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ADDITIVE AND MULTIPLICATIVE STRUCTURES OF $C^{\star}$-SETS

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Abstract

It is known that for an IP* set A in $(\mathbb{N},+)$ and a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$, there exists a sum subsystem $\langle y_n\rangle_{n=1}^{\infty}$ of $\langle x_n\rangle_{n=1}^{\infty}$ such that $FS(\langle y_n\rangle_{n=1}^{\infty}) \cup FP(\langle y_n\rangle_{n=1}^{\infty}) \subseteq A$. Similar types of results have also been proved for central* sets where the sequences have been considered from the class of minimal sequences. In this present work, we shall prove some analogous results for C*-sets for a more general class of sequences.

1. Introduction

A famous Ramsey-theoretic result is Hindman's Theorem:

Theorem 1.1. *Given a finite coloring of $\mathbb{N} = \bigcup_{i=1}^{r} A_i$, there exists a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$ and $i \in \{1, 2, \ldots, r\}$ such that*

$$FS(\langle x_n\rangle_{n=1}^{\infty}) = \left\{ \sum_{n\in F} x_n : F \in \mathcal{P}_f(\mathbb{N}) \right\} \subseteq A_i,$$

where for any set X, $\mathcal{P}_f(X)$ is the set of all finite nonempty subsets of X.

A strongly negative answer to a combined additive and multiplicative version of Hindman's Theorem was presented in [12, Theorem 2.11]. Given a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$, let

$$PS(\langle x_n\rangle_{n=1}^{\infty}) = \{x_m + x_n : m, n \in \mathbb{N}, m \neq n\}$$

and

$$PP(\langle x_n\rangle_{n=1}^{\infty}) = \{x_m \cdot x_n : m, n \in \mathbb{N}, m \neq n\}.$$

Theorem 1.2. *There exists a finite partition $\mathcal{R}$ of $\mathbb{N}$ with no one-to-one sequence $\langle x_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that $PS(\langle x_n\rangle_{n=1}^{\infty}) \cup PP(\langle x_n\rangle_{n=1}^{\infty})$ is contained in one cell of the partition $\mathcal{R}$.*

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The original proof of Theorem 1.1 was combinatorial in nature. But later, using the algebraic structure of $\beta\mathbb{N}$, a very elegant proof of Hindman's Theorem was established by Galvin and Glazer, which they never published. A proof of the Theorem 1.1, that uses the algebraic structure of $\beta\mathbb{N}$ was first presented in [6, Theorem 10.3]. One can also see the proof in [14, Corollary 5.10].

Let us first give a brief description of the algebraic structure of βS_d for a discrete semigroup $(S,\cdot)$. We take the points of βS_d to be the ultrafilters on S, identifying the principal ultrafilters with the points of S and thus pretending that $S \subseteq \beta S_d$. Given a set $A \subseteq S$, let us define the subsets of βS_d by the following formula:

$$c\ell A = \overline{A} = \{p \in \beta S_d : A \in p\}.$$

Then the set $\{c\ell A \subseteq \beta S_d : A \subseteq \mathbb{N}\}$ forms a basis for the closed sets of βS_d as well as for the open sets. The operation " $\cdot$ " on S can be extended to the Stone-Čech compactification βS_d of S, so that $(\beta S_d, \cdot)$ becomes a compact right topological semigroup (meaning that for any $p \in \beta S_d$, the function $\rho_p : \beta S_d \to \beta S_d$, defined by $\rho_p(q) = q \cdot p$, is continuous) with S contained in its topological center (meaning that for any $x \in S$, the function $\lambda_x : \beta S_d \to \beta S_d$, defined by $\lambda_x(q) = x \cdot q$, is continuous). A nonempty subset I of a semigroup T is called a *left ideal of S* if $TI \subset I$, a *right ideal* if $IT \subset I$, and a *two-sided ideal* (or simply an *ideal*) if it is both a left and a right ideal. A *minimal left ideal* is a left ideal that does not contain any proper left ideal. Similarly, we can define a *minimal right ideal* and the *smallest ideal*.

Numakura proved in [18] (as remarked in [10, Lemma 8.4]), that any compact Hausdorff right topological semigroup T contains idempotents and therefore has a smallest two-sided ideal

$$\begin{aligned} K(T) &= \bigcup\{L : L \text{ is a minimal left ideal of } T\} \\ &= \bigcup\{R : R \text{ is a minimal right ideal of } T\}. \end{aligned}$$

Given a minimal left ideal L and a minimal right ideal R, it easily follows that $L \cap R$ is a group and thus in particular contains an idempotent. If p and q are idempotents in T, we write $p \leq q$ if and only if $p \cdot q = q \cdot p = p$. An idempotent is minimal with respect to this relation if and only if it is a member of the smallest ideal $K(T)$ of T.

Given $p, q \in \beta S$ and $A \subseteq S$, the set $A \in p\cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$, where $x^{-1}A = \{y \in S : x \cdot y \in A\}$. See [14] for an elementary introduction to the algebra of βS and for any unfamiliar details.

A set $A \subseteq \mathbb{N}$ is called an IP* set if it belongs to every idempotent in $\beta\mathbb{N}$. Given a sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}$, we let $FP(\langle x_n\rangle_{n=1}^\infty)$ be the product analogue of finite sums. Given a sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}$, we say that $\langle y_n\rangle_{n=1}^\infty$ is a *sum subsystem* of $\langle x_n\rangle_{n=1}^\infty$, provided there is a sequence $\langle H_n\rangle_{n=1}^\infty$ of nonempty finite subsets of $\mathbb{N}$, such that $\max H_n < \min H_{n+1}$ and $y_n = \sum_{t\in H_n} x_t$ for each $n \in \mathbb{N}$. The following

Theorem [5, Theorem 2.6] shows that IP* sets have substantially rich multiplicative structures.

Theorem 1.3. *Let $\langle x_n\rangle_{n=1}^\infty$ be a sequence in $\mathbb{N}$ and A be an IP* set in $(\mathbb{N},+)$. Then there exists a sum subsystem $\langle y_n\rangle_{n=1}^\infty$ of $\langle x_n\rangle_{n=1}^\infty$ such that*

$$FS(\langle y_n\rangle_{n=1}^\infty)\cup FP(\langle y_n\rangle_{n=1}^\infty)\subseteq A.$$

Let us recall the definition of central set [14, Definition 4.42].

Definition 1.4. Let S be a semigroup and $C\subseteq S$. Then C is said to be *central* if and only if there is some idempotent $p\in K(\beta S)$ such that $C\in p$.

The algebraic structure of the smallest ideal of βS plays a significant role in Ramsey Theory. It is known that any central subset of $(\mathbb{N},+)$ is guaranteed to have a substantial amount of additive combinatorial structures. But Theorem 16.27 of [14] shows that central sets in $(\mathbb{N},+)$ need not admit any multiplicative structure at all. On the other hand, in [5, Theorem 2.4] we see that sets which belong to every minimal idempotent of $\beta\mathbb{N}$, called central* sets, must have significant multiplicative structures. In fact, central* sets in any semigroup $(S,\cdot)$ are defined to be those sets which meet every central set.

Theorem 1.5. *If A is a central* set in $(\mathbb{N},+)$, then it is also central in $(\mathbb{N},\cdot)$.*

In case of central* sets, a result similar to 1.3 was proved in [7, Theorem 2.4] for a restricted class of sequences called minimal sequences. Recall that a sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}$ is said to be a minimal sequence if

$$\bigcap_{m=1}^\infty c\ell FS(\langle x_n\rangle_{n=m}^\infty)\cap K(\beta\mathbb{N})\neq\emptyset.$$

Theorem 1.6. *Let $\langle y_n\rangle_{n=1}^\infty$ be a minimal sequence and let A be a central* set in $(\mathbb{N},+)$. Then there exists a sum subsystem $\langle x_n\rangle_{n=1}^\infty$ of $\langle y_n\rangle_{n=1}^\infty$ such that*

$$FS(\langle x_n\rangle_{n=1}^\infty)\cup FP(\langle x_n\rangle_{n=1}^\infty)\subseteq A.$$

A similar result was proved in a different setup in [9].

The original Central Sets Theorem was proved by Furstenberg in [10, Theorem 8.1] (using a different but equivalent definition of central sets). However, the most general version of Central Sets Theorem is in [8]. We state it here only for the case of a commutative semigroup.

Theorem 1.7. *Let $(S,\cdot)$ be a commutative semigroup, and let $\mathcal{T}={}^{\mathbb{N}}S$ be the set of all sequences in S. Let C be a central subset of S. Then there exist functions $\alpha:\mathcal{P}_f(\mathcal{T})\to S$ and $H:\mathcal{P}_f(\mathcal{T})\to\mathcal{P}_f(\mathbb{N})$ such that*

1. *if* $F, G \in \mathcal{P}_f(\mathcal{T})$ *and* $F \subsetneq G$ *then* $\max H(F) < \min H(G)$, *and*

2. *whenever* $m \in \mathbb{N}$, $G_1, G_2, \ldots, G_m \in \mathcal{P}_f(\mathcal{T})$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m$, *and for each* $i \in \{1, 2, \ldots, m\}$, $f_i \in G_i$, *one has*

$$\prod_{i=1}^{m} \big(\alpha(G_i) \cdot \prod_{t \in H(G_i)} f_i(t)\big) \in C.$$

Recently, a lot of attention has been paid to those sets which satisfy the conclusion of the latest Central Sets Theorem.

Definition 1.8. Let $(S, \cdot)$ be a commutative semigroup, and let $\mathcal{T} = {}^{\mathbb{N}}S$ be the set of all sequences in S. A subset C of S is said to be a C-set if there exist functions $\alpha : \mathcal{P}_f(\mathcal{T}) \to S$ and $H : \mathcal{P}_f(\mathcal{T}) \to \mathcal{P}_f(\mathbb{N})$ such that

1. if $F, G \in \mathcal{P}_f(\mathcal{T})$ and $F \subsetneq G$, then $\max H(F) < \min H(G)$ and

2. whenever $m \in \mathbb{N}$, $G_1, G_2, \ldots, G_m \in \mathcal{P}_f(\mathcal{T})$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m$, and for each $i \in \{1, 2, \ldots, m\}$, $f_i \in G_i$, one has

$$\prod_{i=1}^{m} \big(\alpha(G_i) \cdot \prod_{t \in H(G_i)} f_i(t)\big) \in C.$$

We now present some notation from [8].

Definition 1.9. Let $(S, \cdot)$ be a commutative semigroup, and let $\mathcal{T} = {}^{\mathbb{N}}S$ be the set of all sequences in S.

1. A subset A of S is said to be a J-set if for every $F \in \mathcal{P}_f(\mathcal{T})$ there exist $a \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for all $f \in F$,
   $a \cdot \prod_{t \in H} f(t) \in A$.

2. $J(S) = \{p \in \beta S : (\forall A \in p)(A \text{ is a } J\text{-set})\}$.

Theorem 1.10. *Let* $(S, \cdot)$ *be a discrete commutative semigroup and* A *be a subset of* S. *Then* A *is a* J*-set if and only if* $J(S) \cap c\ell A \neq \emptyset$.

Proof. Since the collection of J-sets forms a partition regular family, the theorem follows from [14, Therem 3.11]. □

The following is a consequence of [8, Theorem 3.8]. The easy proof for the commutative case can be found in [15, Theorem 2.5].

Theorem 1.11. *Let* $(S, +)$ *be a commutative semigroup and let* $\mathcal{T} = {}^{\mathbb{N}}S$ *be the set of all sequences in* S, *and let* $A \subseteq S$. *Then* A *is a* C*-set if and only if there is an idempotent* $p \in c\ell A \cap J(S)$.

We conclude these introductory discussions with the following [8, Theorem 3.5].

Theorem 1.12. *If* $(S, \cdot)$ *be a discrete commutative semigroup then* $J(S)$ *is a closed two-sided ideal of* βS *and* $c\ell K(\beta S) \subset J(S)$*.*

2. $C^{\star}$-set

We have already discussed IP* and central* sets before; let us now introduce the notion of $C^{\star}$-set.

Definition 2.1. Let $(S, \cdot)$ be a discrete commutative semigroup. A set $A \subseteq S$ is said to be a C*-set if it is a member of all the idempotents of $J(S)$.

It is clear from the definition of C*-set that

$$\text{IP}^{\star}\text{-set} \Rightarrow \text{C}^{\star}\text{-set} \Rightarrow \text{central}^{\star}\text{-set}.$$

Remark 2.2. It is shown in [13] that there exists a set $A \subset \mathbb{N}$ which is a C-set in $(\mathbb{N}, +)$, but its upper Banach density (defined below) vanishes. Since A is a C-set in $(\mathbb{N}, +)$, there exists an idempotent $p \in J(\mathbb{N})$ such that $A \in p$. But, as the upper Banach density of A is zero, it is not a central set in $(\mathbb{N}, +)$, so it is not contained in any minimal idempotent of $\beta\mathbb{N}$. Hence $\mathbb{N} \setminus A$ is a member of all the minimal idempotents in $\beta\mathbb{N}$. Therefore $\mathbb{N} \setminus A$ is a central*-set but not a C*-set as $\mathbb{N} \setminus A \notin p$.

Definition 2.3. Let $(S, +)$ be a discrete commutative semigroup. A sequence $\langle x_n \rangle_{n=1}^{\infty}$ is said to be an *almost minimal sequence* if

$$\bigcap_{m=1}^{\infty} c\ell FS(\langle x_n \rangle_{n=m}^{\infty}) \cap J(S)) \neq \emptyset.$$

To provide an example of an *almost minimal sequence* which is not minimal, let us recall the following notion of "Banach density". The author is grateful to Prof. Neil Hindman for his help in constructing this example.

Definition 2.4. Let $A \subset \mathbb{N}$. Then

1. $d^*(A) = \sup\{\alpha \in \mathbb{R} : (\forall\, k \in \mathbb{N})(\exists n > k)(\exists a \in \mathbb{N})(|A \cap \{a+1, a+2, \ldots, a+n\}| \geq \alpha \cdot n)\}$.
2. $\triangle^* = \{p \in \beta\mathbb{N} : (\forall A \in p)(d^*(A) > 0)\}$.

$d^*(A)$ is said to be the upper Banach density of A.

It follows from [14, Theorem 20.5 and 20.6] that $\triangle^*$ is a closed two-sided ideal of $(\beta\mathbb{N}, +)$, so that $c\ell K(\beta\mathbb{N}) \subset \triangle^*$.

Let us now recall the following Theorem from [1], which we shall require to construct example of an almost minimal sequence which is not minimal.

Theorem 2.5. *Let* $\langle x_n\rangle_{n=1}^{\infty}$ *be a sequence in* $\mathbb{N}$*, such that for all* $n \in \mathbb{N}$ *we have* $x_{n+1} > \sum_{t=1}^{n} x_t$*, and* $T = \bigcap_{m=1}^{\infty} c\ell FS(\langle x_n\rangle_{n=m}^{\infty})$*. Then the following conditions are equivalent:*

1. $T \cap K(\beta\mathbb{N}) \neq \emptyset$;
2. $T \cap c\ell K(\beta\mathbb{N}) \neq \emptyset$;
3. *the set* $\{x_{n+1} - \sum_{t=1}^{n} x_t : n \in \mathbb{N}\}$ *is bounded.*

Proof. See [1, Theorem 2.1]. □

We already know that a subset A of $\mathbb{N}$ is central if it belongs to some idempotent of $K(\beta\mathbb{N})$. Further, the members of the ultrafilters of $K(\beta\mathbb{N})$ are piecewise syndetic. Replacing piecewise syndeticity with positive upper Banach density leads to the class of *essential idempotents*: viz. $q \in \beta\mathbb{N}$ is an *essential idempotent* if it is an idempotent ultrafilter, all of whose elements have positive upper Banach density-that is, $q \in \triangle^*$. By [4, Theorem 2.8], a set $S \subseteq \mathbb{N}$ is a D-set if it is contained in some essential idempotent. The authors proved in [3, Theorem 11] that D-sets also satisfy the conclusion of the original Central Sets Theorem and are in particular J-sets.

Now let $d \in \mathbb{N}$, and let us define a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$ by the following formula: $x_{n+1} = \sum_{t=1}^{n} x_t + \lfloor \frac{n+(d-1)}{d} \rfloor$. Then the set $\{x_{n+1} - \sum_{t=1}^{n} x_t : n \in \mathbb{N}\}$ being unbounded, we have $\bigcap_{m=1}^{\infty} c\ell FS(\langle x_n\rangle_{n=m}^{\infty}) \cap K(\beta\mathbb{N}) = \emptyset$. Therefore, the sequence $\langle x_n\rangle_{n=1}^{\infty}$ is not minimal. But by [1, Lemma 2.20], we have $\bigcap_{m=1}^{\infty} c\ell FS(\langle x_n\rangle_{n=m}^{\infty}) \cap \triangle^* \neq \emptyset$. Since $\bigcap_{m=1}^{\infty} c\ell FS(\langle x_n\rangle_{n=m}^{\infty}) \cap \triangle^*$ is a compact subsemigroup of $\beta\mathbb{N}$, we can choose an idempotent p in $\bigcap_{m=1}^{\infty} c\ell FS(\langle x_n\rangle_{n=m}^{\infty}) \cap \triangle^*$. In particular, $FS(\langle x_n\rangle_{n=1}^{\infty}) \in p$. Therefore, by the above discussion, $FS(\langle x_n\rangle_{n=1}^{\infty})$ satisfies the conclusion of the original Central Sets Theorem and is in particular a J-set. Hence by the foregoing Theorem 2.7, we have $\bigcap_{m=1}^{\infty} c\ell FS(\langle x_n\rangle_{n=m}^{\infty}) \cap J(\mathbb{N})) \neq \emptyset$. This implies that this sequence $\langle x_n\rangle_{n=1}^{\infty}$ is *almost minimal*.

Question 2.6. Does there exist a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$ with the property that $FS(\langle x_n\rangle_{n=1}^{\infty})$ is a C-set but its Banach density is zero?

The author is thankful to the anonymous referee for her/his help in simplifying the following proof.

Theorem 2.7. *In the semigroup* $(\mathbb{N}, +)$*, the following conditions are equivalent:*

(a) $\langle x_n\rangle_{n=1}^{\infty}$ *is an almost minimal sequence;*

(b) $FS(\langle x_n\rangle_{n=1}^{\infty})$ *is a* J*-set;*

(c) *there is an idempotent in* $\bigcap_{m=1}^{\infty} c\ell FS(\langle x_n\rangle_{n=m}^{\infty}) \cap J(\mathbb{N})) \neq \emptyset$.

Proof. $(a) \Rightarrow (b)$: The proof follows from the definition of an almost minimal sequences.

$(b) \Rightarrow (c)$: Let $FS(\langle x_n\rangle_{n=1}^{\infty})$ be a J-set. Then by Theorem 1.10, we have $J(\mathbb{N}) \cap c\ell FS(\langle x_n\rangle_{n=1}^{\infty}) \neq \emptyset$. We choose $p \in J(\mathbb{N}) \cap c\ell FS(\langle x_n\rangle_{n=1}^{\infty})$. By [14, Lemma 5.11], $\bigcap_{m=1}^{\infty} c\ell FS(\langle x_n\rangle_{n=m}^{\infty})$ is a subsemigroup of $\beta\mathbb{N}$. As a consequence of [8, Theorem 3.5], it follows that $J(\mathbb{N})$ is a subsemigroup of $\beta\mathbb{N}$. Also, $J(\mathbb{N})$ being closed, it is a compact subsemigroup of $\beta\mathbb{N}$. Therefore, it suffices to show that for each m, $\bigcap_{m=1}^{\infty} c\ell FS(\langle x_n\rangle_{n=m}^{\infty}) \cap J(\mathbb{N}) \neq \emptyset$, because then it must contain an idempotent. For this, it in turn suffices to show that, for any $m \in \mathbb{N}$ we have $c\ell FS(\langle x_n\rangle_{n=m}^{\infty}) \cap J(\mathbb{N}) \neq \emptyset$. So let $m \in \mathbb{N}$ be given. Then $FS(\langle x_n\rangle_{n=1}^{\infty}) = FS(\langle x_n\rangle_{n=m}^{\infty}) \cup FS(\langle x_n\rangle_{n=1}^{m-1}) \cup \bigcup\{t + FS(\langle x_n\rangle_{n=m}^{\infty}) : t \in FS(\langle x_n\rangle_{n=1}^{m-1})\}$. Hence we must have one of the following:

1. $FS(\langle x_n\rangle_{n=1}^{m-1}) \in p$;
2. $FS(\langle x_n\rangle_{n=m}^{\infty}) \in p$;
3. $t + FS(\langle x_n\rangle_{n=m}^{\infty}) \in p$ for some $t \in FS(\langle x_n\rangle_{n=1}^{m-1})$.

Clearly (1) does not hold, because in that case p becomes a member of $\mathbb{N}$, that is a principal ultrafilter, while $p \in \beta\mathbb{N} \setminus \mathbb{N}$. If (2) holds, then we are done. So assume that (3) holds. Then for some $t \in FS(\langle x_n\rangle_{n=1}^{m-1})$, we have that $t + FS(\langle x_n\rangle_{n=m}^{\infty}) \in p$. We choose some $q \in c\ell FS(\langle x_n\rangle_{n=m}^{\infty})$ so that $t + q = p$. For every $F \in q$, we have $t \in \{n \in \mathbb{N} : -n + (t + F) \in q\}$ so that $t + F \in p$. Since J-sets in $(\mathbb{N}, +)$ are translation invariant, F becomes a J-set. Thus $q \in J(\mathbb{N}) \cap c\ell FS(\langle x_n\rangle_{n=m}^{\infty})$.

$(c) \Rightarrow (a)$: This case is obvious. □

Lemma 2.8. *If A is a C-set in $(\mathbb{N}, +)$ then for any $n \in \mathbb{N}$, nA and $n^{-1}A$ are also C-sets, where $n^{-1}A = \{m \in \mathbb{N} : n \cdot m \in A\}$.*

Proof. [17, Lemma 8.1]. □

Lemma 2.9. *If A is a $C^\star$-set in $(\mathbb{N}, +)$ then $n^{-1}A$ is also a $C^\star$-set for any $n \in \mathbb{N}$.*

Proof. Let A be a $C^\star$-set and $t \in \mathbb{N}$. To prove that $t^{-1}A$ is a $C^\star$-set, it is sufficient to show that for any C-set C, we have $C \cap t^{-1}A \neq \emptyset$. Since C is a C-set, tC is also a C-set, so that $A \cap tC \neq \emptyset$. Choose $n \in tC \cap A$ and $k \in C$ such that $n = tk$. Therefore $k = n/t \in t^{-1}A$. Hence $C \cap t^{-1}A \neq \emptyset$. □

Theorem 2.10. *Let $\langle x_n\rangle_{n=1}^{\infty}$ be an almost minimal sequence and A be a $C^\star$-set in $(\mathbb{N}, +)$. Then there exists a sum subsystem $\langle y_n\rangle_{n=1}^{\infty}$ of $\langle x_n\rangle_{n=1}^{\infty}$ such that*

$$FS(\langle y_n\rangle_{n=1}^{\infty}) \cup FP(\langle y_n\rangle_{n=1}^{\infty}) \subseteq A.$$

Proof. Since $\langle x_n \rangle_{n=1}^{\infty}$ is an almost minimal sequence, by Theorem 2.7, we can find some idempotent $p \in J(\mathbb{N})$ such that $FS(\langle x_n \rangle_{n=m}^{\infty}) \in p$, for each $m \in \mathbb{N}$. Again, since A is a $C^{\star}$-set in $(\mathbb{N}, +)$, from Lemma 2.9, it follows that, $s^{-1}A \in p$, for every $s \in \mathbb{N}$. Let $A^{\star} = \{s \in A : -s + A \in p\}$. Then by [14, Lemma 4.14], we have $A^{\star} \in p$. Then we can choose $y_1 \in A^{\star} \cap FS(\langle x_n \rangle_{n=1}^{\infty})$. Inductively, let $m \in \mathbb{N}$ and $\langle y_i \rangle_{i=1}^{m}$, $\langle H_i \rangle_{i=1}^{m}$ in $\mathcal{P}_f(\mathbb{N})$ be chosen with the following properties:

1. for all $i \in \{1, 2, \ldots, m-1\}$, $\max H_i < \min H_{i+1}$;
2. if $y_i = \sum_{t \in H_i} x_t$ then $\sum_{t \in H_m} x_t \in A^{\star}$ and $FP(\langle y_i \rangle_{i=1}^{m}) \subseteq A^{\star}$.

We observe that $\{\sum_{t \in H} x_t : H \in \mathcal{P}_f(\mathbb{N}), \min H > \max H_m\} \in p$. Let us set $B = \{\sum_{t \in H} x_t : H \in \mathcal{P}_f(\mathbb{N}),\ \min H > \max H_m\}$, $E_1 = FS(\langle y_i \rangle_{i=1}^{m})$ and $E_2 = FP(\langle y_i \rangle_{i=1}^{m})$. Now consider

$$D = B \cap A^{\star} \cap \bigcap_{s \in E_1} (-s + A^{\star}) \cap \bigcap_{s \in E_2} (s^{-1}A^{\star}).$$

Then $D \in p$. Choose $y_{m+1} \in D$ and $H_{m+1} \in \mathcal{P}_f(\mathbb{N})$ such that $\min H_{m+1} > \max H_m$. Putting $y_{m+1} = \sum_{t \in H_{m+1}} x_t$, it shows that the induction can be continued and this eventually proves the theorem. $\square$

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