THE $(r_1, \ldots, r_p)$-BELL POLYNOMIALS

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Abstract

In a previous paper, Mihoubi et al. introduced the $(r_1, \ldots, r_p)$-Stirling numbers and the $(r_1, \ldots, r_p)$-Bell polynomials and gave some of their combinatorial and algebraic properties. These numbers and polynomials generalize, respectively, the r-Stirling numbers of the second kind introduced by Broder and the r-Bell polynomials introduced by Mező. In this paper, we prove that the $(r_1, \ldots, r_p)$-Stirling numbers of the second kind are log-concave. We also give generating functions and generalized recurrences related to the $(r_1, \ldots, r_p)$-Bell polynomials.

1. Introduction

In 1984, Broder [2] introduced and studied the r-Stirling number of the second kind $\left\{ {n \atop k} \right\}_r$, which counts the number of partitions of the set $[n] = \{1, 2, \ldots, n\}$ into k non-empty subsets such that the r first elements are in distinct subsets. In 2011, Mező [8] introduced and studied the r-Bell polynomials. In 2012, Mihoubi et al. [12] introduced and studied the $(r_1, \ldots, r_p)$-Stirling number of the second kind $\left\{ {n \atop k} \right\}_{r_1,\ldots,r_p}$, which counts the number of partitions of the set $[n]$ into k non-empty subsets such that the elements of each of the p sets $R_1 := \{1, \ldots, r_1\}$, $R_2 := \{r_1 + 1, \ldots, r_1 + r_2\}$, $\ldots$, $R_p := \{r_1 + \cdots r_{p-1} + 1, \ldots, r_1 + \cdots + r_p\}$ are in distinct subsets.

This work is motivated by the study of the r-Bell polynomials [8] and the $(r_1, \ldots, r_p)$-Stirling numbers of the second kind [12], in which we may establish

- the log-concavity of the $(r_1, \ldots, r_p)$-Stirling numbers of the second kind,

- generalized recurrences for the $(r_1,\ldots,r_p)$-Bell polynomials, and
- the ordinary generating functions of these numbers and polynomials.

To begin, by the symmetry of the $(r_1,\ldots,r_p)$-Stirling numbers with respect to $r_1,\ldots,r_p$, let us suppose that $r_1\leq r_2\leq\cdots\leq r_p$, and throughout this paper we use the following notation and definitions

$$\begin{aligned}\mathbf{r}_p &:= (r_1,\ldots,r_p)\,, \quad |\mathbf{r}_p| := r_1+\cdots+r_p,\\ P_t(z;\mathbf{r}_p) &:= (z+r_p)^t\,(z+r_p)^{\underline{r_1}}\cdots(z+r_p)^{\underline{r_{p-1}}}\,, \quad t\in\mathbb{R},\\ B_n(z;\mathbf{r}_p) &:= \sum_{k=0}^{n+|\mathbf{r}_{p-1}|}\left\{{n+|\mathbf{r}_p| \atop k+r_p}\right\}_{\mathbf{r}_p} z^k, \quad n\geq 0\end{aligned}$$

and $\mathbf{e}_i$ denotes the i-th vector of the canonical basis of $\mathbb{R}^p$.
In [12], the following were proved:

$$B_n(z;\mathbf{r}_p) = \exp(-z)\sum_{k\geq 0}P_n(k;\mathbf{r}_p)\frac{z^k}{k!}, \tag{1}$$

$$P_n(z;\mathbf{r}_p) = \sum_{k=0}^{n+|\mathbf{r}_{p-1}|}\left\{{n+|\mathbf{r}_p| \atop k+r_p}\right\}_{\mathbf{r}_p} z^{\underline{k}}. \tag{2}$$

For later use we define the following numbers

$$a_k(\mathbf{r}_{p-1}) = (-1)^{|\mathbf{r}_{p-1}|-k}\sum_{|\mathbf{j}_{p-1}|=k}\begin{bmatrix} r_1\\ j_1\end{bmatrix}\cdots\begin{bmatrix} r_{p-1}\\ j_{p-1}\end{bmatrix}, \quad |\mathbf{j}_{p-1}| = j_1+\cdots+j_{p-1},$$

where $\begin{bmatrix} n\\ k\end{bmatrix}$ are the absolute Stirling numbers of the first kind.
Upon using the known identity

$$(u)^{\underline{r}} = \sum_{j=0}^{r}(-1)^{r-j}\begin{bmatrix} r\\ j\end{bmatrix}u^j$$

we may state that we have

$$\sum_{k=0}^{|\mathbf{r}_{p-1}|}a_k(\mathbf{r}_{p-1})\,u^k = (u)^{\underline{r_1}}\cdots(u)^{\underline{r_{p-1}}}. \tag{3}$$

In our contribution, we give more properties for the $\mathbf{r}_p$-Stirling numbers and the $\mathbf{r}_p$-Bell polynomials. The paper is organized as follows. In the next section we prove that the sequence $\left(\left\{{n+|\mathbf{r}_p| \atop k+r_p}\right\}_{\mathbf{r}_p};\ 0\leq k\leq n+|\mathbf{r}_{p-1}|\right)$ is strongly log-concave and we give an approximation of $\left\{ {n+|\mathbf{r}_p| \atop k+r_p} \right\}_{\mathbf{r}_p}$ when $n \to \infty$ for a fixed k. In the third section we write $B_n(z;\mathbf{r}_p)$ in the basis $\{B_{n+k}(z;r_p) : 0 \le k \le |\mathbf{r}_{p-1}|\}$ and $B_{n+m}(z;\mathbf{r}_p)$ in the family of bases $\left\{ z^j B_m(z;\mathbf{r}_p + j\mathbf{e}_p) : 0 \le j \le n \right\}$. As consequences, we also give some identities for the $\mathbf{r}_p$-Stirling numbers. In the fourth section we give the ordinary generating functions of the $\mathbf{r}_p$-Stirling numbers of the second kind and the $\mathbf{r}_p$-Bell polynomials.

2. Log-Concavity of the $\mathbf{r}_p$-Stirling Numbers

In this section we discuss the real roots of the polynomial $B_n(z;\mathbf{r}_p)$, the log-concavity of the sequence $\left(\left\{ {n+|\mathbf{r}_p| \atop k+r_p} \right\}_{\mathbf{r}_p},\ 0 \le k \le n + |\mathbf{r}_{p-1}| \right)$, the greatest maximizing index of $\left\{ {n \atop k} \right\}_{\mathbf{r}_p}$ and we give an approximation of $\left\{ {n+|\mathbf{r}_p| \atop m+r_p} \right\}_{\mathbf{r}_p}$ when n tends to infinity. The case $p = 1$ was studied by Mező [9] and another study is done by Zhao [15] for a large class of the Stirling numbers.

In what follows, for illustration or if the order of $r_1, \ldots, r_p$ is unknown, we write the polynomial $B_n(z;\mathbf{r}_p)$ as $B_n(z;r_1,\ldots,r_p)$ for which $r_1,\ldots,r_p$ are taken in any order.

Theorem 1. *The roots of the polynomial $B_n(z;\mathbf{r}_p)$ are real and non-positive.*

To prove this theorem, we use the following lemma.

Lemma 2. *Let j, p be nonnegative integers and set*

$$B_n^{(j)}(z;\mathbf{r}_p) := \exp(-z) \frac{d^j}{dz^j} \left(z^{r_p} \exp(z) B_n(z;\mathbf{r}_p) \right),$$

where $r_0 := 0$ and $B_n(z;\mathbf{r}_0) := B_n(z) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} z^k$. Then, we have

$$B_n^{(j)}(z;\mathbf{r}_p) = z^{r_p - j} B_n(z;r_1,\ldots,r_p,j) \quad \text{if } j < r_p,$$
$$B_n^{(j)}(z;\mathbf{r}_p) = B_n(z;r_1,\ldots,r_p,j) \quad \text{if } j \ge r_p,$$

with $\deg B_n^{(j)} = n + |\mathbf{r}_p|$. In particular, we have $B_n^{(r_{p+1})}(z;\mathbf{r}_p) = B_n(z;\mathbf{r}_{p+1})$.

Proof. The definition of $B_n^{(j)}(z;\mathbf{r}_p)$ and the identity (1) show that we have

$$\begin{aligned}
&\exp(z)\, B_n^{(j)}(z;\mathbf{r}_p)\\
&= \frac{d^j}{dz^j}\left(\sum_{k\geq 0} P_n(k;\mathbf{r}_p)\,\frac{z^{k+r_p}}{k!}\right)\\
&= \sum_{k\geq \max(0,j-r_p)} (k+r_p)^n\,(k+r_p)^{\underline{r_1}}\cdots(k+r_p)^{\underline{r_{p-1}}}\,(k+r_p)^{\underline{j}}\,\frac{z^{k+r_p-j}}{k!}.
\end{aligned}$$

Then, for $0\leq j<r_p$ we obtain

$$\begin{aligned}
\exp(z)\, B_n^{(j)}(z;\mathbf{r}_p) &= \sum_{k\geq 0}(k+r_p)^n\,(k+r_p)^{\underline{r_1}}\cdots(k+r_p)^{\underline{r_{p-1}}}\,(k+r_p)^{\underline{j}}\,\frac{z^{k+r_p-j}}{k!}\\
&= z^{r_p-j}\exp(z)\, B_n(z;r_1,\ldots,r_p,j)
\end{aligned}$$

and for $j\geq r_p$ we obtain

$$\begin{aligned}
\exp(z)\, B_n^{(j)}(z;\mathbf{r}_p) &= \sum_{k\geq j-r_p}(k+r_p)^n\,(k+r_p)^{\underline{r_1}}\cdots(k+r_p)^{\underline{r_{p-1}}}\,(k+r_p)^{\underline{j}}\,\frac{z^{k+r_p-j}}{k!}\\
&= \sum_{k\geq 0}(k+j)^n\,(k+j)^{\underline{r_1}}\cdots(k+j)^{\underline{r_{p-1}}}\,(k+j)^{\underline{r_p}}\,\frac{z^{k}}{k!}\\
&= \exp(z)\, B_n(z;r_1,\ldots,r_p,j).
\end{aligned}$$

It is obvious that we have $\deg B_n^{(j)} = n+|\mathbf{r}_p|$ and for $j=r_{p+1}\geq r_p$ we obtain $B_n^{(r_{p+1})}(z;\mathbf{r}_p)=B_n(z;r_1,\ldots,r_p,r_{p+1})=B_n(z;\mathbf{r}_{p+1})$. □

Proof of Theorem 1. We will show by induction on p that the roots of the polynomials $B_n(z;\mathbf{r}_p)$ are real and non-positive. Indeed, for $p=0$ the classical Bell polynomial $B_n(z;\mathbf{r}_0)=B_n(z)$ has only real non-positive roots and for $p=1$ the polynomial $B_n(z;\mathbf{r}_1)$ is the r_1-Bell polynomial introduced in [8] and has only real non-positive roots. Assume, for $1\leq r_1\leq r_2\leq\cdots\leq r_p$, that the roots of the polynomial $B_n(z;\mathbf{r}_p)$ are real and negative, denoted by $z_1,\ldots,z_{n+|\mathbf{r}_{p-1}|}$ with $0>z_1\geq\cdots\geq z_{n+|\mathbf{r}_{p-1}|}$. We will prove that the polynomial $B_n^{(j)}(z;\mathbf{r}_p)$ has only real non-positive roots and we conclude that the polynomial $B_n(z;\mathbf{r}_{p+1})=B_n^{(r_{p+1})}(z;\mathbf{r}_p)$ (see Lemma 2) has only real non-positive roots.

Firstly, we examine the polynomials $B_n^{(j)}(z;\mathbf{r}_p)$ for $j<r_p$. Indeed, the above statements show that the function

$$f_n(z;\mathbf{r}_p) := \exp(z)\, B_n^{(0)}(z;\mathbf{r}_p) = z^{r_p}\exp(z)\, B_n(z;\mathbf{r}_p)$$

vanishes at $z_0, z_1, \ldots, z_{n+|\mathbf{r}_{p-1}|}$ with $z_0 = 0 > z_1 \geq \cdots \geq z_{n+|\mathbf{r}_{p-1}|}$ and $z_0 = 0$ is of multiplicity r_p. Lemma 2 gives

$$\frac{d}{dz}\left(f_n\left(z;\mathbf{r}_p\right)\right) = \exp\left(z\right) B_n^{(1)}\left(z;\mathbf{r}_p\right) = z^{r_p-1}\exp\left(z\right) B_n\left(z;r_1,\ldots,r_{p-1},r_p,1\right)$$

and by applying Rolle's theorem to the function $f_n\left(z;\mathbf{r}_p\right)$ we conclude that its derivative $\frac{d}{dz}\left(f_n\left(z;\mathbf{r}_p\right)\right)$ vanishes at some points $x_1,\ldots,x_{n+|\mathbf{r}_{p-1}|}$ with $0 > x_1 \geq z_1 \geq x_2 \geq \cdots \geq x_{n+|\mathbf{r}_{p-1}|} \geq z_{n+|\mathbf{r}_{p-1}|}$. Consequently, the polynomial $B_n^{(1)}\left(z;\mathbf{r}_p\right)$ vanishes at $x_1,\ldots,x_{n+|\mathbf{r}_{p-1}|}$ and at $x_0 = 0$ (with multiplicity $r_p - 1$). The number of these roots is $\left(n + |\mathbf{r}_{p-1}|\right) + \left(r_p - 1\right) = n + |\mathbf{r}_p| - 1$. Because $B_n^{(1)}\left(z;\mathbf{r}_p\right)$ is of degree $n+|\mathbf{r}_p|$ (see Lemma 2), it must have exactly $n+|\mathbf{r}_p|$ finite roots; the missing one, denoted by $x_{n+|\mathbf{r}_{p-1}|+1}$, cannot be complex. By the fact that the coefficients of z^k in $B_n\left(z;r_1,\ldots,r_{p-1},r_p,1\right)$ are positive, the root $x_{n+|\mathbf{r}_{p-1}|+1}$ must be negative too. So, the polynomial $B_n^{(1)}\left(z;\mathbf{r}_p\right)$ has $n + |\mathbf{r}_{p-1}| + 1$ real negative roots and $z = 0$ is a root with multiplicity $r_p - 1$. Similarly, we apply Rolle's theorem to the function $\frac{d}{dz}\left(f_n\left(z;\mathbf{r}_p\right)\right)$ to conclude that the polynomial $B_n^{(2)}\left(z;\mathbf{r}_p\right)$ has $n+|\mathbf{r}_{p-1}|+2$ real negative roots and $z = 0$ is a root with multiplicity $r_p - 2$, and so on. So, the polynomials $B_n^{(0)}\left(z;\mathbf{r}_p\right),\ B_n^{(1)}\left(z;\mathbf{r}_p\right),\ldots,\ B_n^{(r_p-1)}\left(z;\mathbf{r}_p\right)$ have only real non-positive roots.

Secondly, we examine the polynomials $B_n^{(j)}\left(z;\mathbf{r}_p\right)$ for $r_p \leq j \leq r_{p+1}$. Indeed, we have $B_n^{(r_p)}\left(0;\mathbf{r}_p\right) \neq 0$ and consider the function

$$\frac{d^{r_p-1}}{dz^{r_p-1}} f_n\left(z;\mathbf{r}_p\right) = \exp\left(z\right) B_n^{(r_p-1)}\left(z;\mathbf{r}_p\right) = z\exp\left(z\right) B_n\left(z;r_1,\ldots,r_{p-1},r_p,r_{p-1}\right).$$

As it is shown above, this function has $n + |\mathbf{r}_p| - 1$ real negative roots and the root $z = 0$, then Rolle's theorem shows that its derivative

$$\frac{d^{r_p}}{dz^{r_p}} f_n\left(z;\mathbf{r}_p\right) = \exp\left(z\right) B_n^{(r_p)}\left(z;\mathbf{r}_p\right) = \exp\left(z\right) B_n\left(z;r_1,\ldots,r_{p-1},r_p,r_p\right)$$

has at least $n + |\mathbf{r}_p| - 1$ real negative roots. This means that the polynomial

$$B_n^{(r_p)}\left(z;\mathbf{r}_p\right) = B_n\left(z;r_1,\ldots,r_{p-1},r_p,r_p\right)$$

has at least $n + |\mathbf{r}_p| - 1$ real negative roots and because it is of degree $n + |\mathbf{r}_p|$, the missing one cannot be complex. By the fact that the coefficients of z^k in $B_n\left(z;r_1,\ldots,r_{p-1},r_p,r_p\right)$ are positive, this root must be negative too. So, the polynomial $B_n^{(r_p)}\left(z;\mathbf{r}_p\right)$ has $n+|\mathbf{r}_p|$ real negative roots. Similarly, apply Rolle's theorem to $\frac{d^{r_p}}{dz^{r_p}} f_n\left(z;\mathbf{r}_p\right)$ and conclude that the polynomial $B_n^{(r_p+1)}\left(z;\mathbf{r}_p\right)$ has $n + |\mathbf{r}_p|$ real negative roots and so on. So, the polynomials $B_n^{(r_p)}\left(z;\mathbf{r}_p\right),\ldots,\ B_n^{(r_{p+1})}\left(z;\mathbf{r}_p\right)$ vanish only at negative numbers. Then, the polynomial $B_n\left(z;\mathbf{r}_{p+1}\right) = B_n^{(r_{p+1})}\left(z;\mathbf{r}_p\right)$ (see Lemma 2) has only real negative roots. $\square$

Upon using Newton's inequality [6, p. 52], which is given by

Theorem 3. *(Newton's inequality) Let a_0, $a_1, \ldots,$ a_n be real numbers. If all the zeros of the polynomial $P(x) = \sum_{k=0}^{n} a_i x^i$ are real, then the coefficients of P satisfy*

$$a_i^2 \geq \left(1+\frac{1}{i}\right)\left(1+\frac{1}{n-i}\right)a_{i+1}a_{i-1}, \quad 1 \leq i \leq n-1,$$

we may state that:

Corollary 4. *The sequence $\left\{ \left\{ {n+|\mathbf{r}_p| \atop k+r_p} \right\}_{\mathbf{r}_p},\ 0 \leq k \leq n+|\mathbf{r}_{p-1}| \right\}$ is strongly log-concave (and thus unimodal).*

This property shows that the sequence $(\left\{ {n \atop k} \right\}_{\mathbf{r}_p}, 0 \leq k \leq n)$ admits an index $K \in \{0,1,\ldots,n\}$ for which $\left\{ {n \atop K} \right\}_{\mathbf{r}_p}$ is the maximum of $\left\{ {n \atop k} \right\}_{\mathbf{r}_p}$. An application of Darroch's inequality [3] will help us to localize this index.

Theorem 5. *(Darroch's inequality) Let a_0, $a_1, \ldots,$ a_n be real numbers. If all the zeros of the polynomial $P(x) = \sum_{k=0}^{n} a_i x^i$ are real and negative and $P(1) > 0$, then the value of k for which a_k is maximized is within one of $P'(1)/P(1)$.*

The following corollary gives a small interval for this index.

Corollary 6. *Let $K_{n,\mathbf{r}_p}$ be the greatest maximizing index of $\left\{ {n \atop k} \right\}_{\mathbf{r}_p}$. We have*

$$\left| K_{n+|\mathbf{r}_p|,\mathbf{r}_p} - \left(\frac{B_{n+1}\left(1;\mathbf{r}_p\right)}{B_n\left(1;\mathbf{r}_p\right)} - (r_p+1) \right) \right| < 1.$$

Proof. Since the sequence $\left\{ {n+|\mathbf{r}_p| \atop k+r_p} \right\}_{\mathbf{r}_p}$ is strongly log-concave, there exists an index $K_{n+|\mathbf{r}_p|,\mathbf{r}_p}$ for which $\left\{ {n+|\mathbf{r}_p| \atop r_p} \right\}_{\mathbf{r}_p} < \cdots < \left\{ {n+|\mathbf{r}_p| \atop K_{n+|\mathbf{r}_p|,\mathbf{r}_p}} \right\}_{\mathbf{r}_p} > \cdots > \left\{ {n+|\mathbf{r}_p| \atop n+r_p} \right\}_{\mathbf{r}_p}$. Then, on applying Theorem 1 and Darroch's theorem, we obtain

$$\left| K_{n+|\mathbf{r}_p|,\mathbf{r}_p} - \frac{\frac{d}{dz}B_n\left(z;\mathbf{r}_p\right)\big|_{z=1}}{B_n\left(1;\mathbf{r}_p\right)} \right| < 1.$$

It remains to use the first identity given in [12, Corollary 12] by$z\frac{d}{dz}\left(B_n\left(z;\mathbf{r}_p\right)\right) = B_{n+1}\left(z;\mathbf{r}_p\right) - \left(z+r_p\right)B_n\left(z;\mathbf{r}_p\right).$ □

3. Generalized Recurrences and Consequences

In this section, different representations of the polynomial $B_n(z;\mathbf{r}_p)$ in different bases or families of basis are given by Theorems 7 and 10. Indeed, a representation in the basis $\{B_{n+k}(z;r_p):\ 0\le k\le n+|\mathbf{r}_{p-1}|\}$ is given by the following theorem.

Theorem 7. *We have*

$$B_n(z;\mathbf{r}_p)=\sum_{k=0}^{|\mathbf{r}_{p-1}|}a_k(\mathbf{r}_{p-1})\,B_{n+k}(z;r_p),$$

$$B_n(z;\mathbf{r}_{p+q})=\sum_{k=0}^{|\mathbf{r}_{p-1}|}a_k(\mathbf{r}_{p-1})\,B_{n+k}(z;r_p,\ldots,r_{p+q}).$$

Proof. Upon using the fact that $(k+r_p)^{\underline{r_m}}=\sum_{j=0}^{r_m}(-1)^{r_m-j}\left[{r_m\atop j}\right](k+r_p)^j$, we get

$$\begin{aligned}B_n(z;\mathbf{r}_p)&=\exp(-z)\sum_{k\ge0}P_n(k;\mathbf{r}_p)\frac{z^k}{k!}\\&=\exp(-z)\sum_{k\ge0}\sum_{j=0}^{r_m}(-1)^{r_m-j}\begin{bmatrix}r_m\\j\end{bmatrix}\frac{P_0(k;\mathbf{r}_p)}{(k+r_p)^{\underline{r_m}}}(k+r_p)^{n+j}\frac{z^k}{k!}\\&=\sum_{j=0}^{r_m}(-1)^{r_m-j}\begin{bmatrix}r_m\\j\end{bmatrix}B_{n+j}(z;\mathbf{r}_p-r_m\mathbf{e}_m),\quad m=1,2,\ldots,p-1,\end{aligned}$$

and with the same process, we obtain

$$\begin{aligned}B_n(z;\mathbf{r}_p)&=\sum_{j_1=0}^{r_1}\cdots\sum_{j_{p-1}=0}^{r_{p-1}}(-1)^{|\mathbf{r}_{p-1}|-|\mathbf{j}_{p-1}|}\begin{bmatrix}r_1\\j_1\end{bmatrix}\cdots\begin{bmatrix}r_{p-1}\\j_{p-1}\end{bmatrix}B_{n+|\mathbf{j}_{p-1}|}(z;r_p)\\&=\sum_{k=0}^{|\mathbf{r}_{p-1}|}(-1)^{|\mathbf{r}_{p-1}|-k}B_{n+k}(z;r_p)\sum_{|\mathbf{j}_{p-1}|=k}\begin{bmatrix}r_1\\j_1\end{bmatrix}\cdots\begin{bmatrix}r_{p-1}\\j_{p-1}\end{bmatrix}\\&=\sum_{k=0}^{|\mathbf{r}_{p-1}|}a_k(\mathbf{r}_{p-1})\,B_{n+k}(z;r_p).\end{aligned}$$

This implies the first identity of the theorem.

Now, from Lemma 2 we can write

$$\begin{aligned}&\exp(-z)\frac{d^{r_{p+1}}}{dz^{r_{p+1}}}\left(z^{r_p}\exp(z)\,B_n(z;\mathbf{r}_p)\right)\\&\qquad=\sum_{k=0}^{|\mathbf{r}_{p-1}|}a_k(\mathbf{r}_{p-1})\exp(-z)\frac{d^{r_{p+1}}}{dz^{r_{p+1}}}\left(z^{r_p}\exp(z)\,B_{n+k}(z;r_p)\right),\end{aligned}$$

which gives by utilizing Lemma 2: $B_n(z;\mathbf{r}_{p+1}) = \sum_{k=0}^{|\mathbf{r}_{p-1}|} a_k(\mathbf{r}_{p-1}) B_{n+k}(z; r_p, r_{p+1})$.
We can repeat this process q times to obtain the second identity of the theorem. □

So, the $\mathbf{r}_p$-Stirling numbers admit an expression in terms of the usual r-Stirling numbers given by the following corollary.

Corollary 8. *We have*

$$\left\{ {n+|\mathbf{r}_p| \atop k+r_p} \right\}_{\mathbf{r}_p} = \sum_{j=0}^{|\mathbf{r}_{p-1}|} \left\{ {n+j+r_p \atop k+r_p} \right\}_{r_p} a_j(\mathbf{r}_{p-1}).$$

Proof. Using Theorem 7, the polynomial $B_n(z;\mathbf{r}_p)$ can be written as follows:

$$\begin{aligned}\sum_{j=0}^{|\mathbf{r}_{p-1}|} a_j(\mathbf{r}_{p-1}) B_{n+j}(z;r_p) &= \sum_{j=0}^{|\mathbf{r}_{p-1}|} a_j(\mathbf{r}_{p-1}) \sum_{k=0}^{n+j} \left\{ {n+j+r_p \atop k+r_p} \right\}_{r_p} z^k \\ &= \sum_{k=0}^{n+|\mathbf{r}_{p-1}|} z^k \sum_{j=0}^{|\mathbf{r}_{p-1}|} a_j(\mathbf{r}_{p-1}) \left\{ {n+j+r_p \atop k+r_p} \right\}_{r_p}\end{aligned}$$

and since $B_n(z;\mathbf{r}_p) = \sum_{k=0}^{n+|\mathbf{r}_{p-1}|} \left\{ {n+|\mathbf{r}_p| \atop k+r_p} \right\}_{\mathbf{r}_p} z^k$, the identity follows by identification. □

In [12], we proved the following:

$$\sum_{n\geq 0} B_n(z;\mathbf{r}_p) \frac{t^n}{n!} = B_0(z\exp(t);\mathbf{r}_p)\exp(z(\exp(t)-1)+r_p t).$$

The following theorem gives more details on the exponential generating function of the $\mathbf{r}_p$-Bell polynomials and will be used later.

Theorem 9. *We have*

$$\begin{aligned}\sum_{n\geq 0} B_{n+m}(z;\mathbf{r}_p) \frac{t^n}{n!} &= B_m(z\exp(t);\mathbf{r}_p)\exp(z(\exp(t)-1)+r_p t) \\ &= \sum_{k=0}^{|\mathbf{r}_{p-1}|} a_k(\mathbf{r}_{p-1}) \frac{d^{m+k}}{dt^{m+k}} (\exp(z(\exp(t)-1)+r_p t)).\end{aligned}$$

Proof. Use (1) to get

$$\begin{aligned}\sum_{n\geq 0} B_{n+m}\left(z;\mathbf{r}_p\right)\frac{t^n}{n!} &= \sum_{n\geq 0}\left(\exp\left(-z\right)\sum_{k\geq 0} P_0\left(k;\mathbf{r}_p\right)\left(k+r_p\right)^{n+m}\frac{z^k}{k!}\right)\frac{t^n}{n!}\\ &= \exp\left(-z\right)\sum_{k\geq 0} P_0\left(k;\mathbf{r}_p\right)\left(k+r_p\right)^{m}\frac{z^k\exp\left(\left(k+r_p\right)t\right)}{k!}\\ &= B_m\left(z\exp\left(t\right);\mathbf{r}_p\right)\exp\left(z\left(\exp\left(t\right)-1\right)+r_pt\right).\end{aligned}$$

For the second part of the theorem, use Theorem 7 to obtain

$$\begin{aligned}\sum_{n\geq 0} B_{n+m}\left(z;\mathbf{r}_p\right)\frac{t^n}{n!} &= \sum_{k=0}^{|\mathbf{r}_{p-1}|} a_k\left(\mathbf{r}_{p-1}\right)\sum_{n\geq 0} B_{n+m+k}\left(z;r_p\right)\frac{t^n}{n!}\\ &= \sum_{k=0}^{|\mathbf{r}_{p-1}|} a_k\left(\mathbf{r}_{p-1}\right)\frac{d^{m+k}}{dt^{m+k}}\left(\sum_{n\geq 0} B_n\left(z;r_p\right)\frac{t^n}{n!}\right)\\ &= \sum_{k=0}^{|\mathbf{r}_{p-1}|} a_k\left(\mathbf{r}_{p-1}\right)\frac{d^{m+k}}{dt^{m+k}}\left(\exp\left(z\left(\exp\left(t\right)-1\right)+r_pt\right)\right).\end{aligned}$$

□

Using combinatorial arguments, Spivey [13] established the following identity:

$$B_{n+m} = \sum_{k=0}^{n}\sum_{j=0}^{m}\left\{ {m \atop j} \right\}\binom{n}{k} j^{n-k} B_k,$$

where B_n is the n-th Bell number, i.e., the number of ways to partition a set of n elements into non-empty subsets. After that, Belbachir et al. [1] and Gould et al. [7] showed, using different methods, that the polynomial $B_{n+m}\left(z\right) = B_{n+m}\left(z;\mathbf{0}\right)$ admits a recurrence relation related to the family $\left\{z^i B_j\left(z\right)\right\}$ as follows:

$$B_{n+m}\left(z\right) = \sum_{k=0}^{n}\sum_{j=0}^{m}\left\{ {m \atop j} \right\}\binom{n}{k} j^{n-k} z^j B_k\left(z\right). \tag{4}$$

Recently, Xu [14] gave a recurrence relation on a large family of Stirling numbers and Mihoubi et al. [11] extended the relation (4) to r-Bell polynomials as follows:

$$B_{n+m,r}\left(x\right) = \sum_{k=0}^{n}\sum_{j=0}^{m}\left\{ {m+r \atop j+r} \right\}_r\binom{n}{k} j^{n-k} x^j B_{k,r}\left(x\right). \tag{5}$$

Other recurrence relations are given by Mező [10]. The following theorem generalizes the Identities (4) and (5), and the Carlitz's identities [4, 5] given by

$$B_{n+m}(1;r) = \sum_{k=0}^{m} \left\{ \begin{matrix} m+r \\ k+r \end{matrix} \right\}_r B_n(1;k+r),$$
$$B_n(1;r+s) = \sum_{k=0}^{s} \left[\begin{matrix} s+r \\ k+r \end{matrix} \right]_r (-1)^{s-k} B_{n+k}(1;r),$$

and shows that $B_{n+m}(z;\mathbf{r}_p)$ admits r-Stirling recurrence coefficients in the families of basis

$$\left\{ z^j B_m(z;\mathbf{r}_p + j\mathbf{e}_p) : 0 \le j \le n \right\},$$
$$\left\{ z^j B_{m+i}(z;r+j) : 0 \le i \le |\mathbf{r}_{p-1}|,\ 0 \le j \le n \right\},$$

where $B_n(1;r)$ is the number of ways to partition a set of n elements into non-empty subsets such that the r first elements are in different subsets.

Theorem 10. *We have*

$$B_{n+m}(z;\mathbf{r}_p) = \sum_{j=0}^{n} \left\{ \begin{matrix} n+r_p \\ j+r_p \end{matrix} \right\}_{r_p} z^j B_m(z;\mathbf{r}_p + j\mathbf{e}_p),$$
$$B_{n+m}(z;\mathbf{r}_p) = \sum_{i=0}^{|\mathbf{r}_{p-1}|} \sum_{j=0}^{n} \left\{ \begin{matrix} n+r_p \\ j+r_p \end{matrix} \right\}_{r_p} a_i(\mathbf{r}_{p-1}) z^j B_{m+i}(z;r_p + j),$$
$$z^n B_m(z;\mathbf{r}_p + n\mathbf{e}_p) = \sum_{j=0}^{n} \left[\begin{matrix} n+r_p \\ j+r_p \end{matrix} \right]_{r_p} (-1)^{n-j} B_{m+j}(z;\mathbf{r}_p).$$

Proof. Let $T_m(z;\mathbf{r}_p) := \sum_{n\ge 0} \left(\sum_{j=0}^{n} \left\{ \begin{smallmatrix} n+r_p \\ j+r_p \end{smallmatrix} \right\}_{r_p} z^j B_m(z;\mathbf{r}_p + j\mathbf{e}_p) \right) \frac{t^n}{n!}$. The second identity given in [12, Corollary 12] by

$$\exp(z) B_m(z;\mathbf{r}_p + \mathbf{e}_p) = \frac{d}{dz} \left(\exp(z) B_m(z;\mathbf{r}_p) \right)$$

can be used to get

$$\exp(z) B_m(z;\mathbf{r}_p + j\mathbf{e}_p) = \frac{d^j}{dz^j} \left(\exp(z) B_m(z;\mathbf{r}_p) \right). \qquad (6)$$

Identity (6) and the exponential generating function of the r-Stirling numbers (see [2])

$$\sum_{n\ge j} \left\{ \begin{matrix} n+r_p \\ j+r_p \end{matrix} \right\}_{r_p} \frac{t^n}{n!} = \frac{1}{j!} \left(\exp(t) - 1 \right)^j \exp(r_p t)$$

prove that

$$T_m(z;\mathbf{r}_p)=\sum_{j\geq 0}B_m(z;\mathbf{r}_p+j\mathbf{e}_p)\,z^j\frac{1}{j!}(\exp(t)-1)^j\exp(r_pt)$$
$$=\exp(r_pt-z)\sum_{j\geq 0}\frac{d^j}{dz^j}(\exp(z)\,B_m(z;\mathbf{r}_p))\frac{(z(\exp(t)-1))^j}{j!}.$$

Now, by the Taylor-Maclaurin expansion we have

$$\sum_{j\geq 0}\frac{d^j}{dz^j}(\exp(z)\,B_m(z;\mathbf{r}_p))\frac{(u-z)^j}{j!}=\exp(u)\,B_m(u;\mathbf{r}_p).$$

So, this identity and Theorem 9 show that we have

$$T_m(z;\mathbf{r}_p)=\exp(r_pt-z)\exp(z\exp(t))\,B_m(z\exp(t);\mathbf{r}_p)=\sum_{n\geq 0}B_{n+m}(z;\mathbf{r}_p)\frac{t^n}{n!}.$$

By comparing the coefficients of t^n in the two expressions of $T_m(z;\mathbf{r}_p)$, the first identity of this theorem follows. The second identity follows by replacing $B_m(z;\mathbf{r}_p+j\mathbf{e}_p)$, as given by its expression in Theorem 7 by

$$\sum_{i=0}^{|\mathbf{r}_{p-1}|}a_i(\mathbf{r}_{p-1})\,B_{m+i}(z;j+r_p).$$

For the third identity, let $A:=\sum_{j=0}^{n}\begin{bmatrix}n+r_p\\ j+r_p\end{bmatrix}_{r_p}(-1)^{n-j}B_{m+j}(z;\mathbf{r}_p)$. We use Identity (1) and the known identity $(k+r_p)^{\underline{n}}=\sum\limits_{j=0}^{n}\begin{bmatrix}n+r_p\\ j+r_p\end{bmatrix}_{r_p}k^j$ (see [2]) to obtain

$$A=\exp(-z)\sum_{k\geq 0}P_m(k;\mathbf{r}_p)\frac{z^k}{k!}\sum_{j=0}^{n}\begin{bmatrix}n+r_p\\ j+r_p\end{bmatrix}_{r_p}(-1)^{n-j}(k+r_p)^j$$
$$=(-1)^n\exp(-z)\sum_{k\geq 0}P_m(k;\mathbf{r}_p)\frac{z^k}{k!}\sum_{j=0}^{n}\begin{bmatrix}n+r_p\\ j+r_p\end{bmatrix}_{r_p}(-k-r_p)^j$$
$$=(-1)^n\exp(-z)\sum_{k\geq 0}P_m(k;\mathbf{r}_p)\frac{z^k}{k!}(-k-r_p+r_p)^{\overline{n}}$$
$$=\exp(-z)\sum_{k\geq n}P_m(k;\mathbf{r}_p)\,k^{\underline{n}}\frac{z^k}{k!}$$
$$=z^n\exp(-z)\sum_{k\geq 0}P_m(k+n;\mathbf{r}_p)\frac{z^k}{k!}$$
$$=z^nB_m(z;\mathbf{r}_p+n\mathbf{e}_p).\qquad\square$$

As consequences of Theorem 10, some identities for the $\mathbf{r}_p$-Stirling numbers of the second kind can be deduced as is shown by the following corollary.

Corollary 11. *We have*

$$\sum_{i=0}^{k}\left\{\begin{matrix} m+|\mathbf{r}_p| \\ i+r_p \end{matrix}\right\}_{\mathbf{r}_p}\left\{\begin{matrix} n+r_p \\ k-i+r_p \end{matrix}\right\}_{r_p}=\left\{\begin{matrix} n+m+|\mathbf{r}_p| \\ k+r_p \end{matrix}\right\}_{\mathbf{r}_p},$$

$$\sum_{j=0}^{n}\left\{\begin{matrix} m+j+|\mathbf{r}_p| \\ k+n+r_p \end{matrix}\right\}_{\mathbf{r}_p}\begin{bmatrix} n+r_p \\ j+r_p \end{bmatrix}_{r_p}(-1)^{n-j}=\left\{\begin{matrix} m+|\mathbf{r}_p|+n \\ k+r_p+n \end{matrix}\right\}_{\mathbf{r}_p+n\mathbf{e}_p},$$

$$\sum_{j=0}^{n}\left\{\begin{matrix} m+j+|\mathbf{r}_p| \\ k+r_p \end{matrix}\right\}_{\mathbf{r}_p}\begin{bmatrix} n+r_p \\ j+r_p \end{bmatrix}_{r_p}(-1)^{n-j}=0, \quad k<n.$$

Proof. From the first identity of Theorem 10 we have

$$B_{n+m}(z;\mathbf{r}_p)=\sum_{j=0}^{n}\left\{\begin{matrix} n+r_p \\ j+r_p \end{matrix}\right\}_{r_p} z^j B_m(z;\mathbf{r}_p+j\mathbf{e}_p)$$

which can be written as

$$\begin{aligned}\sum_{k=0}^{n+m+|\mathbf{r}_{p-1}|}\left\{\begin{matrix} n+m+|\mathbf{r}_p| \\ k+r_p \end{matrix}\right\}_{\mathbf{r}_p} z^k &= \sum_{j=0}^{n}\left\{\begin{matrix} n+r_p \\ j+r_p \end{matrix}\right\}_{r_p} z^j \sum_{i=0}^{m+|\mathbf{r}_{p-1}|}\left\{\begin{matrix} m+|\mathbf{r}_p| \\ i+r_p \end{matrix}\right\}_{\mathbf{r}_p} z^i \\ &= \sum_{k=0}^{n+m+|\mathbf{r}_{p-1}|} z^k \sum_{i=0}^{k}\left\{\begin{matrix} m+|\mathbf{r}_p| \\ i+r_p \end{matrix}\right\}_{\mathbf{r}_p}\left\{\begin{matrix} n+r_p \\ k-i+r_p \end{matrix}\right\}_{r_p}.\end{aligned}$$

Then, the desired identity follows by comparing the coefficients of z^k in the last expansion. Using the definition of $B_n(z;\mathbf{r}_p)$ and the third identity of Theorem 10, the second and the third identities of the corollary follow from the definition

$$B_m(z;\mathbf{r}_p+n\mathbf{e}_p)=\sum_{k=0}^{m+|\mathbf{r}_{p-1}|}\left\{\begin{matrix} m+|\mathbf{r}_p|+n \\ k+r_p+n \end{matrix}\right\}_{\mathbf{r}_p+n\mathbf{e}_p} z^k$$

and the expansion

$$\begin{aligned}B_m(z;\mathbf{r}_p+n\mathbf{e}_p) &= z^{-n}\sum_{j=0}^{n}\begin{bmatrix} n+r_p \\ j+r_p \end{bmatrix}_{r_p}(-1)^{n-j} B_{m+j}(z;\mathbf{r}_p) \\ &= z^{-n}\sum_{j=0}^{n}\begin{bmatrix} n+r_p \\ j+r_p \end{bmatrix}_{r_p}(-1)^{n-j}\sum_{k=0}^{m+j+|\mathbf{r}_{p-1}|}\left\{\begin{matrix} m+j+|\mathbf{r}_p| \\ k+r_p \end{matrix}\right\}_{\mathbf{r}_p} z^k \\ &= \sum_{k=0}^{m+n+|\mathbf{r}_{p-1}|} z^{k-n}\sum_{j=0}^{n}\left\{\begin{matrix} m+j+|\mathbf{r}_p| \\ k+r_p \end{matrix}\right\}_{\mathbf{r}_p}\begin{bmatrix} n+r_p \\ j+r_p \end{bmatrix}_{r_p}(-1)^{n-j} \\ &= \sum_{k=-n}^{m+|\mathbf{r}_{p-1}|} z^k\sum_{j=0}^{n}\left\{\begin{matrix} m+j+|\mathbf{r}_p| \\ k+n+r_p \end{matrix}\right\}_{\mathbf{r}_p}\begin{bmatrix} n+r_p \\ j+r_p \end{bmatrix}_{r_p}(-1)^{n-j}.\end{aligned}$$

□

4. Ordinary Generating Functions

The *ordinary generating function* of the r-Stirling numbers of the second kind [2] is given by

$$\sum_{n\geq k}\left\{ {n+r \atop k+r} \right\}_r t^n = t^k \prod_{j=0}^{k} \left(1-(r+j)\,t\right)^{-1}. \tag{7}$$

An analogous result for the $\mathbf{r}_p$-Stirling numbers is given by the following theorem.

Theorem 12. *Let*

$$\widetilde{B}_n\left(z;\mathbf{r}_p\right) := \sum_{k=0}^{n} \left\{ {n+|\mathbf{r}_p| \atop k+|\mathbf{r}_p|} \right\}_{\mathbf{r}_p} z^k.$$

Then, we have

$$\sum_{n\geq k}\left\{ {n+|\mathbf{r}_p| \atop k+|\mathbf{r}_p|} \right\}_{\mathbf{r}_p} t^n = t^{k+|\mathbf{r}_{p-1}|}\,\underline{\left(\frac{1}{t}\right)^{r_1}}\cdots\underline{\left(\frac{1}{t}\right)^{r_{p-1}}}\prod_{j=0}^{k+|\mathbf{r}_{p-1}|}\left(1-(r_p+j)\,t\right)^{-1},$$

$$\sum_{n\geq 0}\widetilde{B}_n\left(z;\mathbf{r}_p\right)t^n = \underline{\left(\frac{1}{t}\right)^{r_1}}\cdots\underline{\left(\frac{1}{t}\right)^{r_{p-1}}}\sum_{k\geq|\mathbf{r}_{p-1}|}\frac{z^{k-|\mathbf{r}_{p-1}|}t^k}{\prod_{j=0}^{k}\left(1-(r_p+j)\,t\right)}.$$

Proof. Use Corollary 8 to obtain

$$\begin{aligned}\sum_{n\geq k}\left\{ {n+|\mathbf{r}_p| \atop k+|\mathbf{r}_p|} \right\}_{\mathbf{r}_p} t^n &= \sum_{n\geq k}\left\{ {n+|\mathbf{r}_p| \atop k+|\mathbf{r}_{p-1}|+r_p} \right\}_{\mathbf{r}_p} t^n \\ &= \sum_{j=0}^{|\mathbf{r}_{p-1}|} a_j\left(\mathbf{r}_{p-1}\right)t^{-j}\sum_{n\geq k}\left\{ {n+j+r_p \atop k+|\mathbf{r}_{p-1}|+r_p} \right\}_{r_p} t^{n+j} \\ &= \sum_{j=0}^{|\mathbf{r}_{p-1}|} a_j\left(\mathbf{r}_{p-1}\right)t^{-j}\sum_{n\geq k+j}\left\{ {n+r_p \atop k+|\mathbf{r}_{p-1}|+r_p} \right\}_{r_p} t^{n},\end{aligned}$$

and because $\left\{ {n+r_p \atop k+|\mathbf{r}_{p-1}|+r_p} \right\}_{r_p} = 0$ for $n=k,\ldots,k+|\mathbf{r}_{p-1}|-1$, we get

$$\sum_{n\geq k}\left\{ {n+|\mathbf{r}_p| \atop k+|\mathbf{r}_p|} \right\}_{\mathbf{r}_p} t^n = \left(\sum_{n\geq k+|\mathbf{r}_{p-1}|}\left\{ {n+r_p \atop k+|\mathbf{r}_{p-1}|+r_p} \right\}_{r_p} t^n\right)\left(\sum_{j=0}^{|\mathbf{r}_{p-1}|} a_j\left(\mathbf{r}_{p-1}\right)t^{-j}\right).$$

The first generating function of the theorem follows by using (3) and (7). For the second one, use the definition of $\widetilde{B}_n\left(z;\mathbf{r}_p\right)$ and the last expansion. □

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