THE GENERALIZED TRIBONACCI NUMBERS WITH NEGATIVE SUBSCRIPTS

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Abstract

Generalized tribonacci numbers R_n are defined through the recurrence

$$R_{n+1} = aR_n + bR_{n-1} + cR_{n-2}.$$

A generating matrix of three tribonacci sequences with negative subscripts is defined and used to establish identities connecting these sequences which is analogous to the matrix of Shannon and Horadam. We derive an explicit formula for the generalized tribonacci numbers with negative subscripts.

1. Introduction

Let $\mathcal{L}(a, b, c)$ be the set of all third-order recurrent sequences $\{R_n\}_{n\in\mathbb{Z}}$ satisfying the relation

$$R_n = aR_{n-1} + bR_{n-2} + cR_{n-3},$$

where a, b, c are positive integers.

Three generalized tribonacci sequences $\{J_n\}, \{K_n\}$ and $\{L_n\}$ in $\mathcal{L}(a, b, c)$ are uniquely determined by taking special values $n = 0, 1, 2$, namely,

(*i*) $J_0 = 0, J_1 = 1$ and $J_2 = a$,

(*ii*) $K_0 = 1, K_1 = 0$ and $K_2 = b$,

(*iii*) $L_0 = 0, L_1 = 0$ and $L_2 = c$.

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These sequences have been studied by many researchers (for more details see [5]-[7]). If $a = b = c = 1$, then $\{J_n\}$ is a sequence of the classical tribonacci numbers, say $\{T_n\}$. It can be written R_n as a linear combination of J_n, K_n and L_n, namely,

$$R_{n+1} = R_2 J_n + R_1 K_n + R_0 L_n,$$

and the following relations are easily proved, see [4], [5] or [7],

$$J_{n+1} = aJ_n + K_n, \quad K_{n+1} = bJ_n + cJ_{n-1}, \quad \text{and} \quad L_{n+1} = cJ_n. \tag{1}$$

In 1972, Shannon and Horadam [7] constructed the 3×3 matrix and computed the n^{th} power of this matrix

$$\begin{bmatrix} a & b & c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n = \begin{bmatrix} J_{n+2} & K_{n+2} & L_{n+2} \\ J_{n+1} & K_{n+1} & L_{n+1} \\ J_n & K_n & L_n \end{bmatrix}$$

and they showed that

$$(a+b+c-1)\sum_{i=1}^{n} J_i = J_{n+3} + (1-a)J_{n+2} + (1-a-b)J_{n+1} - 1,$$

$$J_{n+1} = \sum_{i=0}^{[n/2]} \sum_{j=0}^{[n/3]} \binom{i+j}{j} \binom{n-i-2j}{i+j} a^{n-2i-3j} b^i c^j;$$

see also [1], [2], [5], [6].

The sequences $\{J_n\}$, $\{K_n\}$ and $\{L_n\}$ can be defined for negative values of n by using the definition of any recurrent relation and initial conditions. The first few terms of them are shown in the following table.

n	J_n	K_n	L_n
-1	0	0	1
-2	$1/c$	$-a/c$	$-b/c$
-3	$-b/c^2$	$(ab+c)/c^2$	$(b^2-ac)/c^2$
-4	$(b^2-ac)/c^3$	$(a^2c-ab^2-bc)/c^3$	$(c^2+2abc-b^3)/c^3$

In this article we construct certain matrices for J_{-n}, K_{-n}, L_{-n} and partial sums of J_{-n} to derive interesting identities involving these numbers. We also derive an expansion of J_{-n} in a partial sum of binomial coefficients.

2. Matrix Representations

For $n \in \mathbb{N}$, define the 3×3 matrices A and C_n as follows

$$A = \begin{bmatrix} -b/c & -a/c & 1/c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad C_n = \begin{bmatrix} L_{-n-1} & K_{-n-1} & J_{-n-1} \\ L_{-n} & K_{-n} & J_{-n} \\ L_{-n+1} & K_{-n+1} & J_{-n+1} \end{bmatrix}.$$

Theorem 1. *For all $n \in \mathbb{N}$, we have $A^n = C_n$.*

Proof. (Induction on n) Using the above table (section 1), one can see that $A^1 = C_1$. Assume $A^n = C_n$ holds for $n > 1$. By our assumption and a matrix multiplication, we get $A^{n+1} = A^n A = C_n A$, which, by using all equations in (1), satisfies $A^{n+1} = C_n A = C_{n+1}$. Thus, complete the proof. □

Since $A^{n+m} = A^n A^m$, equating the $(2,1), (2,2)$ and $(2,3)$-entries on both sides of this matrix equation, we get the following corollary.

Corollary 1. *For $m, n \in \mathbb{N}$, we get the relation*

$$R_{-m-n} = L_{-m}R_{-n-1} + K_{-m}R_{-n} + J_{-m}R_{-n+1},$$

where R_n is J_n, K_n or L_n.

Taking $n = 2$ and $m = 2$ in Corollary 1, we get

$$R_{-m-2} = R_{-3}L_{-m} + R_{-2}K_{-m} + R_{-1}J_{-m},$$

$$R_{-n-2} = \frac{1}{c}(-bR_{-n-1} - aR_{-n} + R_{-n+1}),$$

respectively.

Next, for $n \in \mathbb{N}$, we define the 4×4 matrices B and D_n as follows

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/c & & & \\ 0 & & A & \\ 0 & & & \end{bmatrix} \quad \text{and} \quad D_n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ S_{-n-1} & & & \\ S_{-n} & & C_n & \\ S_{-n+1} & & & \end{bmatrix},$$

where $S_{-n} = \sum_{i=1}^{n} J_{-i}$ and $S_0 = 0$.

Theorem 2. *For all $n \in \mathbb{N}$, we have $B^n = D_n$.*

Proof. Since $L_{-n+1} = cJ_{-n}$, we can write $S_{-n-1} = S_{-n} + \frac{1}{c}L_{-n}$. Combining the above identity and the result of Theorem 1, we write $D_n = D_{n-1}B$. By using induction on n, the result still holds. □

Corollary 2. *For all positive integers m, n, we have*

$$S_{-m-n-1} = S_{-m-1} + L_{-m-1}S_{-n-1} + K_{-m-1}S_{-n} + J_{-m-1}S_{-n+1}. \tag{2}$$

Proof. Since $D^{m+n} = D^m D^n$, we have the equation (2) by equating the $(2,1)$-entry on both sides of this matrix equation. □

Now we derive an explicit formula for partial sums of J_{-n}. We have the following theorem.

Theorem 3. *For $n \geq 1$, we have*

$$\sum_{i=1}^{n} J_{-i} = \frac{1}{a+b+c-1}(1 - cJ_{-n-1} + (a-1)J_{-n} - J_{-n+1}). \tag{3}$$

Proof. Put $m = 1$ in the equation (2) of Corollary 2, we get

$$cS_{-n-2} = 1 - aS_{-n} - bS_{-n-1} + S_{-n+1}.$$

This equation equivalent to

$$(a+b+c-1)S_{-n+1} = 1 - cJ_{-n-2} + J_{-n-1}(-b-c) + J_{-n}(-a-b-c).$$

By the definition of J_{-n}, we can rewrite the last equation to obtain (3). $\square$

Taking $a = b = c = 1$ in identity (3), we obtain

$$\sum_{i=1}^{n} T_{-i} = \frac{1}{2}(1 - T_{-n-1} - T_{-n+1}).$$

3. Expansions

Definition 1. *Let n, i be non-negative integers with $n \geq i$. Denote*

$$\mathcal{B}(n,i) = \sum_{j=0}^{\lfloor (n+i)/3 \rfloor} \binom{i}{j}\binom{n-j}{i} a^{i-j} b^{n-i-j} c^{j}.$$

It is easy to see that we can write $\mathcal{B}(n,i)$ in the form of the recursive recurrence as

$$\mathcal{B}(n,i) = b\mathcal{B}(n-1,i) + a\mathcal{B}(n-1,i-1) + c\mathcal{B}(n-2,i-1). \tag{4}$$

Theorem 4. *For non-negative integer n, we have*

$$J_{-n-2} = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{(-1)^{n-i}}{c^{n-i+1}} \mathcal{B}(n-i,i). \tag{5}$$

Proof. We see that

$$J_{-2} = c^{-1}, \quad J_{-3} = -bc^{-2} \quad \text{and} \quad J_{-4} = (b^2 - ac)c^{-3}.$$

By induction on n, assume that identity (5) holds for all $n = 0, 1, 2, ..., k-1$. By the definition of J_k, the identity (4) and the inductive hypothesis, we get

$$\begin{aligned}
cJ_{-k-2} &= J_{-k+1} - aJ_{-k} - bJ_{-k-1} \\
&= \sum_{i=0}^{\lfloor (k-3)/2 \rfloor} \frac{(-1)^{k-3-i}}{c^{k-i-2}} \mathcal{B}(k-i-3, i) - a \sum_{i=0}^{\lfloor (k-2)/2 \rfloor} \frac{(-1)^{k-2-i}}{c^{k-i-1}} \mathcal{B}(k-i-2, i) \\
&\quad - b \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} \frac{(-1)^{k-1-i}}{c^{k-i}} \mathcal{B}(k-i-1, i) \\
&= \sum_{i=1}^{\lfloor (k-1)/2 \rfloor} \frac{(-1)^{k-2-i}}{c^{k-i-1}} \mathcal{B}(k-i-2, i-1) - a \sum_{i=1}^{\lfloor k/2 \rfloor} \frac{(-1)^{k-1-i}}{c^{k-i}} \mathcal{B}(k-i-1, i-1) \\
&\quad - \frac{(-1)^{k-1}b}{c^k} \mathcal{B}(k-1, 0) - b \sum_{i=1}^{\lfloor (k-1)/2 \rfloor} \frac{(-1)^{k-1-i}}{c^{k-i}} \mathcal{B}(k-i-1, i) \\
&= \begin{cases} \frac{(-1)^k}{c^k} \mathcal{B}(k, 0) - \displaystyle\sum_{i=1}^{\lfloor (k-1)/2 \rfloor} \frac{(-1)^{k-1-i}}{c^{k-i}} \mathcal{B}(k-i, i) & ; \quad k \text{ is odd} \\ \frac{(-1)^k}{c^k} \mathcal{B}(k, 0) - \displaystyle\sum_{i=1}^{\lfloor (k-1)/2 \rfloor} \frac{(-1)^{k-1-i}}{c^{k-i}} \mathcal{B}(k-i, i) + \left(\frac{-a}{c}\right)^{k/2} & ; \; k \text{ is even} \end{cases} \\
&= \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{(-1)^{k-i}}{c^{k-i}} \mathcal{B}(k-i, i),
\end{aligned}$$

so

$$J_{-k-2} = \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{(-1)^{k-i}}{c^{k-i+1}} \mathcal{B}(k-i, i).$$

Showing that (5) holds for $n = k$, thereby proving the theorem. $\square$

We can rewrite (5) in terms of binomial coefficients by using Definition 1.

Corollary 3. *For non-negative integer n, we have*

$$J_{-n-2} = \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^{\lfloor n/3 \rfloor} (-1)^{n-i} \binom{i}{j} \binom{n-i-j}{i} a^{i-j} b^{n-2i-j} c^{i+j-n-1}.$$

Taking $a = b = c = 1$, we get an explicit formula for the tribonacci numbers, so

$$T_{-n-2} = \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^{\lfloor n/3 \rfloor} (-1)^{n-i} \binom{i}{j} \binom{n-i-j}{i}. \tag{6}$$

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