Lemma 3.2c). Hence β' has the same basis as β. Repeat this step as long as there is a proper coset represented in two long blocks. Eventually we reach a standard augmentation with the same basis as β.

So it is enough to show that every two standard augmentations β_1, β_2 have the same basis. The terms of α in $2G$ belong to each one of β_1 and β_2. The terms in an even $2G$-coset U form a multiset with even cardinality which is partitioned into pairs, both in β_1 and β_2. Call the two partitions $\mathcal{P}_1$ and $\mathcal{P}_2$. By Lemma 3.4 $\mathcal{P}_2$ can be obtained from $\mathcal{P}_1$ through a sequence of swaps. They yield all pairs in β_2 with terms from U. Consider the terms $u_1, \ldots, u_{2m-1}$ in an odd $2G$-coset U. In both β_1 and β_2 all of them but one are divided into pairs. Let the respective pairs be $p_1', \ldots, p_{m-1}'$ and $p_1'', \ldots, p_{m-1}''$, and let u_1, u_2 be the unpaired terms in β_1, β_2. Then u_1 is in a long block $B_1 = u_1 B$ of β_1 where $B|\alpha$ is a subsequence with sum in U, $|B| \geq 2$. Set $p_m' = \{u_1, B\}$, $p_m'' = \{u_2, B\}$. Then $\mathcal{P}_1 = \{p_1', \ldots, p_m'\}$ and $\mathcal{P}_2 = \{p_1'', \ldots, p_m''\}$ are partitions of the $2m$-element multiset $\{u_1, \ldots, u_{2m-1}, B\}$ into pairs. Again, $\mathcal{P}_2$ can be obtained from $\mathcal{P}_1$ by a sequence of swaps (Lemma 3.4). As a result all pairs in β_2 with terms from U are obtained, and also a long block $B_1' = u_2 B$ that contains the only term $u_2 \in U$ involved in a long block of β_2. Clearly $|B_1'| = |B_1|$.

Carry out the swaps described, for all proper $2G$-cosets. No swap changes the number of long blocks in which a $2G$-coset is represented. Therefore all $2G$-factorizations throughout the process are standard augmentations of α. They share the same basis by Lemma 2.5 which applies with $w = 2$ like above. Furthermore the term representing an odd coset U in a long block can be replaced by another term from U only through swaps involving U itself, not any other coset.

In summary, the swaps lead to a standard augmentation β_0 with the same basis as β_1 and with the same pairs as β_2. The long blocks in β_0 combined contain the same terms of α as the long blocks in β_2 combined. The swaps do not change the number and the lengths of long blocks. In particular β_0 and β_1 have the same number w_1 of long blocks and hence also the same length.

Let β_2 have w_2 long blocks; one may assume $w_1 \leq w_2$. Replace the long blocks in β_0 by the ones in β_2 to obtain β_2. Apply Lemma 2.5 one more time, with $w = w_1$, to conclude that β_0 and β_2 have the same basis. The conditions $2|\beta_0| - k \geq 4$ and $w_1 \leq 2|\beta_0| - k - 2$ are ensured by Lemma 3.2c. So β_1 and β_2 have the same basis. □

Now we observe that α has a term of rather special nature.

Lemma 3.6. *Let g be the common basis element of all augmentations of α. There exists a term $a \in \alpha$ with order $2k$ such that $2a = g$.*

Proof. Let β be an augmentation in which the multiplicity of g is a minimum. By Lemma 3.2b β has at least 2^r terms g that are not present in α; they are sums of pairs or long blocks. At least 2^{r+1} terms of α are needed for their formation, and there are $2^r - 1$ proper $2G$-cosets. Hence some proper coset U is involved at least 3 times in blocks with sum g. At least one of its participations is in a pair by the definition of an augmentation. A pair with a term in U has both of its terms in U, so β contains a pair $u_1 u'$ and a block $u_2 B$ with sums g, where the terms u_1, u_2, u' and the sum $S(B) = u''$ belong to U. We claim that $u_1 = u_2 = a$ and $2a = g$.

Swap u_1 and u_2 to obtain the blocks $u_2 u'$ and $u_1 B$. A new augmentation results, with the same basis $\{g\}$. The minimum choice of β implies that the new blocks have sums g, like $u_1 u'$ and $u_2 B$. So $u_2 + u' = u_1 + u'' = g = u_1 + u' = u_2 + u''$, yielding $u_1 = u_2$, $u' = u''$. A symmetric argument gives $u' = u_2$, $u_1 = u''$. Hence $u_1 = u_2 = u' = u'' = a$ where a is a term of α from U, and $2a = g$. Clearly $\mathrm{ord}(a) = 2k$ because $2G = \langle g \rangle \subset \langle a \rangle$, $a \notin 2G$, $|2G| = k$ and $exp(G) = 2k$. Therefore the term a meets the requirements. □

Henceforth we write $x_a(B)$ for an $\langle a \rangle$-block B instead of $x_a(S(B))$. The empty subsequence $\emptyset | \alpha$ is regarded as an $\langle a \rangle$-block with $x_a(\emptyset) = 0$.

Lemma 3.7. *There exists a term $a \in \alpha$ of order $2k$ with the following properties:*

a) If β is an augmentation of α then $\sum_{t\in\beta} x_a(t) = 2k$ and $0 < x_a(t) < k$ for each $t \in \beta$.

b) If $B|\alpha$ is a $2G$-block and $B = B_1 \cdots B_m$ is a standard partition of B then $x_a(B) = x_a(B_1) + \cdots + x_a(B_m)$.

c) If $B|\alpha$ is a minimal $2G$-block or a minimal $\langle a\rangle$-block then $0 < x_a(B) < k$.

Proof. Let $a \in \alpha$ be a term like in Lemma 3.6. If U is the proper $2G$-coset containing a then $\langle a\rangle = 2G \cup U$ is a cyclic subgroup of G with maximum order $2k$.

a) Since $\{g\}$ is the basis of β, we have $\sum_{t\in\beta} x_g(t) = k$. Because $2|\beta| - k \geq 4$ (Lemma 3.2c), Lemma 2.4 applies and gives $0 < x_g(t) \leq k - |\beta| + 1$ for each $t \in \beta$. Now we use $2|\beta| - k \geq 4$ again to obtain $0 < 2x_g(t) \leq k - 2 < k$. On the other hand $g = 2a$, so $t = x_g(t)g = (2x_g(t))a$. As $2x_g(t) \in (0, k)$, the a-coordinate of t is $x_a(t) = 2x_g(t)$. Hence the relations $\sum_{t\in\beta} x_g(t) = k$ and $0 < 2x_g(t) < k$ can be written respectively as $\sum_{t\in\beta} x_a(t) = 2k$ and $0 < x_a(t) < k$.

b) By Lemma 3.3 the sums of $B_1, \ldots, B_m$ are terms of an augmentation. Then (a) implies that the sum of their a-coordinates is at most $2k$. This is enough to conclude that $x_a(B) = x_a(B_1) + \cdots + x_a(B_m)$.

c) A minimal $2G$-block B is a standard partition of itself, so there is an augmentation with term $S(B)$ (Lemma 3.3). Hence (a) gives $0 < x_a(B) < k$.

Let B be a minimal $\langle a\rangle$-block. Then $S(B) \in \langle a\rangle = 2G \cup U$, and we can assume $S(B) \in U$. (If $S(B) \in 2G$ then B is also a minimal $2G$-block.) Observe that $a \in B$ implies $B = a$ by the minimality of B, and the claim is trivially true. So suppose that $a \notin B$ and form the block $B' = Ba$. Now $S(B') \in 2G$ and it is immediate that B' is a minimal $2G$-block. Therefore $0 < x_a(B') < k$ by the previous case, i.e., $S(B') \in \{a, 2a, \ldots, (k-1)a\}$. Because $S(B') = S(B) + a$ and $S(B) \neq 0$, it follows that $S(B) \in \{a, 2a, \ldots, (k-2)a\}$. Thus $0 < x_a(B) < k$ holds true. □

We proceed to the main result of the section.

Proposition 3.8. *Let $G = C_2^{r-1} \oplus C_{2k}$ where $r \geq 2$, and let α be a minimal zero-sum sequence in G with length $|\alpha| \geq k + \left\lceil \frac{3r-1}{r+1}(2^r - 1) \right\rceil + 1$. There exists a term a of α with order $2k$ such that every $\langle a\rangle$-factorization of α has basis $\{a\}$.*

Proof. Let $a \in \alpha$ be a term with the properties from Lemma 3.7. Consider any $\langle a\rangle$-factorization γ of α and set $S = \sum_{t\in\gamma} x_a(t)$. We construct an augmentation δ of α such that $\sum_{t\in\delta} x_a(t) = S$. Then Lemma 3.7a will yield $S = 2k$. Denote by U again the proper $2G$-coset containing a; then $\langle a\rangle = 2G \cup U$.

The terms of γ are sums of minimal $\langle a\rangle$-blocks, they are either in $2G$ or in U. Moreover there is an even number of blocks with sums in U. Partition them into pairs, and let A_1, A_2 be one such pair. The product A_1A_2 is a $2G$-block. Let $A_1A_2 = B_1 \cdots B_m$ be a standard partition, with $B_1, \ldots, B_m$ minimal $2G$-blocks.

Replace A_1, A_2 by $B_1, \dots, B_m$. We have $x_a(A_1A_2) = x_a(B_1) + \cdots + x_a(B_m)$ by Lemma 3.7b. On the other hand $x_a(A_1A_2) = x_a(A_1) + x_a(A_2)$ as $x_a(A_i) \in (0, k)$ by Lemma 3.7c, $i = 1, 2$. Hence $x_a(A_1) + x_a(A_2) = x_a(B_1) + \cdots + x_a(B_m)$.

By doing the same with every pair of blocks with sums in U we obtain a $2G$-factorization β of α such that $\sum_{t\in\beta} x_a(t) = S$. (The initial $\langle a\rangle$-blocks with sum in $2G$ are minimal $2G$-blocks; they are unchanged.) Suppose that two long blocks B', B'' of β contain terms u', u'' from the same proper $2G$-coset. Form a standard partition $B'B'' = B_1 \cdots B_m$ of the $2G$-block $B'B''$ in which one of the blocks B_i is the pair $u'u''$. Like above we deduce $x_a(B'B'') = x_a(B_1)+\cdots+x_a(B_m)$ from Lemma 3.7b and $x_a(B'B'') = x_a(B') + x_a(B'')$ from Lemma 3.7c. It follows that $x_a(B') + x_a(B'') = x_a(B_1) + \cdots + x_a(B_m)$. Therefore replacing B', B'' by $B_1, \dots, B_m$ yields a new $2G$-factorization of α, with the same value of S as β and with more pairs.

Repeat the same step as long as a proper $2G$-coset is represented in more than one long block. Since the number of pairs cannot increase indefinitely, the process terminates. The resulting $2G$-factorization δ is a (standard) augmentation of α and satisfies $\sum_{t\in\delta} x_a(t) = S$, as desired. $\square$

All subsequent work rests on the existence of a term like in Proposition 3.8. Such a term has other substantial properties, the ones from the previous two lemmas. Some of them are summarized in the first statement of the next section.

4. Additional General Properties

We proceed to consider selected additional structural properties of long minimal zero-sum sequences over $C_2^{r-1} \oplus C_{2k}$. The selection is determined by our next goal, the longest minimal zero-sum sequences over the rank-5 group $C_2^4 \oplus C_{2k}$. So we could have treated only the case $r = 5$, but keeping the natural generality is desirable. Most conclusions are interesting on their own right and can be developed further.

Proposition 4.1. *Let $G = C_2^{r-1} \oplus C_{2k}$ where $r \geq 2$, and let α be a minimal zero-sum sequence in G with length $|\alpha| \geq k + \left\lceil \frac{3r-1}{r+1}(2^r - 1) \right\rceil + 1$. There exists an order-$2k$ term a of α with the following properties:*

a) Every $\langle a\rangle$-decomposition of α has basis $\{a\}$.

b) If $B|\alpha$ is a minimal $\langle a\rangle$-block then $0 < x_a(B) < k$.

c) If $B|\alpha$ is an $\langle a\rangle$-block and $B = B_1 \cdots B_m$ is an $\langle a\rangle$-decomposition of B then $x_a(B) = x_a(B_1) + \cdots + x_a(B_m)$.

d) If $B|\alpha$ and B' are $\langle a\rangle$-blocks such that $B'|B$ then $x_a(B') \leq x_a(B)$, with equality if and only if $B' = B$.

e) Every $\langle a\rangle$-block $B|\alpha$ with $x_a(B) = 1$ is minimal.

Proof. Let $a \in \alpha$ be a term like in Proposition 3.8. We show that it has the stated properties. Claim (b) is a part of Lemma 3.7c. We prove claim (c) which implies (a), (d) and (e). Suppose first that $B = B_1 \cdots B_m$ is an $\langle a\rangle$-factorization of B. Take any $\langle a\rangle$-factorization of the complementary block αB^{-1} and add $B_1, \ldots, B_m$ to obtain an $\langle a\rangle$-factorization of α. The a-coordinates of its blocks have sum $2k$ (Proposition 3.8). Hence $\sum_{i=1}^m x_a(B_i) \in (0, 2k]$ (for $B \neq \emptyset$) and so $x_a(B) = \sum_{i=1}^m x_a(B_i)$.

For the general case, factorize each B_i into a product of minimal $\langle a\rangle$-blocks $B_{i1}, \ldots, B_{is_i}$. By the above, $x_a(B_i) = x_a(B_{i1}) + \cdots + x_a(B_{is_i})$ for each $i = 1, \ldots, m$, so $\sum_{i=1}^m x_a(B_i) = \sum_{i=1}^m \sum_{j=1}^{s_i} x_a(B_{ij})$. The B_{ij}'s form an $\langle a\rangle$-factorization of B. Hence $x_a(B) = \sum_{i=1}^m \sum_{j=1}^{s_i} x_a(B_{ij})$ by the previous case, and the claim follows. $\square$

For the rest of the section let α be again a minimal zero-sum sequence over $G = C_2^{r-1} \oplus C_{2k}$, $r \geq 2$, that satisfies $|\alpha| \geq k + \left\lceil \frac{3r-1}{r+1}(2^r - 1)\right\rceil + 1$. Fix a term $a \in \alpha$ like in Proposition 4.1.

Most of the proofs that follow require k to be relatively large as compared to r: the modest $k > r^2$ suffices for the purpose. It is amply ensured by the length condition above, which is condition (3.1). Recall that (3.1) implies (3.2), an inequality much stronger than $k > r^2$ for $r \geq 2$.

We repeatedly use the fact that a minimal $\langle a\rangle$-block in α has length at most r, due to $G/\langle a\rangle \cong C_2^{r-1}$. A multitude of $\langle a\rangle$-factorizations is considered, so "$\langle a\rangle$-block," "$\langle a\rangle$-factorization" and "$\langle a\rangle$-decomposition" are usually abbreviated to "block," "factorization" and "decomposition." However *decomposition* also keeps its general meaning, a partition of a sequence into arbitrary disjoint subsequences. The context excludes ambiguity. Write $\bar{t}$ for the $\langle a\rangle$-coset containing $t \in G$ and $u \sim v$ if $\overline{u} = \overline{v}$. For a sequence $\gamma = \prod t_i$ in G denote by $\bar{\gamma}$ the sequence $\prod \bar{t_i}$ in $G/\langle a\rangle$.

Our exposition uses two non-conventional notions which we introduce now.

The δ-quantity. Let $B|\alpha$ be a block and $X|B$ a proper nonempty subsequence. Then $X' = BX^{-1}$ is also proper and nonempty; sometimes we say that $B = XX'$ is a proper decomposition of B. As $S(X)$ and $S(X')$ are in the same $\langle a\rangle$-coset, they differ by a multiple of a.

Hence there is a unique integer $\delta(X) \in [0, k]$ such that $S(X') = S(X) + \delta(X)a$ or $S(X) = S(X') + \delta(X)a$. Write $\delta(X)$ instead of the precise but cumbersome $\delta_B(X)$; no confusion will arise. Naturally $\delta(X) = \delta(X')$. If, e.g., $S(X') = S(X) + \delta(X)a$, then $S(X) + S(X') = x_a(B)a$ leads to the relations $2S(X') = \big(x_a(B) + \delta(X)\big)a$, $2S(X) = \big(x_a(B) - \delta(X)\big)a$. As $2S(X) \in 2G$ and $2a$ generates $2G$, we see that $\delta(X)$ and $x_a(B)$ are of the same parity.

It follows that there is an element e in the $\langle a\rangle$-coset $\overline{S(X)}$ such that $2e = 0$ and

$$\{S(X), S(X')\} = \left\{e + \tfrac{1}{2}\big(x_a(B) + \delta(X)\big)a, e + \tfrac{1}{2}\big(x_a(B) - \delta(X)\big)a\right\}. \tag{4.1}$$

If $\overline{S(X)}$ is a proper $\langle a\rangle$-coset then $\mathrm{ord}(e) = 2$ (e is one of the two elements with order 2 in $\overline{S(X)}$). If $\overline{S(X)} = \langle a\rangle$ then $e = ka$ (ka is the only order-2 element in $\langle a\rangle$)

or $e = 0$ (and $\operatorname{ord}(e) = 1$). Let us remark however that the exceptional $e = 0$ practically does not occur in our exposition.

Indeed we consider mostly proper decompositions $B = XX'$ of minimal blocks B. In this case $\overline{S(X)} \neq \langle a \rangle$, so the respective element e from (4.1) has order 2.

We also define the *lower member* X^* of the decomposition $B = XX'$ (of the pair X, X'). Namely let X^* be X or X' according as $S(X') = S(X) + \delta(X)a$ or $S(X) = S(X') + \delta(X)a$. By (4.1) $S(X^*) = e + \frac{1}{2}\big(x_a(B) - \delta(X)\big)a$. Note that if $\delta(X) = 0$ then either one of X and X' can be taken as X^*.

The defect. For every $\langle a \rangle$-block $B|\alpha$ define $d(B) = |B| - x_a(B)$ and call $d(B)$ the *defect* of B. The defect is additive: For each $\langle a \rangle$-decomposition $B = \prod_{i=1}^m B_i$ of B one has $d(B) = \sum_{i=1}^m d(B_i)$ (Proposition 4.1c).

In particular the entire α is an $\langle a \rangle$-block with defect $d(\alpha) = |\alpha| - x_a(\alpha) = |\alpha| - 2k$; equivalently $|\alpha| = 2k + d(\alpha)$.

While Proposition 4.1 represents the main idea of the approach, the following statement is its main technical tool.

Lemma 4.2. *Let $B_1, \ldots, B_m$ be disjoint blocks in α with $\sum_{i=1}^m x_a(B_i) < k$ and $B_i = X_i X_i'$ proper decompositions, $i = 1, \ldots, m$, such that $\sum_{i=1}^m \overline{S(X_i)} = \overline{0}$. Then:*

a) The product of the lower members $X_1^*, \ldots, X_m^*$ is an $\langle a \rangle$-block dividing $B_1 \cdots B_m$ with a-coordinate

$$x_a(X_1^* \cdots X_m^*) = \frac{1}{2}\left(\sum_{i=1}^m x_a(B_i) - \sum_{i=1}^m \delta(X_i)\right).$$

In addition $\sum_{i=1}^m \delta(X_i) \leq \sum_{i=1}^m x_a(B_i) - 2$.

b) For each $i = 1, \ldots, m$ there exists an element $e_i \in \overline{S(X_i)}$ such that

$$2e_i = 0, \quad \{S(X_i), S(X_i')\} = \{e_i + \tfrac{1}{2}\big(x_a(B_i) - \delta(X_i)\big)a, e_i + \tfrac{1}{2}\big(x_a(B_i) + \delta(X_i)\big)a\},$$

and $e_1, \ldots, e_m$ satisfy $e_1 + \cdots + e_m = 0$.

Proof. Denote $\delta(X_i) = q_i$, $i = 1, \ldots, m$. Since $\delta(X_i) = \delta(X_i')$, one may assume that $X_i = X_i^*$ for each $i = 1, \ldots, m$. In other words $S(X_i') = S(X_i) + q_i a$ for all i. Let $W_i = X_1' \cdots X_i' X_{i+1} \cdots X_m$ for $1 \leq i \leq m-1$ and $W_0 = \prod_{i=1}^m X_i$, $W_m = \prod_{i=1}^m X_i'$. Every W_i is a block due to $\sum_{i=1}^m \overline{S(X_i)} = \overline{0}$, and $W_0 = \prod_{i=1}^m X_i^*$.

a) Note that $x_a(W_i) < k$ for all $i = 0, \ldots, m$. Indeed $B_1 \cdots B_m$ is the product of the blocks W_i and $W_i' = X_1 \cdots X_i X_{i+1}' \cdots X_m'$ for $1 \leq i \leq m-1$, hence Proposition 4.1c yields $x_a(W_i) + x_a(W_i') = x_a(B_1 \cdots B_m) = \sum_{i=1}^m x_a(B_i) < k$. In addition we obtain analogously $x_a(W_0) + x_a(W_m) = \sum_{i=1}^m x_a(B_i) < k$.

Observe next that $S(W_i) - S(W_{i-1}) = S(X_i') - S(X_i) = q_i a$ for $i = 1, \ldots, m$, meaning that $x_a(W_i) \equiv x_a(W_{i-1}) + q_i \pmod{2k}$. Given $x_a(W_i) < k$ and $q_i \in [0, k]$ for all admissible i, the congruence turns into equality: $x_a(W_i) = x_a(W_{i-1}) + q_i$. Summation over $i = 1, \ldots, m$ yields $x_a(W_m) - x_a(W_0) = \sum_{i=1}^m q_i = \sum_{i=1}^m \delta(X_i)$.

It follows that $x_a(W_0) = \frac{1}{2}\bigl(\sum_{i=1}^m x_a(B_i) - \sum_{i=1}^m \delta(X_i)\bigr)$, as required. Also, since the positive integers $x_a(W_0)$ and $x_a(W_m)$ add up to $\sum_{i=1}^m x_a(B_i)$, their difference $x_a(W_m) - x_a(W_0) = \sum_{i=1}^m \delta(X_i)$ is at most $\sum_{i=1}^m x_a(B_i) - 2$.

b) Let $S(X_i) = e_i + \frac{1}{2}\bigl(x_a(B_i) + \epsilon_i\delta(X_i)\bigr)a$, with $e_i \in \overline{S(X_i)}$ such that $2e_i = 0$ and $\epsilon_i \in \{-1, 1\}$, $i = 1, \ldots, m$. Such a representation exists by (4.1), which also gives $S(X_i') = e_i + \frac{1}{2}\bigl(x_a(B_i) - \epsilon_i\delta(X_i)\bigr)a$. So it suffices to show $\sum_{i=1}^m e_i = 0$. Clearly $\sum_{i=1}^m e_i \in \{0, ka\}$ as $\sum_{i=1}^m e_i \in \langle a\rangle$, $2\sum_{i=1}^m e_i = 0$. If $\sum_{i=1}^m e_i = ka$ then $S(X_1 \cdots X_m) = sa$ where $s = k + \frac{1}{2}\bigl(\sum_{i=1}^m x_a(B_i) + \sum_{i=1}^m \epsilon_i\delta(X_i)\bigr)$. The inequality $\sum_{i=1}^m \delta(X_i) \le \sum_{i=1}^m x_a(B_i) - 2$ from (a) and the hypothesis $\sum_{i=1}^m x_a(B_i) < k$ imply $s \in (k, 2k)$; in particular $x_a(X_1 \cdots X_m) = s$. However the block $X_1 \cdots X_m$ divides $B_1 \cdots B_m$, hence $s = x_a(X_1 \cdots X_m) \le x_a(B_1 \cdots B_m) = \sum_{i=1}^m x_a(B_i) < k$ (Proposition 4.1d). The contradiction proves $\sum_{i=1}^m e_i = 0$. □

We often use Lemma 4.2 in the case where one $x_a(B_i)$ equals 2 or 3, and the remaining ones are equal to 1. The justification of the next corollary is straightforward. All claims follow directly from Lemma 4.2 and the fact that $x_a(B_i)$ and $\delta(X_i)$ have the same parity.

Corollary 4.3. *Under the assumptions of Lemma 4.2 suppose that $x_a(B_i) = 1$ for $i = 1, \ldots, m-1$ and $x_a(B_m) \in \{2, 3\}$. Then there exist elements $e_i \in \overline{S(X_i)}$ with $2e_i = 0$, $i = 1, \ldots, m$, such that $\sum_{i=1}^m e_i = 0$ and:*

a) $\delta(X_i) = 1$ and $\{S(X_i), S(X_i')\} = \{e_i, e_i + a\}$ for $i = 1, \ldots, m-1$;

b) If $x_a(B_m) = 2$ then $\delta(X_m) = 0$ and $S(X_m) = S(X_m') = e_m + a$; if $x_a(B_m) = 3$ then $\delta(X_m) = 1$ and $\{S(X_m), S(X_m')\} = \{e_m + a, e_m + 2a\}$.

We are about to see that substantial structural properties can be expressed conveniently in terms of the δ-quantity and the defect. Let us mention already now that minimal blocks with positive defects are particularly interesting.

Lemma 4.4. *Each minimal block $B|\alpha$ with positive defect has a term $b \in B$ such that $\delta(b) \ge x_a(B)$.*

Proof. Suppose that a minimal block $B = b_1 \cdots b_m$ in α satisfies $d(B) > 0$ and $\delta(b_i) < x_a(B)$ for all i. Applying (4.1) with $X = b_i$, $X' = Bb_i^{-1}$ yields an $e_i \in \overline{b_i}$ with $\mathrm{ord}(e_i) = 2$ and $b_i = e_i + \frac{1}{2}\bigl(x_a(B) \pm \delta(b_i)\bigr)a$ for a suitable choice of sign. Hence $b_i = e_i + x_i a$ with x_i an integer in $\bigl(0, x_a(B)\bigr)$ because $\frac{1}{2}\bigl(x_a(B) \pm \delta(b_i)\bigr) \in \bigl(0, x_a(B)\bigr)$ by $\delta(b_i) < x_a(B)$. Adding up gives $x_a(B)a = \sum_{i=1}^m b_i = \sum_{i=1}^m e_i + \left(\sum_{i=1}^m x_i\right)a$, and we have $\sum_{i=1}^m e_i \in \{0, ka\}$ due to $\sum_{i=1}^m e_i \in \langle a\rangle$, $2\sum_{i=1}^m e_i = 0$.

Now $x_a(B) < |B|$ since $d(B) = |B| - x_a(B) > 0$, and $m = |B| \le r$ since B is minimal. These imply $\sum_{i=1}^m x_i < |B|x_a(B) < |B|^2 \le r^2$. Hence $\sum_{i=1}^m x_i < k$ due to $k > r^2$. Thus $x_a(B)a = \left(\epsilon + \sum_{i=1}^m x_i\right)a$ with $\epsilon \in \{0, k\}$ and $\sum_{i=1}^m x_i \in (0, k)$, which yields $\epsilon + \sum_{i=1}^m x_i = x_a(B)$. Next, $x_a(B) \in (0, k)$ (Proposition 4.1b), so $\epsilon = 0$ and $\sum_{i=1}^m x_i = x_a(B)$. Therefore $|B| = m \le x_a(B)$, contradicting $d(B) > 0$. □

Lemma 4.5. *Let $B_1, \dots, B_m | \alpha$ be disjoint minimal blocks with positive defects and $X_i | B_i$ proper nonempty subsequences satisfying $\delta(X_i) \geq x_a(B_i)$, $i = 1, \dots, m$. Then $\overline{S(X_1)}, \dots, \overline{S(X_m)}$ are independent elements in $G/\langle a \rangle$.*

Proof. If $\overline{S(X_1)}, \dots, \overline{S(X_m)}$ are not independent, the sequence $\prod_{i=1}^{m} \overline{S(X_i)}$ has a nonempty minimal zero-sum subsequence γ, say $\gamma = \prod_{i=1}^{n} \overline{S(X_i)}$ where $0 < n \leq m$. Because $\sum_{i=1}^{n} \overline{S(X_i)} = \overline{0}$, the main assumption of Lemma 4.2 is satisfied for the proper decompositions $B_i = X_i X_i'$, $i = 1, \dots, n$. We show that $\sum_{i=1}^{n} x_a(B_i) < k$ is satisfied too. Since γ is minimal, its length does not exceed the Davenport constant r of $G/\langle a \rangle \cong C_2^{r-1}$. Hence $n = |\gamma| \leq r$. Since $x_a(B_i) < |B_i| \leq r$ for $i = 1, \dots, n$ (each B_i is minimal with $d(B_i) > 0$), we have $\sum_{i=1}^{n} x_a(B_i) < \sum_{i=1}^{n} |B_i| \leq nr \leq r^2$. Now $k > r^2$ gives $\sum_{i=1}^{n} x_a(B_i) < k$. So the inequality in Lemma 4.2a holds true, yielding $\sum_{i=1}^{n} \delta(X_i) \leq \sum_{i=1}^{n} x_a(B_i) - 2$. However this is false as $\delta(X_i) \geq x_a(B_i)$ for all $i = 1, \dots, n$. $\square$

There follows the non-evident observation that a sufficiently long minimal zero-sum sequence in G has few disjoint blocks with positive defects.

Corollary 4.6. *a) There are at most $r - 1$ disjoint blocks with positive defects in α.*

b) Let $B_1, \dots, B_m$ be disjoint minimal blocks with positive defects in α. Then $x_a(B_1) + \cdots + x_a(B_m) < k$.

Proof. a) Let $B_1, \dots, B_m | \alpha$ be disjoint blocks with $d(B_i) > 0$, $i = 1, \dots, m$. Each B_i factorizes into minimal blocks, and at least one of the factors has positive defect because $d(\cdot)$ is additive. By choosing one such factor for every $i = 1, \dots, m$ we can assume that the B_i's are minimal. For each $i = 1, \dots, m$ some $b_i \in B_i$ satisfies $\delta(b_i) \geq x_a(B_i)$ (Lemma 4.4). Then $\overline{b_1}, \dots, \overline{b_m}$ are independent elements in $G/\langle a \rangle$ (Lemma 4.5). Hence m does not exceed the rank $r - 1$ of $G/\langle a \rangle \cong C_2^{r-1}$.

b) We have $m < r$ due to (a), and $d(B_i) > 0$ means $x_a(B_i) < |B_i|$, $i = 1, \dots, m$. So $\sum_{i=1}^{m} x_a(B_i) < \sum_{i=1}^{m} |B_i| \leq mr < r^2$; now refer to $k > r^2$ again. $\square$

Remark 4.7. *By Corollary 4.6b the condition $\sum_{i=1}^{m} x_a(B_i) < k$ in Lemma 4.2 and Corollary 4.3 holds for minimal blocks with positive defects. So in further applications of these statements to such blocks we omit checking $\sum_{i=1}^{m} x_a(B_i) < k$.*

In the sequel an *(l, s)-block* means a minimal block B with length l and sum sa where $l > s$; thus $d(B) > 0$ is assumed. The phrase "B is an (l, s)-block" is shortened to "$B = (l, s)$" wherever convenient. We write $(*, s)$-block or $(l, *)$-block if l or s is irrelevant. Furthermore a *unit block* means a product of $(*, 1)$-blocks.

Lemma 4.8. *a) For each unit block $U | \alpha$ the subgroup $\langle \overline{U} \rangle$ of $G/\langle a \rangle$ has rank $d(U)$. Consequently $d(U) \leq r - 1$.*

b) Let $U | \alpha$ be a unit block, $B | \alpha U^{-1}$ a minimal block with positive defect and $X | B$ a proper nonempty subsequence such that $\delta(X) \geq x_a(B)$. Then $\overline{S(X)} \notin \langle \overline{U} \rangle$.

Proof. a) Let $U = U_1 \cdots U_m$ be a product of $(*, 1)$-blocks U_i. Remove one term from each $\overline{U_i}$ and note that the remaining $\sum_{i=1}^{m}(|U_i| - 1)$ terms are independent in $G/\langle a\rangle$. If not there exist $U_{i_1}, \dots, U_{i_s}$, $s \le m$, and proper nonempty $X_{i_j}|U_{i_j}$ such that $\sum_{j=1}^{s}\overline{S(X_{i_j})} = \overline{0}$. However $\delta(X_{i_j}) \ge x_a(U_{i_j}) = 1$ for all j as $\delta(X_{i_j})$ is odd. Hence $\overline{S(X_{i_1})}, \dots, \overline{S(X_{i_s})}$ are independent elements in $G/\langle a\rangle$ by Lemma 4.5, which is a contradiction. So the remaining $\sum_{i=1}^{m}(|U_i| - 1)$ terms are independent indeed, and this is the maximum number of independent elements in $\langle\overline{U}\rangle$ (an independent set must be missing at least one element from each $\overline{U_i}$). Hence $\langle\overline{U}\rangle$ is a subgroup of rank $\sum_{i=1}^{m}(|U_i| - 1)$. The sum equals $\sum_{i=1}^{m} d(U_i) = d(U)$ as $x_a(U_i) = 1$ for all i. In particular $d(U)$ does not exceed the rank $r - 1$ of $G/\langle a\rangle \cong C_2^{r-1}$.

b) Let $\overline{S(X)} \in \langle\overline{U}\rangle$ for a proper nonempty $X|B$ with $\delta(X) \ge x_a(B)$. Then there is a product $U_1 \cdots U_m|U$ of $(*, 1)$-blocks U_i and proper nonempty $X_i|U_i$ such that $\overline{S(X)}+\sum_{i=1}^{m}\overline{S(X_i)} = \overline{0}$. However this is false: Lemma 4.5 applies as $\delta(X_i) \ge x_a(U_i)$ for all i and $\delta(X) \ge x_a(B)$, hence $\overline{S(X)}, \overline{S(X_1)}, \dots, \overline{S(X_m)}$ are independent. □

There follows a direct consequence of Lemma 4.4 and Lemma 4.8.

Corollary 4.9. *If U is a unit block in α with $d(U) = r - 1$ then αU^{-1} is not divisible by blocks with positive defect.*

More subtle applications of the δ-quantity involve lower members of proper decompositions. Such is the spirit of the concluding lemmas.

Lemma 4.10. *Let a $(*, 1)$-block $U|\alpha$ satisfy $\delta(u) = 1$ for each $u \in U$. Then there exist a term $u \in U$ and an order-2 element $e \in G$ such that $u = e+a$, $S(Uu^{-1}) = e$. Thus Uu^{-1} is the lower member of the proper decomposition $U = (u)(Uu^{-1})$.*

Proof. By (4.1) the condition $\delta(u) = 1$ implies $\{u, S(Uu^{-1})\} = \{e, e + a\}$ for some $e \in G$ with order 2. There is a term u of the form $e + a$, or else multiplying $\sum_{u\in U} u = a$ by 2 yields the impossible $2a = 0$. The claim follows. □

Lemma 4.11. *a) Let $U = (l, 1)$ and $B = (m, 2)$ be disjoint blocks in α such that $\overline{u} \in \langle\overline{B}\rangle$ for every term $u \in U$. Then the product UB is divisible by a $(*, 1)$-block V with $d(V) > d(U)$. Moreover if $m \ge 5$ then $d(V) > d(U)$ can be strengthened to $d(V) > d(U) + 1$.*

b) Let $U = (l, 1)$ and $B = (m, 3)$ be disjoint blocks in α such that $\overline{u} \in \langle\overline{B}\rangle$ for every term $u \in U$. Suppose that the product UB is not *divisible by a unit block V with $d(V) > d(U)$. Then $l = 2$ and UB is divisible by an $(m, 2)$-block.*

Proof. In both (a) and (b) for each $u \in U$ there is a proper nonempty $X|B$ such that $u \sim S(X)$; let $X' = BX^{-1}$. Consider the proper decompositions $U = (u)(Uu^{-1})$ and $B = XX'$. Because $x_a(B) \in \{2, 3\}$, Corollary 4.3a implies $\delta(u) = 1$. Hence by Lemma 4.10 u can be chosen so that $u = e + a$ and $S(Uu^{-1}) = e$ for an order-2 element $e \in G$. Then Uu^{-1} is the lower member of the pair u, Uu^{-1}.

Next, in (a) we have $\delta(X) = 0$ and $S(X) = S(X') = e + a$ by Corollary 4.3b. Similarly $\delta(X) = 1$ and $\{S(X), S(X')\} = \{e + a, e + 2a\}$ hold in (b). By symmetry let $|X| \geq |X'|$ in (a) and $S(X) = e + a$, $S(X') = e + 2a$ in (b). Then the pair X, X' has lower member X; in (a) one can assume so because $\delta(X) = 0$.

Let $V = (Uu^{-1})X$ be the product of the lower members of the same decompositions $U = (u)(Uu^{-1})$, $B = XX'$. This is a block with sum $e+(e+a) = a$ and length $l' = l - 1 + |X|$. Note that $l' > 1$ since $l > 1$, so V is an $(l', 1)$-block dividing UB.

In (a) we obtain $l' \geq l - 1 + \lceil m/2 \rceil \geq l + 1$ as $|X| \geq \lceil m/2 \rceil$ and also $m \geq 3$ by $d(B) > 0$. So $d(V) \geq l > d(U)$, as needed. Likewise if $m \geq 5$ then $d(V) > d(U) + 1$.

The assumption in (b) implies $d(V) \leq d(U)$, i.e., $l' \leq l$. Then $|X| = 1$ and $|X'| = m - 1$. Consider the block $C = (Uu^{-1})X'$, with sum $e + (e + 2a) = 2a$ and length $l+m-2$. We claim that C is minimal. Otherwise it has an $\langle a \rangle$-decomposition with at least two factors. Since $x_a(C) = 2$, it follows from Proposition 4.1, parts (c) and (e), that the factors are exactly two and each one is a $(*, 1)$-block. Then their product C is a unit block dividing UB with $d(C) > d(U)$, contrary to the assumption. Indeed $d(C) = l + m - 4 > l - 1 = d(U)$ since $m \geq 4$ by $d(B) > 0$.

Because $\overline{t} \in \langle \overline{B} \rangle$ for all $t \in C$ and C is minimal, it follows that $|C| \leq |B|$, i.e., $l + m - 2 \leq m$. This gives $l = 2$ and $|C| = m$. Therefore $C|UB$ is an $(m, 2)$-block, completing the proof. Let us remark that $UBC^{-1} = (e+a)^2$ is a block with length 2 and sum $2a$; its defect is 0. $\square$

The last statement concerns $(r, 3)$-blocks where $r \geq 5$ is the rank of G.

Corollary 4.12. *Suppose that G has rank $r \geq 5$. Let $U_1 = (2, 1)$, $U_2 = (2, 1)$ and $B = (r, 3)$ be disjoint blocks in α. Then the product U_1U_2B is divisible by a unit block V with $d(V) > d(U_1U_2)$.*

Proof. The claim is evident if U_1B is divisible by a unit block V with $d(V) > d(U_1)$. If not we apply Lemma 4.11b to U_1 and B. This is possible since $\langle \overline{B} \rangle = G/\langle a \rangle$. So there is an $(r, 2)$-block C that divides U_1B. Now U_2 and C are disjoint and $\langle \overline{C} \rangle = G/\langle a \rangle$, hence Lemma 4.11a applies. Its last part yields a $(*, 1)$-block V with $d(V) > d(U_2) + 1 = d(U_1U_2)$ that divides U_2C. Clearly V divides U_1U_2B. $\square$

5. Longest Minimal Zero-Sum Sequences in $C_2^4 \oplus C_{2k}$

5.1. General Observations

Despite what appears to be an extensive preparation, the Davenport problems for the groups $C_2^{r-1} \oplus C_{2k}$ are still out of reach, even for relatively small $r \geq 5$. The next two general lemmas are necessary to start out an argument in this direction. The first one is a basic property of all longest minimal zero-sum sequences. The second one is related to Proposition 4.1d.

Lemma 5.1. *Let G be a finite abelian group and α a minimal zero-sum sequence of maximum length in G. For each term $t \in \alpha$ and each element $g \in G$ there is a subsequence of α that contains t and has sum g. In particular $\Sigma(\alpha) = G$.*

Proof. Replace $t \in \alpha$ by $t - g$ and g. The resulting sequence α' has sum 0 and is not minimal since α is longest possible. Take a proper nonempty zero-sum subsequence $\beta|\alpha'$. Notice that its complementary sequence $\alpha'\beta^{-1}$ is also a proper nonempty zero-sum subsequence. The minimality of α implies that β and $\alpha'\beta^{-1}$ each contains exactly one of $t - g$ and g. We may assume $t - g \in \beta$, $g \notin \beta$; then $\beta = \gamma(t - g)$ where $\gamma|\alpha t^{-1}$. (The extreme cases $\gamma = \emptyset$ and $\gamma = \alpha t^{-1}$ correspond to $g = t$ and $g = 0$ respectively.) Now γt is a subsequence of α with the stated properties. Indeed it contains t, and $S(\beta) = 0$ rewrites as $S(\gamma t) = g$. $\square$

Lemma 5.2. *Let G be a finite abelian group and $a \in G$ an element with order m. Suppose that an $\langle a\rangle$-block β in G satisfies the following condition:*

> *For every $\langle a\rangle$-block $\beta'|\beta$ the inequality $x_a(\beta') \le x_a(\beta)$ holds, with equality if and only if $\beta' = \beta$.*

Then $\beta a^{m-x_a(\beta)}$ is a minimal zero-sum sequence in G, of length $m + |\beta| - x_a(\beta)$.

Proof. Denoting $x_a(\beta) = s$ and $\alpha = \beta a^{m-s}$, we have $S(\alpha) = sa+(m-s)a = 0$. Each zero-sum subsequence $\alpha'|\alpha$ has the form $\alpha' = \beta' a^t$ where $\beta'|\beta$ and $0 \le t \le m - s$. Clearly $S(\beta') \in \langle a\rangle$, so β' is an $\langle a\rangle$-block. Then $0 \le x_a(\beta') \le s$ by the assumption of the lemma, also $0 \le x_a(a^t) = t \le m-s$. Hence $0 \le x_a(\beta')+x_a(a^t) \le m$. In addition if $\alpha' \ne \emptyset$ then $x_a(\beta')+x_a(a^t) > 0$ (the summands are not both zero). The inequality $0 < x_a(\beta') + x_a(a^t) \le m$ shows that $x_a(\beta') + x_a(a^t) = x_a(\beta' a^t) = x_a(\alpha')$. Since $S(\alpha') = 0$, we obtain $m = x_a(\alpha') = x_a(\beta') + x_a(a^t) \le s + (m - s) = m$, implying $x_a(\beta') = s = x_a(\beta)$ and $x_a(a^t) = m - s$. These equalities hold respectively only if $\beta' = \beta$, by the equality claim in the assumption, and $t = m - s$. Thus $\alpha' = \alpha$, proving that α is a minimal zero-sum sequence. $\square$

5.2. The Approach

We mentioned in Section 4 that a long minimal zero-sum sequence α in $C_2^{r-1} \oplus C_{2k}$ can be regarded as an $\langle a\rangle$-block with defect $d(\alpha) = |\alpha| - x_a(\alpha) = |\alpha| - 2k$; thus $|\alpha| = 2k + d(\alpha)$. Here $a \in \alpha$ is a term like in Proposition 4.1. Hence $|\alpha|$ is a maximum if and only if so is $d(\alpha)$. From this viewpoint finding the Davenport constant of $C_2^{r-1} \oplus C_{2k}$ is equivalent to maximizing the defect $d(\alpha)$ over all (sufficiently long) minimal zero-sum sequences α. Solving the inverse Davenport problem reduces to a characterization of the equality cases.

To argue in this spirit with the tools at hand, we need them to apply (at least) to longest minimal zero-sum sequences.

In this subsection G denotes the group $C_2^{r-1} \oplus C_{2k}$ where $r \geq 3$ and

$$k \geq \left\lceil \tfrac{3r-1}{r+1}(2^r - 1) \right\rceil - r + 2. \tag{5.1}$$

Let α be a minimal zero-sum sequence of maximum length over G, and let $a \in \alpha$ be a term like in Proposition 4.1. In the sequel we call such a term distinguished.

The restriction (5.1) does ensure that all earlier conclusions hold for longest minimal zero-sum sequences α over G. Indeed they are known to satisfy the standard inequality $|\alpha| \geq \mathsf{D}^*(G) = 2k + r - 1$ which, together with (5.1), implies the assumption $|\alpha| \geq k + \left\lceil \frac{3r-1}{r+1}(2^r - 1) \right\rceil + 1$ in Section 4. In particular a distinguished term exists. Observe that (5.1) is stronger than the implicit constraint (3.2) in Section 3. Note also that $|\alpha| \geq 2k + r - 1$ rewrites as $d(\alpha) \geq r - 1$ in terms of the defect.

We go back to the abbreviations "block" and "factorization" for "$\langle a \rangle$-block" and "$\langle a \rangle$-factorization" wherever convenient. However from now on "decomposition" means a proper decomposition of a block as it is defined in Section 4 together with the δ-quantity.

Combined with our general structural results, Lemma 5.2 yields a strong consequence for longest minimal zero-sum sequences.

Corollary 5.3. *Every $\langle a \rangle$-block in α has nonnegative defect.*

Proof. Let $B|\alpha$ be an $\langle a \rangle$-block. By Proposition 4.1d the assumption of Lemma 5.2 holds for its complementary $\langle a \rangle$-block $C = \alpha B^{-1}$. So, since $\operatorname{ord}(a) = 2k$, there exists a minimal zero-sum sequence over G of length $l = 2k + |C| - x_a(C)$. We have $l = |\alpha| - d(B)$ due to $|B| + |C| = |\alpha|$ and $x_a(B) + x_a(C) = x_a(\alpha) = 2k$ (Proposition 4.1a). Because $|\alpha|$ is longest possible, it follows that $|\alpha| - d(B) \leq |\alpha|$, implying $d(B) \geq 0$. $\square$

The true significance of Corollary 5.3 is to be seen shortly. Let us mention first several easy-to-obtain implications. All terms of α in the subgroup $\langle a \rangle$ are equal to a. Every proper $\langle a \rangle$-coset contains at most two distinct terms of the sequence. Because $|B| \leq r$ for every minimal $\langle a \rangle$-block $B|\alpha$, the conclusion $x_a(B) \leq |B|$ of Corollary 5.3 (i.e., $d(B) \geq 0$) shows that $0 < x_a(B) \leq r$. For k large with respect to r, which is the case with our setting, this inequality is much stronger than the general $0 < x_a(B) < k$ (Proposition 4.1b).

Here is an informal description of a possible approach to the Davenport problems. Let $\mathcal{F}$ be an arbitrary factorization of α. All blocks in it have nonnegative defects (Corollary 5.3). At most $r - 1$ of them have positive defects (Corollary 4.6), so the remaining ones have defect 0. If $B_1, \ldots, B_m$ are the blocks with positive defect then $d(\alpha) = \sum_{i=1}^m d(B_i)$ because $d(\cdot)$ is additive. The blocks with defect 0 are irrelevant to $d(\alpha)$ and hence to the Davenport constant $\mathsf{D}(G) = |\alpha| = 2k + d(\alpha)$.

The combined length of $B_1, \dots, B_m$ is less than r^2, and r^2 is much less than the total length $|\alpha| \geq k + \left\lceil \frac{3r-1}{r+1}(2^r - 1) \right\rceil + 1$ under the constraint (5.1). We observe that $\mathsf{D}(G)$ is determined by a very small part of sequence. This part and its structure depend on the factorization $\mathcal{F}$ while $d(\alpha)$ does not. Therefore one can choose $\mathcal{F}$ as desired. In general it is not easy to deal with the multitude of diverse factorizations. We focus mostly on ones with the following extremal property.

Definition 5.4. *A factorization of α is* canonical *if the product of all $(*, 1)$-blocks in it has maximum defect.*

Fix the notation $W_{\mathcal{F}}$ for the product of all $(*, 1)$-blocks in a factorization $\mathcal{F}$. The defects $d(W_{\mathcal{F}})$ are equal (and maximal) for all canonical factorizations $\mathcal{F}$ of α. Denote their common value by $d^*(\alpha)$.

The following properties are meant to simplify the remainder of the exposition. The first one is somewhat cumbersome, yet its formulation enables us to avoid multiple repetitions of the same standard reasoning.

Let $\mathcal{F}$ be a canonical factorization of α. Then:

(i) The complementary block $\alpha W_{\mathcal{F}}^{-1}$ of $W_{\mathcal{F}}$ is not divisible by a unit block. More generally let $B_1, \dots, B_m$ be blocks in $\mathcal{F}$, and let d be the combined defect of the $(*, 1)$-blocks among them. Then the product $B_1 \cdots B_m$ is not divisible by a unit block V with defect $d(V) > d$.

(ii) $2 \leq d(W_{\mathcal{F}}) \leq r - 1$; consequently $d^*(\alpha) = d(W_{\mathcal{F}}) \in \{2, \dots, r-1\}$.

(iii) If $r = 5$ then $\mathcal{F}$ contains neither a $(5, 2)$- nor a $(5, 3)$-block.

Here is the justification.

(i): We prove the general form of the claim. Let $B_1 \cdots B_m$ be divisible by a unit block $V = V_1 \cdots V_n$ where the V_i's are $(*, 1)$-blocks, and let $V' = (B_1 \cdots B_m)V^{-1}$. Replace $B_1, \dots, B_m$ by $V_1, \dots, V_n$ and the factors in any factorization of V'. We obtain a new factorization of α. The new value of $d(W_{\mathcal{F}})$ does not exceed the old one as $\mathcal{F}$ is canonical. It follows that $d(V) \leq d$.

(ii): The right inequality follows from Lemma 4.8a. For the left one take a term $x \in \alpha$ that is not in $\langle a \rangle$. Apply Lemma 5.1 by choosing g to be the distinguished term a. There exists a subsequence U of α that contains x and has sum a. Because U is an $\langle a \rangle$-block with $x_a(U) = 1$, it is a minimal $\langle a \rangle$-block (Proposition 4.1e). By Definition 5.4 then each canonical factorization has $(*, 1)$-blocks, hence $W_{\mathcal{F}} \neq \emptyset$. Suppose that $d(W_{\mathcal{F}}) = 1$. Then $W_{\mathcal{F}}$ is a single $(2, 1)$-block: two terms x, x' with sum a from the same proper $\langle a \rangle$-coset. Let $y \notin \langle a \rangle \cup (x + \langle a \rangle)$ be another term. Such a term exists as $\Sigma(\alpha) = G$ (Lemma 5.1) and $r \geq 3$. Lemma 5.1 yields a $(*, 1)$-block $V|\alpha$ containing y, like with x above. Clearly $|V| \leq |W_{\mathcal{F}}| = 2$, so $|V| = 2$. Thus V is a $(2, 1)$-block contained in the coset $y + \langle a \rangle$ and hence disjoint with $W_{\mathcal{F}}$. This contradicts property (i).

(iii): Let on the contrary $\mathcal{F}$ contain a $(5, s)$-block B with $s \in \{2, 3\}$, and let $U|W_{\mathcal{F}}$ be a $(*, 1)$-block. Such a block exists by $d(W_{\mathcal{F}}) \geq 2$ (property (ii)). For $s = 2$ Lemma 4.11a shows that the product UB is divisible by a $(*, 1)$-block V with $d(V) > d(U)$, contradicting (i). Lemma 4.11b rejects $s = 3$ in the same way if $|U| > 2$. So assume that $s = 3$ and $W_{\mathcal{F}}$ contains only $(2, 1)$-blocks. Note that there are at least two of them by property (ii). Let U_1 and U_2 be such blocks. Corollary 4.12 states that the product U_1U_2B is divisible by a unit block V with $d(V) > d(U_1U_2)$, which yields a contradiction like before.

In summary, one can undertake the Davenport problems for $C_2^{r-1} \oplus C_{2k}$ by studying the blocks with positive defects in suitably chosen factorizations. A basic result about the "excessive Davenport inequality" $\mathsf{D}(G) > \mathsf{D}^*(G)$ is available already now.

> *Suppose that $\mathsf{D}(G) > \mathsf{D}^*(G)$ holds for a group $G = C_2^{r-1} \oplus C_{2k}$ where k and $r \geq 3$ satisfy $k \geq \left\lceil \frac{3r-1}{r+1}(2^r - 1) \right\rceil - r + 2$. Let α be a longest minimal zero-sum sequence in G with distinguished term a. Then $d^*(\alpha) \leq r - 2$ and $d(\alpha W_{\mathcal{F}}^{-1}) \geq 2$ for each canonical $\langle a \rangle$-factorization $\mathcal{F}$ of α.*

Indeed if $d^*(\alpha) > r-2$ then $d(W_{\mathcal{F}}) = d^*(\alpha) = r-1$ by property (ii). Corollary 4.9 and Corollary 5.3 imply that the only blocks with nonzero defect in $\mathcal{F}$ are the factors of $W_{\mathcal{F}}$. By the additivity of $d(\cdot)$ then $d(\alpha) = d(W_{\mathcal{F}}) = r - 1$. However this yields $\mathsf{D}(G) = |\alpha| = 2k + d(\alpha) = 2k + r - 1 = \mathsf{D}^*(G)$, contradicting $\mathsf{D}(G) > \mathsf{D}^*(G)$.

Note that $\mathsf{D}(G) > \mathsf{D}^*(G)$ implies $d(\alpha) \geq r$. So the second claim follows from $d^*(\alpha) \leq r - 2$, already proven, and $d(\alpha) = d(W_{\mathcal{F}}) + d(\alpha W_{\mathcal{F}}^{-1}) = d^*(\alpha) + d(\alpha W_{\mathcal{F}}^{-1})$.

We use this general observation in the solution of the direct Davenport problem for rank $r = 5$ (Theorem 5.8).

5.3. Rank Five

There is a variety of ways to solve the Davenport problems for the groups $C_2^4 \oplus C_{2k}$. Some of them differ considerably from the scheme followed here. Our conclusions hold for $k \geq 70$, a restriction obtained from the general constraint (5.1) for $r = 5$.

> *Henceforth let $G = C_2^4 \oplus C_{2k}$ where $k \geq 70$, and let α be a minimal zero-sum sequence of maximum length in G with distinguished term a.*

The preparatory work is not over yet. To say the least, the old example showing that $\mathsf{D}(C_2^4 \oplus C_{2k}) > \mathsf{D}^*(C_2^4 \oplus C_{2k})$ applies only to odd $k \geq 3$. No example to this effect turned up for even k, which suggests that the even case is different. So parity considerations are likely to emerge at some point. There is no trace of them so far.

Our approach rests on three lemmas concerning rank 5. They explain the presence of several specific statements in Section 4. The lemmas are lengthy but provide more structural insight than some shorter arguments. Naturally we rely on the conclusions from Section 4 and also on properties (i)–(iii) from Subsection 5.2.

Lemma 5.5. *Let a canonical factorization $\mathcal{F}$ of α contain a $(4,2)$-block B. Then $\mathcal{F}$ contains neither an (l,s)-block with $s \geq 2$ apart from B nor a $(2,1)$-block.*

Proof. Suppose on the contrary that $\mathcal{F}$ contains a block $C \neq B$ which is either an (l,s)-block with $s \geq 2$ or a $(2,1)$-block. By Lemma 4.4 C has a term c such that $\delta(c) \geq s$; here $s \geq 2$ if $C = (l,s)$ with $s \geq 2$ and $s = 1$ if $C = (2,1)$.

We claim that $\overline{c} \notin \langle \overline{B} \rangle$. Otherwise $c \sim S(X)$ for some proper nonempty $X|B$. If $C = (l,s)$ with $s \geq 2$, apply Lemma 4.2a to the decompositions $B = XX'$, $C = YY'$ where $X' = BX^{-1}$, $Y = c$, $Y' = Cc^{-1}$. This yields a block A dividing BC with $x_a(A) = \frac{1}{2}\big(2 + s - \delta(X) - \delta(c)\big)$. Property (i) implies $x_a(A) \geq 2$ which leads to a contradiction in view of $\delta(c) \geq s$, $\delta(X) \geq 0$. For $C = (2,1)$ Corollary 4.3 provides an order-2 element e such that $C = e(e+a)$ and $S(X) = S(X') = e + a$. There is no loss of generality in assuming $|X| \geq 2$, so we obtain a (minimal) block $A = eX$ dividing BC, with sum a and length ≥ 3. Hence $d(A) \geq 2 > d(C)$ which contradicts property (i). In conclusion $\overline{c} \notin \langle \overline{B} \rangle$, implying $\langle \overline{Bc} \rangle = G/\langle a \rangle$ (since $\langle \overline{B} \rangle$ has index 2 in $G/\langle a \rangle \cong C_2^4$).

Let $U|\alpha(BC)^{-1}$ be an $(m,1)$-block in $\mathcal{F}$. Such a block exists since $d(W_{\mathcal{F}}) \geq 2$ (property (ii)). Take a proper nonempty subsequence $X|U$ and note that $S(X) \not\sim c$ by Lemma 4.8b. So, because $\langle \overline{Bc} \rangle = G/\langle a \rangle$, one of the following relations holds:

$$\text{(a) } \overline{S(X)} + \overline{S(Y)} = \overline{0}, \qquad\qquad \text{(b) } \overline{S(X)} + \overline{S(Y)} + \overline{c} = \overline{0},$$

with $Y|B$ proper and nonempty. Apply the inequality in Lemma 4.2a to the respective decompositions $U = XX'$, $B = YY'$, and, in case (b), $C = ZZ'$ where $Z = c$, $Z' = Cc^{-1}$. In both (a) and (b) the inequality implies $\delta(X) = 1$, $\delta(Y) = 0$, and also $\delta(Z) = s$ in (b). Hence the lower members of the pairs form a $(*,1)$-block. In (a) this is $V_1 = X^*Y^*|UB$, with $x_a(V_1) = \frac{1}{2}(1 + 2 - 1 - 0) = 1$. In (b) we have $V_2 = X^*Y^*Z^*|UBC$, with $x_a(V_2) = \frac{1}{2}(1 + 2 + s - 1 - 0 - s) = 1$. Either one of Y and Y' can be taken as the lower member of the pair Y, Y' as $\delta(Y) = 0$. Thus we can assume $|Y^*| = \max(|Y|, |Y'|) \geq 2$, so that $|V_1| = |X^*| + |Y^*| \geq |X^*| + 2$ in (a) and $|V_2| = |X^*| + |Y^*| + |Z^*| \geq |X^*| + 3$ in (b).

We proved that $\delta(X) = 1$ for all proper nonempty $X|U$, hence $\delta(u) = 1$ for each term $u \in U$. By Lemma 4.10 $u \in U$ can be chosen so that $u = e + a$, $S(Uu^{-1}) = e$ for some $e \in G$ with $\mathrm{ord}(e) = 2$. Apply the above to the subsequence $X = u|U$. The lower member of the pair X, X' is $X^* = Uu^{-1}$, with $|X^*| = m - 1$.

In case (a) we obtain a block $V_1|UB$ with $x_a(V_1) = 1$, $|V_1| \geq (m-1)+2 = m+1$ and $d(V_1) \geq m > d(U)$. The latter contradicts property (i). Similarly in case (b) there is a block $V_2|UBC$ with $x_a(V_2) = 1$, $|V_2| \geq (m-1)+3 = m+2$ and $d(V_2) \geq m+1$. However the $(*,1)$-blocks among U, B and C have combined defect $d \leq m$ ($d = m-1$ if $C = (l,s)$ with $s \geq 2$ and $d = m$ if $C = (2,1)$). We reach a contradiction with property (i) again. The proof is complete. □

Lemma 5.6. *Suppose that there exists a canonical factorization of α that contains two blocks of the form $(s+1, s)$ with $s \in \{2,3,4\}$. Then α has a canonical factorization that contains a $(4,2)$-block.*

Proof. Let a canonical factorization $\mathcal{F}$ of α contain two blocks B and C of the form $(s+1, s)$ with $s \in \{2,3,4\}$. Consider three cases. In each one of them we find a $(4,2)$-block A that divides the product BC. To complete the proof, it remains to replace B and C by A and the factors in any factorization of the complementary block $A' = BCA^{-1}$. Let us remark that $d(A') = 0$ because $d(BC) = d(A) = 2$.

`Case 1:` B is a $(5,4)$-block and C an $(s+1, s)$-block with $s \in \{2,3,4\}$. Choose $c \in C$ with $\delta(c) \geq s$ (Lemma 4.4) and a proper nonempty $X|B$ with $c \sim S(X)$. Apply Lemma 4.2a to the decompositions $B = XX'$, $C = YY'$ where $Y = c$. The product $A|BC$ of their lower members satisfies $x_a(A) = \frac{1}{2}\big(4 + s - \delta(X) - \delta(c)\big)$. Now $x_a(A) \geq 2$ by property (i), implying $\delta(X) = 0$, $\delta(c) = s$, $x_a(A) = 2$. Since $\delta(X) = 0$, either one of X and X' can be taken as the lower member X^*. So assume $|X^*| = \max(|X|, |X'|) \geq 3$ to obtain $|A| \geq |Y^*| + 3 \geq 4$. On the other hand $A' = BCA^{-1}$ has length $s - |A| + 6$ and sum $(s+2)a$. Since $d(A') \geq 0$ (Corollary 5.3), it follows that $|A| \leq 4$. In conclusion $|A| = 4$, so $A = (4,2)$.

`Case 2:` B is a $(4,3)$-block and C an $(s+1, s)$-block with $s \in \{2,3\}$. There exists $X|C$ with $|X| = 2$ and $\overline{S(X)} \in \langle \overline{B} \rangle$. This is so if C has two terms c_1, c_2 with $\overline{c_1}, \overline{c_2}$ in $\langle \overline{B} \rangle$. If not there are terms $c_1, c_2 \in C$ with $\overline{c_1}, \overline{c_2} \notin \langle \overline{B} \rangle$ and $X = c_1 c_2$ has the stated property ($\langle \overline{B} \rangle$ is an index-2 subgroup of $G/\langle a \rangle$). Take a proper nonempty $Y|B$ such that $S(X) \sim S(Y)$ and apply Lemma 4.2a to the decompositions $B = YY'$, $C = XX'$. Their lower members yield a block $A = X^*Y^*|BC$ with $x_a(A) = \frac{1}{2}\big(3 + s - \delta(Y) - \delta(X)\big)$. We have $x_a(A) \geq 2$ by property (i) again. Now $\delta(Y) \geq 1$ because $x_a(B) = 3$ is odd, which implies $\delta(X) \leq s - 2$. It follows from $s \in \{2,3\}$ that $\delta(X) = s - 2$, $\delta(Y) = 1$. Let $Y^* = Y$ without loss of generality.

If $s = 2$ then $\delta(X) = 0$, $\delta(Y) = 1$, so there is an order-2 element $e \in G$ such that $S(X) = S(X') = e + a$, $S(Y) = e + a$, $S(Y') = e + 2a$ (Lemma 4.2b). If $|Y| = 3$ then $A = XY$ is a $(5,2)$-block; let $A' = BCA^{-1}$. Replace B and C by A and the factors in a factorization of A'. Since $W_{\mathcal{F}}$ is not affected, we obtain a new canonical factorization containing a $(5,2)$-block, and this is impossible by property (iii). The same reasoning rejects $|Y| = 1$ in which case XY' is a $(5,3)$-block. Hence $|Y| = 2$; then $A = XY$ is a $(4,2)$-block that divides BC, as needed.

We argue similarly for $s = 3$ where $\delta(X) = \delta(Y) = 1$ and there is an order-2 element $e \in G$ such that $S(X) = e + a = S(Y)$, $S(X') = e + 2a = S(Y')$ (there is no loss of generality here as $|X| = |X'| = 2$). If $|Y| = 3$ then XY is a $(5,2)$-block; if $|Y| = 1$ then XY' is a $(5,3)$-block. Neither one is possible for the same reasons like with $s = 2$. Hence $|Y| = 2$, so that $A = XY$ is a $(4,2)$-block dividing BC.

Case 3: B and C are $(3,2)$-blocks. We claim that $\langle\overline{B}\rangle \cap \langle\overline{C}\rangle \neq \{\overline{0}\}$. Once this is proven, take decompositions $B = XX'$, $C = YY'$ with $S(X) \sim S(Y)$. Their lower members form a block $A|BC$ with $x_a(A) = \frac{1}{2}(4 - \delta(X) - \delta(Y))$ (Lemma 4.2a), and $x_a(A) \geq 2$ by property (i). Hence $\delta(X) = \delta(Y) = 0$, so each one of X, X', Y, Y' can serve as the lower member of its pair. Since $\{|X|, |X'|\} = \{|Y|, |Y'|\} = \{1, 2\}$, one can ensure $|A| = 4$. Also $\delta(X) = \delta(Y) = 0$ implies $x_a(A) = 2$, so A is a $(4,2)$-block.

Suppose that $\langle\overline{B}\rangle \cap \langle\overline{C}\rangle = \{\overline{0}\}$; then $\langle\overline{BC}\rangle = G/\langle a\rangle$. Let $U|W_{\mathcal{F}}$ be an $(l,1)$-block and $X|U$ proper nonempty. We show that $\delta(X) = 1$. This follows from Corollary 4.3a if $\overline{S(X)}$ is in $\langle\overline{B}\rangle$ or in $\langle\overline{C}\rangle$. Otherwise, since $\langle\overline{BC}\rangle = G/\langle a\rangle$, a relation of the form $\overline{S(X)} + \overline{S(Y)} + \overline{S(Z)} = \overline{0}$ holds, with proper nonempty $Y|B$, $Z|C$. Apply Lemma 4.2a to the respective decompositions. The product $A|UBC$ of their lower members satisfies $x_a(A) = \frac{1}{2}(5 - \delta(X) - \delta(Y) - \delta(Z))$, and $\delta(X) + \delta(Y) + \delta(Z) \leq 3$ by the inequality in the lemma. Since $\delta(X)$ is odd and $\delta(Y), \delta(Z)$ are even, we have $\delta(X) \in \{1,3\}$ and $\delta(Y), \delta(Z) \in \{0,2\}$. If $\delta(X) = 3$ the argument gives $\delta(Y) = \delta(Z) = 0$, $x_a(A) = 1$. Thus one can assume $|Y^*| = \max(|Y|, |Y'|) = 2$ and likewise $|Z^*| = 2$, yielding $|A| = |X^*| + |Y^*| + |Z^*| \geq 5$. Now A is minimal by $x_a(A) = 1$. Hence it is a $(5,1)$-block and so $d^*(\alpha) \geq d(A) = 4$. Then $d^*(\alpha) = 4$ by property (ii), implying $d(W_{\mathcal{F}}) = 4$. This contradicts Corollary 4.9 as $\alpha W_{\mathcal{F}}^{-1}$ is divisible by blocks with positive defect.

Hence $\delta(X) = 1$ for each proper nonempty $X|U$, in particular $\delta(u) = 1$ for each $u \in U$. Then there is a term $u \in U$ such that Uu^{-1} is the lower member of the decomposition $U = (u)(Uu^{-1})$ (Lemma 4.10). Apply the above with $X = Uu^{-1}$.

Let $\overline{S(X)} \in \langle\overline{B}\rangle$ or $\overline{S(X)} \in \langle\overline{C}\rangle$, say $\overline{S(X)} \in \langle\overline{B}\rangle$. Then $S(X) \sim S(Y)$ for a proper nonempty $Y|B$. Apply Corollary 4.3 to the decompositions $U = u(Uu^{-1})$, $B = YY'$ to obtain an order-2 element e such that $S(Y) = S(Y') = e + a$, $S(Uu^{-1}) = e$ and $u = e + a$. Choose Y^* so that $|Y^*| = 2$. The block $A = (Uu^{-1})Y^*$ divides UB and $x_a(A) = 1$, $|A| = l + 1$. Hence $d(A) = l > d(U)$, contradicting (i).

Now suppose that $\overline{S(X)} \notin \langle\overline{B}\rangle$ and $\overline{S(X)} \notin \langle\overline{C}\rangle$. Since $\langle\overline{BC}\rangle = G/\langle a\rangle$, we have $\overline{S(X)} + \overline{S(Y)} + \overline{S(Z)} = \overline{0}$ with proper nonempty $Y|B$, $Z|C$. The respective lower members form a block $A|UBC$ with $x_a(A) = \frac{1}{2}(5 - \delta(X) - \delta(Y) - \delta(Z))$, $\delta(X) = 1$. Also $\delta(Y) + \delta(Z) \leq 2$ (Lemma 4.2a) and so $\delta(Y), \delta(Z) \in \{0,2\}$. Now $|X^*| = l - 1$, implying $|A| \geq l + 1$. If one of $\delta(Y), \delta(Z)$ is 2 then $x_a(A) = 1$ and $d(A) > d(U)$, contradicting property (i). Let $\delta(Y) = \delta(Z) = 0$, then $x_a(A) = 2$. Assuming $|Y^*| = 2$, $|Z^*| = 2$ like before, we have $|A| = l + 3$. Observe that A is a minimal block. Otherwise it decomposes into a product of at least two factors whose a-coordinates have sum 2, implying that the factors are two $(*,1)$-blocks. Hence A is a unit block with $d(A) = l + 1 > d(U)$, which contradicts (i) again. The minimality of A gives $l + 3 \leq 5$ and so $l = 2$.

We may assume now that $W_{\mathcal{F}}$ contains only $(2,1)$-blocks, at least two of them by property (ii). Let U_1 and U_2 be such blocks. The previous argument yields a $(5,2)$-block $A|U_1BC$. Apply the last part of Lemma 4.11a to U_2 and A. It provides a

$(*,1)$-block $V|U_2A$ with $d(V) > d(U_2)+1 = d(U_1U_2)$. Because V divides U_1U_2BC, we reach a contradiction with property (i) one more time. The proof is complete. $\square$

The conclusions so far hold for every sufficiently large k regardless of its parity. The next statement reveals (at long last) why only the odd case is "excessive."

Lemma 5.7. *Let a factorization of α contain a $(4,1)$-block $U = \prod_{i=1}^{4} u_i$ and a $(4,2)$-block $B = \prod_{i=1}^{4} b_i$. Then:*

a) $\overline{b_i} \notin \langle \overline{U} \rangle$ and $\overline{u_i} \notin \langle \overline{B} \rangle$ for all $i = 1,2,3,4$;

b) k is odd and there exist order-2 elements $g_1, g_2, g_3, g_4 \in G$ with sum 0 such that $b_i = g_i + \frac{k+1}{2}a$ for $i = 1,2,3,4$.

Proof. a) Since $\sum_{i=1}^{4} \overline{b_i} = \overline{0}$, there is an even number of terms $\overline{b_i}$ in the index-2 subgroup $\langle \overline{U} \rangle$ of $G/\langle a \rangle$. So if $\overline{b_1} \in \langle \overline{U} \rangle$ we may also assume $\overline{b_2} \in \langle \overline{U} \rangle$ and hence $\overline{b_1} + \overline{b_2} \in \langle \overline{U} \rangle$. Let $b_1 \sim S(X)$ with proper nonempty $X|U$. Apply Corollary 4.3b to the decompositions $U = XX'$, $B = (b_1)(b_2b_3b_4)$ to obtain an $e \in G$ with order 2 such that $b_1 = e+a$. Hence $2b_1 = 2a$ and likewise $2b_2 = 2a$, $2(b_1+b_2) = 2a$. However the three equalities cannot hold simultaneously. Thus $\overline{b_i} \notin \langle \overline{U} \rangle$ for $i = 1,2,3,4$.

Next, suppose that $\overline{u_1} \in \langle \overline{B} \rangle$ and let $\overline{u_2} \in \langle \overline{B} \rangle$ like above; then $\overline{u_1} + \overline{u_2} \in \langle \overline{B} \rangle$. Because $\overline{b_i} \notin \langle \overline{U} \rangle$ for all i, up to relabeling one can also assume $u_1 \sim b_1 + b_3$, $u_2 \sim b_2 + b_3$; then $u_1 + u_2 \sim b_1 + b_2$. Apply Corollary 4.3 to the decompositions $U = (u_1)(u_2u_3u_4)$ and $B = (b_1b_3)(b_2b_4)$. There is an $e_1 \in G$ with order 2 such that $\{u_1, u_2+u_3+u_4\} = \{e_1, e_1+a\}$ and $b_1+b_3 = b_2+b_4 = e_1+a$. In our case $u_1 = e_1$. Indeed if $u_2+u_3+u_4 = e_1$ then $u_2u_3u_4b_1b_3$ and $u_1b_2b_4$ are a $(5,1)$- and a $(3,2)$-block respectively. However this contradicts Corollary 4.9. Thus $\operatorname{ord}(u_1) = 2$ and $b_1+b_3 = u_1+a$. Similarly $\operatorname{ord}(u_2) = 2$ and $b_2+b_3 = u_2+a$. Now consider the decompositions $U = (u_1u_2)(u_3u_4)$ and $B = (b_1b_2)(b_3b_4)$. There is an $e \in G$, $\operatorname{ord}(e) = 2$, such that $\{u_1+u_2, u_3+u_4\} = \{e, e+a\}$ and $b_1+b_2 = b_3+b_4 = e+a$. Note that $e = u_1+u_2$ since $\operatorname{ord}(u_1) = \operatorname{ord}(u_2) = 2$, and so $b_1+b_2 = u_1+u_2+a$. But also $b_1+b_3 = u_1+a$, $b_2+b_3 = u_2+a$, and we obtain $2(b_1+b_2+b_3) = 3a$. This is false as $3a \notin 2G$. Hence $\overline{u_i} \notin \langle \overline{B} \rangle$ for $i = 1,2,3,4$.

b) Since all $\overline{b_i}$ are outside the index-2 subgroup $\langle \overline{U} \rangle$ of $G/\langle a \rangle$, for each pair i,j with $1 \le i < j \le 4$ there is a proper nonempty $X|U$ such that $b_i+b_j \sim S(X)$. Apply Corollary 4.3b to the decompositions $U = XX'$, $B = (b_ib_j)(B(b_ib_j)^{-1})$. This yields an element $g \in G$ with order 2 such that $b_i+b_j = g+a$. On the other hand each b_i has a (unique) representation $b_i = g_i + x_ia$ where $g_i \in \overline{b_i}$ has order 2 and $0 \le x_i < k$ $(i = 1,2,3,4)$. So the integers $x_i, x_j \in [0,k)$ satisfy $g_i + g_j + (x_i+x_j)a = g+a$ for appropriately chosen order-2 elements $g_i, g_j, g \in G$. Multiplication by 2 leads to $2(x_i+x_j-1)a = 0$, meaning that $x_i + x_j \equiv 1 \pmod{k}$. Fix $i = 1$ and set $j = 2,3,4$ to deduce that x_2, x_3, x_4 are congruent mod k; by symmetry so are all x_i. Because $x_i \in [0,k)$, all x_i are equal. Their common value $x \in [0,k)$ satisfies the congruence

$2x \equiv 1 \pmod{k}$, which has a solution only if k is odd. In the latter case the unique solution in $[0,k)$ is $x = \frac{k+1}{2}$. Therefore $b_i = g_i + \frac{k+1}{2}a$ for $i = 1,2,3,4$, as stated. In addition $\sum_{i=1}^{4} b_i = 2a$ leads to $\sum_{i=1}^{4} g_i = 0$. □

Finally we are ready to approach the Davenport problems for the groups $C_2^4 \oplus C_{2k}$.

Theorem 5.8. *For each $k \geq 70$ the Davenport constant of the group $C_2^4 \oplus C_{2k}$ is*

$$\mathsf{D}(C_2^4 \oplus C_{2k}) = \begin{cases} 2k+4 = \mathsf{D}^*(C_2^4 \oplus C_{2k}) & \text{if } k \text{ is even;} \\ 2k+5 = \mathsf{D}^*(C_2^4 \oplus C_{2k}) + 1 & \text{if } k \text{ is odd.} \end{cases}$$

Proof. Suppose that a group $C_2^4 \oplus C_{2k}$ with $k \geq 70$ satisfies the "excessive" inequality $\mathsf{D}(C_2^4 \oplus C_{2k}) > \mathsf{D}^*(C_2^4 \oplus C_{2k}) = 2k+4$. Let α be an arbitrary minimal zero-sum sequence of maximum length over this group, and let $a \in \alpha$ be a distinguished term.

To begin with, $\mathsf{D}(C_2^4 \oplus C_{2k}) > \mathsf{D}^*(C_2^4 \oplus C_{2k})$ gives $d(\alpha) > 4$. Recall the concluding remark in Subsection 5.2. For $r = 5$ it implies $d^*(\alpha) \leq 3$; also $d(\alpha W_{\mathcal{F}}^{-1}) \geq 2$ for each canonical $\langle a \rangle$-factorization $\mathcal{F}$ of α. The blocks of $\mathcal{F}$ whose product is $\alpha W_{\mathcal{F}}^{-1}$ are either of the form $(*, s)$ with $s \geq 2$ or with defect 0. Hence $d(\alpha W_{\mathcal{F}}^{-1}) \geq 2$ implies that $\mathcal{F}$ contains blocks $B = (l, s)$ with $2 \leq s < l \leq 5$ and with combined defect at least 2. Property (iii) rejects $B = (5,2)$ and $B = (5,3)$, so every such block is either $B = (4,2)$ or $B = (s+1, s)$ with $s \in \{2,3,4\}$.

We claim that there is a canonical $\langle a \rangle$-factorization of α with a block $B = (4,2)$. Indeed let $\mathcal{F}$ be a canonical $\langle a \rangle$-factorization without a $(4,2)$-block. Its $(*, s)$-blocks with $s \geq 2$ are of the form $(s+1, s)$ where $s \in \{2,3,4\}$, so each one has defect 1. Since their combined defect is at least 2, there are at least two of them. Hence Lemma 5.6 applies and proves the claim.

Let $\mathcal{F}$ be a canonical $\langle a \rangle$-factorization of α with a $(4,2)$-block. By Lemma 5.5 and Corollary 4.9 each remaining block with positive defect in $\mathcal{F}$ is a $(3,1)$- or a $(4,1)$-block. Such a block exists as $W_{\mathcal{F}} \neq \emptyset$ and is unique as $d(W_{\mathcal{F}}) = d^*(\alpha) \leq 3$. Hence $\mathcal{F}$ has exactly two blocks U, B with nonzero defect. There are two alternatives: $U = (3,1), B = (4,2)$ and $U = (4,1), B = (4,2)$, with $d(\alpha) = 4$ and $d(\alpha) = 5$ respectively. Because $d(\alpha) > 4$, the only possibility is the second one.

Now we see how the parity of k makes a difference. Since $\mathcal{F}$ contains a $(4,1)$- and a $(4,2)$-block, Lemma 5.7b shows that k is odd.

In summary, for $k \geq 70$ the inequality $\mathsf{D}(C_2^4 \oplus C_{2k}) > \mathsf{D}^*(C_2^4 \oplus C_{2k})$ implies that k is odd and $d(\alpha) = 5$ for each longest minimal zero-sum sequence α over $C_2^4 \oplus C_{2k}$. Therefore $\mathsf{D}(C_2^4 \oplus C_{2k}) = 2k + d(\alpha) = 2k+5 = \mathsf{D}^*(C_2^4 \oplus C_{2k}) + 1$.

By a standard example $\mathsf{D}(C_2^4 \oplus C_{2k}) \geq \mathsf{D}^*(C_2^4 \oplus C_{2k})$ for all $k \geq 1$, which settles the even case of the theorem. The example of Geroldinger and Schneider in [4, Theorem 4] shows that $\mathsf{D}(C_2^4 \oplus C_{2k}) > \mathsf{D}^*(C_2^4 \oplus C_{2k})$ for odd $k \geq 3$, which completes the proof of the odd case. □

Let us state a conclusion implied by the proof of Theorem 5.8 for odd $k > 70$. Each longest minimal zero-sum sequence over $C_2^4 \oplus C_{2k}$, with a distinguished term a, has a canonical $\langle a\rangle$-factorization $\mathcal{F}$ that contains a $(4,1)$-block and a $(4,2)$-block; all remaining blocks in $\mathcal{F}$ have defect zero. We use this statement to solve the inverse Davenport problem in the odd case.

Theorem 5.9. *Let $k > 70$ be an odd integer. A sequence α over $C_2^4 \oplus C_{2k}$ is a minimal zero-sum sequence of maximum length $\mathsf{D}(C_2^4 \oplus C_{2k}) = 2k+5$ if and only if there exists a basis $\{e_1, e_2, e_3, e_4, a\}$ of $C_2^4 \oplus C_{2k}$, with* $\mathrm{ord}(e_i) = 2$ *for $i = 1,2,3,4$ and* $\mathrm{ord}(a) = 2k$, *such that $\alpha = a^{2k-3}UB$ where*

$$U = \prod_{i=1}^{3} \left(e_i + \tfrac{k+1}{2}a\right) \left(e_1 + e_2 + e_3 + \tfrac{k-1}{2}a\right), \tag{5.2}$$

$$B = \prod_{i=1}^{3} \left(e_i + e_4 + \tfrac{k+1}{2}a\right) \left(e_1 + e_2 + e_3 + e_4 + \tfrac{k+1}{2}a\right). \tag{5.3}$$

Proof. The justification of the sufficiency is standard. For the necessity consider a minimal zero-sum sequence α of maximum length $2k+5$ over $G = C_2^4 \oplus C_{2k}$, with $k > 70$ odd. Let $a \in \alpha$ be a distinguished term. By the remark after Theorem 5.8 there is a canonical $\langle a\rangle$-factorization $\mathcal{F}$ of α in which the only factors with nonzero defect are a $(4,1)$-block $U = \prod_{i=1}^4 u_i$ and a $(4,2)$-block $B = \prod_{i=1}^4 b_i$. We invoke Lemma 5.7. In view of its part (a) suitable relabeling ensures $u_i + u_j \sim b_i + b_j$ for $1 \le i < j \le 4$. By part (b) of the lemma there are order-2 elements $g_1, g_2, g_3, g_4 \in G$ with sum 0 such that $b_i = g_i + \frac{k+1}{2}a$ for $i = 1,2,3,4$. Note that g_1, g_2, g_3, g_4 are uniquely determined.

Consider the decompositions $U = (u_1u_2)(u_3u_4)$ and $B = (b_1b_2)(b_3b_4)$. By Corollary 4.3 there is an order-2 element $e \in G$ such that $\{u_1+u_2, u_3+u_4\} = \{e, e+a\}$ and $b_1 + b_2 = b_3 + b_4 = e + a$. Without loss of generality let $u_3 + u_4 = e$, $u_1 + u_2 = e + a$. Then the blocks $U' = u_3u_4b_1b_2$ and $B' = u_1u_2b_3b_4$ are respectively a $(4,1)$- and a $(4,2)$-block in a new factorization $\mathcal{F}'$, to which we apply Lemma 5.7b. Because the g_i's are unique, the lemma provides elements $e_1, e_2 \in G$ with order 2 such that $e_1 + e_2 = g_1 + g_2$ and $u_1 = e_1 + \frac{k+1}{2}a$, $u_2 = e_2 + \frac{k+1}{2}a$. Again, e_1 and e_2 are unique. Analogous reasoning for the decompositions $U = (u_1u_3)(u_2u_4)$ and $B = (b_1b_3)(b_2b_4)$ yields an order-2 element $e_3 \in G$ such that $e_1 + e_3 = g_1 + g_3$ and $u_3 = e_3 + \frac{k+1}{2}a$. In the application of Corollary 4.3 here we assume $u_2 + u_4 = e$, $u_1 + u_3 = e + a$; up to relabeling, there is no loss of generality. Since $u_i = e_i + \frac{k+1}{2}a$ for $i = 1,2,3$ and $\sum_{i=1}^4 u_i = a$, we also obtain $u_4 = e_1 + e_2 + e_3 + \frac{k-1}{2}a$.

Furthermore $e_1 + e_2 = g_1 + g_2$, $e_1 + e_3 = g_1 + g_3$ and $g_1 + g_2 + g_3 + g_4 = 0$ imply $g_4 - (e_1 + e_2 + e_3) = g_i - e_i$ for $i = 1,2,3$. The common value e_4 of the four differences is a group element with order 2. It is straightforward that e_1, e_2, e_3, e_4

form a basis of G together with a. In summary,

$$u_i = e_i + \tfrac{k+1}{2}a \quad \text{for } i = 1,2,3, \qquad u_4 = e_1 + e_2 + e_3 + \tfrac{k-1}{2}a;$$
$$b_i = e_i + e_4 + \tfrac{k+1}{2}a \quad \text{for } i = 1,2,3, \qquad b_4 = e_1 + e_2 + e_3 + e_4 + \tfrac{k+1}{2}a.$$

Hence (5.2) and (5.3) hold true for U and B. In addition observe that $\delta(u_i)$ and $\delta(b_i)$ are large; more exactly

$$\delta(u_i) = k \quad \text{for } i = 1,2,3, \quad \delta(u_4) = k-2; \qquad \delta(b_i) = k-1 \quad \text{for } i = 1,2,3,4.$$

It remains to show that the terms of α not in UB are all equal to a. Equivalently, all blocks with defect 0 in $\mathcal{F}$ have length 1. Suppose on the contrary that $\mathcal{F}$ contains a block C with $d(C) = 0$ and $|C| \geq 2$; in other words $|C| = x_a(C) \geq 2$. Choose any term $c \in C$. Since $\overline{u_1}, \overline{u_2}, \overline{u_3}, \overline{b_4}$ generate $G/\langle a\rangle$ and $c \notin \langle a\rangle$, there is a relation of the form $\overline{c} + \epsilon_1\overline{u_1} + \epsilon_2\overline{u_2} + \epsilon_3\overline{u_3} + \epsilon_4\overline{b_4} = \overline{0}$ where $\epsilon_i \in \{0,1\}$, $i = 1,2,3,4$, and not all ϵ_i's are 0. We claim that $\epsilon_4 = 0$. Otherwise $\overline{c} + \overline{S(X)} + \overline{b_4} = \overline{0}$ for a proper $X|U$ (possibly $X = \emptyset$), and we refer to Lemma 4.2. Here the sum $\sum_{i=1}^{m} x_a(B_i)$ is $x_a(C) + x_a(U) + x_a(B) = |C| + 3$ or $x_a(C) + x_a(B) = |C| + 2$. Clearly $|C| + 3 < k$ since C is minimal, so the lemma does apply. However the inequality $\sum_{i=1}^{m} \delta(X_i) \leq \sum_{i=1}^{m} x_a(B_i) - 2$ cannot hold in our case as $\delta(b_4) = k-1$ is too large. Therefore $\epsilon_4 = 0$. For analogous reasons there is an even number of 1's among $\epsilon_1, \epsilon_2, \epsilon_3$. Hence $\overline{c}$ is equal to one of the sums $\overline{u_1} + \overline{u_2}, \overline{u_2} + \overline{u_3}, \overline{u_3} + \overline{u_1}$. These are the nonzero elements of a rank-2 subgroup of $G/\langle a\rangle$. It follows that $|C| = x_a(C) \in \{2,3\}$.

If, e.g., $c \sim u_1 + u_2$, apply Corollary 4.3 to the decompositions $U = (u_1u_2)(u_3u_4)$, $C = (c)(Cc^{-1})$ (the condition $x_a(U) + x_a(C) < k$ holds true). There is an order-2 element $e \in G$ such that $\{u_1 + u_2, u_3 + u_4\} = \{e, e+a\}$ and $c \in \{e+a, e+2a\}$. Because $u_1 + u_2 = e_1 + e_2 + (k+1)a$, $u_3 + u_4 = e_1 + e_2 + ka$, we find $e = e_1 + e_2 + ka$. Hence $c = e_1 + e_2 + (k+\epsilon)a$ with $\epsilon \in \{1,2\}$.

Now notice that $A = cu_2u_3b_3b_1$ is a block with sum

$$e_1 + e_2 + (k+\epsilon)a + e_2 + e_3 + (k+1)a + e_3 + e_1 + (k+1)a = (k+2+\epsilon)a.$$

So $x_a(A) = k+2+\epsilon$ as $\epsilon \in \{1,2\}$ and k is large, also $x_a(A) > k$. Because $|A| = 5 < k$, the defect of A is negative which is impossible. The proof is complete. $\square$

Needless to say, up to isomorphism the excessively long sequence from Theorem 5.9 is just the example of Geroldinger and Schneider in [4].

6. Concluding Remarks

The approach-dependent condition $k \geq 70$ in Theorems 5.8 and 5.9 is certainly too restrictive. In all likelihood it can be relaxed significantly (maybe even the trivial $k \geq 3$ suffices). The same remark applies to the general constraints (3.1) and (5.1).

The well-known equalities $\mathsf{D}(C_2^{r-1} \oplus C_{2k}) = \mathsf{D}^*(C_2^{r-1} \oplus C_{2k})$ with $r = 3, 4$ follow directly from the considerations in Sections 4 and 5. Only a slight refinement in Section 4 is needed for the case $r = 4$.

The uniqueness of the extremal sequence in Theorem 5.9 may be due to the fact that it is excessively long (with length greater than $\mathsf{D}^*(G)$). In contrast, there are quite a few longest minimal zero-sum sequences over the groups $C_2^{r-1} \oplus C_{2k}$ whenever $\mathsf{D}(C_2^{r-1} \oplus C_{2k}) = \mathsf{D}^*(C_2^{r-1} \oplus C_{2k})$ holds true—already for $r = 3, 4$ and $r = 5$ with k even.

In principle the inverse Davenport problem for $C_2^{r-1} \oplus C_{2k}$ can be solved completely with the available tools for small $r \geq 3$, but a specific difficulty arises. As r grows, the extremal sequences tend to be too numerous and too diverse to fit into a unified description, at least for certain values of k. (See the concluding comment to this effect in [9, Section 3] where the simplest case $r = 3$ is considered.)

Our approach suggests a way to deal with this issue. We saw that the Davenport constant $\mathsf{D}(C_2^{r-1} \oplus C_{2k})$ is determined by the blocks with positive defect in $\langle a \rangle$-factorizations. The blocks with defect zero are irrelevant. In a sense, longest minimal zero-sum sequences differing only in blocks of zero defect are not substantially different with respect to the Davenport problems. One may accept the point of view that they are to be regarded as the same sequence. With this convention there are much fewer objects to search for. The inverse problem reduces to identifying the possible sets of blocks with positive defects in canonical factorizations. Our rough description can be made more precise which we do not pursue here.

Let us remark in passing that for k large with respect to r the general results obtained above imply $\mathsf{D}(C_2^{r-1} \oplus C_{2k}) \leq 2k + (r-1)^2$. This upper bound improves $\mathsf{D}(C_2^{r-1} \oplus C_{2k}) \leq 2k + 2^{r-1} - 1$ (for the latter see the justification of (3.2) in Section 3). Let α be a longest minimal zero-sum sequence over $C_2^{r-1} \oplus C_{2k}$ and $\mathcal{F}$ a factorization of α. There are at most $r - 1$ blocks with nonzero defects in $\mathcal{F}$ (Corollary 4.6, Corollary 5.3), and these defects do not exceed $r - 1$ (the blocks are minimal). Therefore $d(\alpha) \leq (r-1)^2$ and so $\mathsf{D}(C_2^{r-1} \oplus C_{2k}) \leq 2k + (r-1)^2$.

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