ON THE EXPONENTIAL SUM WITH THE SUM OF DIGITS OF HEREDITARY BASE b NOTATION

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Abstract

Let $b \geq 2$ be an integer and $w_b(n)$ be the sum of digits of the nonnegative integer n written in hereditary base b notation. We give optimal upper bounds for the exponential sum $\sum_{n=0}^{N-1} \exp(2\pi i w_b(n) t)$, where t is a real number. In particular, our results imply that for each positive integer m the sequence $\{w_b(n)\}_{n=0}^{\infty}$ is uniformly distributed modulo m; and that for each irrational real α the sequence $\{w_b(n)\alpha\}_{n=1}^{\infty}$ is uniformly distributed modulo 1.

1. Introduction

Let $b \geq 2$ be an integer. Exponential sums involving $s_b(\cdot)$, the base b sum of digits function, has been studied extensively [3] [12] [6] [1], primarily in relation to problems of uniform distribution. Similar exponential sums have been also investigated with respect to alternative digital expansions. For example, digital expansions arising from linear recurrences [5] [9], complex base number systems [10] [2] and number systems for number fields [14] [11]. We study the exponential sum

$$S_b(t, N) := \sum_{n=0}^{N-1} (w_b(n)\, t),$$

where N is a positive integer, $t \in \mathbb{R} \setminus \mathbb{Z}$ and $w_b(n)$ is the sum of digits of n written in *hereditary base b notation*.

The hereditary base b notation of a nonnegative integer n is obtained as follows: write n in base b, then write all the exponents in base b, etc. until there appear only the numbers $0, 1, \ldots, b$. For example, the hereditary base 3 notation of 4384 is

$$1 \cdot 3^0 + 1 \cdot 3^{2 \cdot 3^0} + 2 \cdot 3^{1 \cdot 3^0 + 2 \cdot 3^{1 \cdot 3^0}}.$$

The hereditary base b notation was used to define Goodstein sequences and prove the related Goodstein theorem [4], which can be considered the first simple example of a statement, true in ZFC set theory, that is unprovable in Peano arithmetic [7].

So we define $w_b(n)$ as the sum of digits of n written in hereditary base b notation, to continue the example:

$$w_3(4384) = 1 + 1 + 2 + 2 + 1 + 2 + 1 = 10.$$

More precisely, the function $w_b(\cdot)$ can be defined recursively as follows:

$$\begin{aligned} &w_b(0) := 0, \\ &w_b(n) := \sum_{j=1}^{u} (w_b(k_j) + a_j) \quad \text{ for } n = \sum_{j=1}^{u} a_j b^{k_j}, \end{aligned}$$

where $u \geq 1$, $0 \leq k_1 < \ldots < k_u$ are integers and $a_1, \ldots, a_u \in \{1, 2, \ldots, b-1\}$.

We give upper bounds for $S_b(t, N)$, distinguishing between rational and irrational t.

Theorem 1.1. *Let $t = \ell/m$, with $1 \leq |\ell| < m$ relatively prime integers.*

(i). If $m \mid b$ then $S_b(t, N) = (w_b(N))\, P_b(t, N)$ for $N \geq 1$, where $P_b(t, \cdot)$ is a periodic function of period b.

(ii). If m is even and $m \mid (b-2)$, then $S_b(t, N) = (w_b(N))\, Q_b(t, N)$ for $N \geq 1$, where $Q_b(t, \cdot)$ is a periodic function of period $b^{k'+1}$ and k' is the least non-negative integer such that $2(w_b(k') + 1)/m$ is an odd integer.

(iii). If $m \mid (b-1)$ then $S_b(t, N) = O_b(\log N)$ for $N > 1$.

(iv). If we are not in cases (i), (ii) or (iii), then for $\varepsilon > 0$ it results

$$S_b(t, N) = O_{b,t,\varepsilon}(N^{C_{b,t}+\varepsilon})$$

for $N \geq 1$, where the constant

$$C_{b,t} := \frac{1}{m} \log_b \left| 1 - (-1)^{m+b\ell} \left(\frac{\sin(\pi(b-1)t)}{\sin(\pi t)} \right)^m \right| \in \,]0, 1[$$

is the best possible.

Theorem 1.2. *If $b \geq 3$ and t is irrational, then for $\varepsilon > 0$ it results*

$$S_b(t, N) = O_{b,t,\varepsilon}(N^{C_{b,t}+\varepsilon})$$

for $N \geq 1$, where the constant

$$C_{b,t} := \max\left(0, \log_b \left| \frac{\sin(\pi(b-1)t)}{\sin(\pi t)} \right| \right) \in [0, 1[$$

is the best possible.

Unfortunately, in the case in which $b = 2$ and t is irrational, we have not been able to prove an upper bound for $S_b(t, N)$ similar to that of Theorem 1.2. We explain later why our arguments are not enough for that. However, for $b \geq 2$ and $t \in \mathbb{R} \setminus \mathbb{Z}$ we have at least the crude upper bound $S_b(t, N) = o(N)$ as $N \to \infty$ (see Lemma 2.4). Therefore, we obtain the following corollaries (see [8, Chap. 1, Theorem 2.1] and [8, Chap. 5, Corollary 1.1]).

Corollary 1.1. *For each positive integer m, the sequence $\{w_b(n)\}_{n=0}^{\infty}$ is uniformly distributed modulo m, i.e.,*

$$\lim_{x \to \infty} \frac{\#\{n \leq x : w_b(n) \equiv r \bmod m\}}{x} = \frac{1}{m} \quad \textit{for all } r = 1, 2, \ldots, m.$$

Corollary 1.2. *For each irrational real α, the sequence $\{w_b(n)\alpha\}_{n=0}^{\infty}$ is uniformly distributed modulo 1, i.e.,*

$$\lim_{x \to \infty} \frac{\#\{n \leq x : (w_b(n)\alpha \bmod 1) \in [a, b]\}}{x} = b - a \quad \textit{for all } 0 \leq a \leq b \leq 1.$$

1.1. Notations

Through the paper, we reserve the variables ℓ, m, n, M, N, u, h, r, k and k_j for integers. We use the Bachmann–Landau symbols O and o, as well as the Vinogradov symbols $\ll$ and $\gg$, with their usual meanings. We write $\mathcal{D}_b := \{1, 2, \ldots, b-1\}$ for the set of nonzero base b digits and $(x) := e^{2\pi i x}$ for the standard additive character. We adopt the usual convention that empty sums and empty products, e.g. $\sum_{n=x}^{y}$ and $\prod_{n=x}^{y}$ with $x > y$, have values 0 and 1, respectively.

2. Preliminaries to Proofs of Theorem 1.1 and 1.2

Proposition 2.1. *If $k \geq 0$, $a \in \mathcal{D}_b$ and $0 \leq n < b^k$, then*

$$w_b(ab^k + n) = w_b(k) + w_b(n) + a.$$

Proof. It is a straightforward consequence of the definition of $w_b(\cdot)$. $\square$

Proposition 2.2. *If $k \geq 0$, $a \in \mathcal{D}_b$ and $0 < N \leq b^k$, then*

$$S_b(t, ab^k + N) = S_b(t, ab^k) + ((w_b(k) + a)\, t)\, S_b(t, N).$$

Proof. Consider that

$$\begin{aligned} S_b(t, ab^k + N) &= S_b(t, ab^k) + \sum_{n=0}^{N-1} (w_b(ab^k + n)\, t) \\ &= S_b(t, ab^k) + ((w_b(k) + a)\, t) \sum_{n=0}^{N-1} (w_b(n)\, t) \\ &= S_b(t, ab^k) + ((w_b(k) + a)\, t)\, S_b(t, N), \end{aligned}$$

where we have applied Proposition 2.1. □

For the next lemma, we define

$$\begin{aligned} E_b(t, a, \theta) &:= 1 + (\theta + \tfrac{1}{2} a\, t) \cdot \frac{\sin(\pi(a-1)t)}{\sin(\pi t)}, \\ F_b(t, a, r) &:= E_b(t, a, rt), \\ f_b(t, a, h) &:= F_b(t, a, w_b(h)), \end{aligned}$$

for $a \in \mathcal{D}_b$, $\theta \in \mathbb{R}$ and $r, h \geq 0$.

Lemma 2.3. *If $k \geq 0$ and $a \in \mathcal{D}_b$ then*

$$S_b(t, ab^k) = f_b(t, a, k) \prod_{h=0}^{k-1} f_b(t, b, h).$$

Moreover, if $N = \sum_{j=1}^{u} a_j b^{k_j}$, where $u \geq 1$, $a_1, \ldots, a_u \in \mathcal{D}_b$ and $0 \leq k_1 < \cdots < k_u$, then

$$S_b(t, N) = (w_b(N)\, t) \sum_{j=1}^{u} \left(-\sum_{v=1}^{j} (w_b(k_v) + a_v)\, t \right) S_b(t, a_j b^{k_j}).$$

Proof. By Proposition 2.2 (with $N = b^k$), we see that

$$S_b(t, (a+1)b^k) = S_b(t, ab^k) + ((w_b(k) + a)\, t)\, S_b(t, b^k).$$

As a consequence, by induction, we obtain

$$\begin{aligned} S_b(t, (a+1)b^k) &= \left(1 + \sum_{j=1}^{a} ((w_b(k) + j)\, t) \right) S_b(t, b^k) \\ &= \left(1 + ((w_b(k) + 1)\, t) \cdot \frac{(a\, t) - 1}{(t) - 1} \right) S_b(t, b^k) \\ &= f_b(t, a+1, k)\, S_b(t, b^k). \end{aligned}$$

But $f_b(t,1,k)=1$, so that $S_b(t,cb^k)=f_b(t,c,k)\,S_b(t,b^k)$ for each $c\in\mathcal{D}_b\cup\{b\}$. In particular, $S_b(t,b^{k+1})=f_b(t,b,k)\,S_b(t,b^k)$ for $c=b$. Thus, since $S_b(t,1)=1$, we find that

$$S_b(t,ab^k)=f_b(t,a,k)\,S_b(t,b^k)=f_b(t,a,k)\prod_{h=0}^{k-1}f_b(t,b,h),$$

and the first part of the claim is proved.

Now, from Proposition 2.2, we get that

$$\begin{aligned}S_b(t,N+1)&=S_b(t,a_ub^{k_u})+((w_b(k_u)+a_u)\,t)\,S_b(t,N+1-a_ub^{k_u}) \qquad (1)\\&=S_b(t,a_ub^{k_u})+((w_b(k_u)+a_u)\,t)\,S_b(t,a_{u-1}b^{k_{u-1}})+\cdots\\&=\sum_{j=1}^{u}\left(\prod_{v=j+1}^{u}((w_b(k_v)+a_v)\,t)\right)S_b(t,a_jb^{k_j})+(w_b(N)\,t).\end{aligned}$$

Subtracting $(w_b(N)\,t)$ from (1) and recalling that $w_b(N)=\sum_{j=1}^{u}(w_b(k_j)+a_j)$, the second part of the claim follows. □

Lemma 2.4. *$S_b(t,N)=o(N)$ as $N\to\infty$.*

Proof. Let $a\in\mathcal{D}_b$ and $k\geq 0$. From Lemma 2.3, we know that

$$\frac{|S_b(t,ab^k)|}{ab^k}=\frac{|f_b(t,a,k)|}{a}\prod_{h=0}^{k-1}\frac{|f_b(t,b,h)|}{b}. \qquad (2)$$

We claim that the right-hand side of (2) tends to 0 as $k\to\infty$. Observe that $|f_b(t,a,k)|\leq a$ and

$$\frac{|f_b(t,b,h)|}{b}\leq\frac{1}{b}\left(1+\left|\frac{\sin(\pi(b-1)t)}{\sin(\pi t)}\right|\right)\leq 1 \qquad (3)$$

for all $h\geq 0$, so that the product in (2) is a nonincreasing function of k. If $b\geq 3$ then the last inequality in (3) is strict, so the claim follows. If $b=2$ then, for all $h\geq 0$,

$$\frac{|f_b(t,b,h)|}{b}=|\cos(\pi(w_b(h)+1)t)|.$$

On the one hand, if t is rational, then the sequence $\{|\cos(\pi rt)|\}_{r=1}^{\infty}$ is periodic and, since t is not an integer, it has infinitely many terms less than 1. On the other hand, if t is irrational, then the sequence $\{|\cos(\pi rt)|\}_{r=1}^{\infty}$ is dense in $[0,1]$. In any case, there exists $\delta<1$ and an infinitude of positive integers r such that $|\cos(\pi rt)|\leq\delta$. Since $w_b(\cdot)$ is surjective, we get that $|f_b(t,b,h)|/b<\delta$ for infinitely many $h\geq 0$, and the claim follows again.

At this point, we know that $|S_b(t,ab^k)|/(ab^k)\to 0$ as $k\to\infty$. Let

$$L:=\limsup_{N\to\infty}\frac{|S_b(t,N)|}{N},$$

so that $L \in [0,1]$. Put $a = a(N)$, $k = k(N)$ and $M = M(N)$ as functions of $N \geq 1$ such that $N = ab^k + M$ and $0 \leq M < b^k$. Then by Proposition 2.2 and the above, we have

$$\begin{aligned} L &= \limsup_{N\to\infty} \frac{|S_b(t, ab^k + M)|}{ab^k + M} \\ &\leq \limsup_{N\to\infty} \frac{|S_b(t, ab^k)|}{ab^k + M} + \limsup_{N\to\infty} \frac{|S_b(t, M)|}{ab^k + M} \\ &\leq \limsup_{k\to\infty} \frac{|S_b(t, ab^k)|}{ab^k} + \frac{1}{2} \limsup_{M\to\infty} \frac{|S_b(t, M)|}{M} \\ &= 0 + \tfrac{1}{2}L, \end{aligned}$$

so that $L = 0$, as desired. □

Proposition 2.5. *We have $F_b(t, b, r) = 0$ for some $r \in \mathbb{Z}$ if and only if $t = \ell/m$, where ℓ and $m \neq 0$ are relatively prime integers such that:*

(a) $m \mid b$ and $m \mid r$; or

(b) m is even, $m \mid (b-2)$, $\frac{1}{2}m \mid (r+1)$ and $2(r+1)/m$ is odd.

Proof. If cases (a) or (b) hold, then we quickly deduce that $F_b(t, b, r) = 0$. On the other hand, if $F_b(t, b, r) = 0$ then necessarily $|\sin(\pi(b-1)t)| = |\sin(\pi t)|$, so that at least one of bt and $(b-2)t$ is an integer. If bt is an integer then $F_b(t, b, r) = 1 - (rt) = 0$, so also rt is an integer. Hence, $t = \ell/m$ for some relatively prime integers ℓ and $m \neq 0$ such that $m \mid b$ and $m \mid r$, this is case (a). If $(b-2)t$ is an integer then $F_b(t, b, r) = 1 + ((r+1)t) = 0$, so that $(r+1)t = u + \frac{1}{2}$ for some integer u. Hence, $t = (2u+1)/(2r+2) = \ell/m$ for some relatively prime integers ℓ and $m \neq 0$ such that m is even, $m \mid (b-2)$ and $2(r+1)/m$ is odd, this is case (b). □

Proposition 2.6. *If t is irrational and $b \geq 3$ then $E_b(t, b, \theta) \neq 0$ for all $\theta \in \mathbb{R}$.*

Proof. As in the proof of Proposition 2.5, if $E_b(t, b, \theta) = 0$ then $|\sin(\pi(b-1)t)| = |\sin(\pi t)|$, so that at least one of bt and $(b-2)t$ is an integer, but this is impossible, since t is irrational and $b \geq 3$. □

3. Proof of Theorem 1.1

First, suppose that k'' is a nonnegative integer such that $f_b(t, b, k'') = 0$. Then, by Lemma 2.3, we have that $S_b(t, ab^k) = 0$ for all $a \in \mathcal{D}_b$ and $k > k''$. Moreover, again by Lemma 2.3, it results that $S_b(t, N) = (w_b(N)\, t)\, R_b(t, N)$ for any $N \geq 1$, where $R_b(t, \cdot)$ is a function depending only on $(N \bmod b^{k''+1})$, so periodic of (not necessarily minimal) period $b^{k''+1}$. Therefore, by Proposition 2.5, the claim follows in case (i), taking $k''=0$, and in case (ii), taking $k''=k'$.

Now suppose we are in case (iii), i.e., $m \mid (b-1)$. Then $f_b(t,b,h)=1$ for all $h \geq 0$, so that from Lemma 2.3 we get $S_b(t,b^k)=1$ for all $k \geq 0$. Taking N as in Lemma 2.3, we find that

$$|S_b(t,N)| \leq \sum_{j=1}^{u} |S_b(t,a_j b^{k_j})| \leq (b-1)\sum_{j=1}^{u} |S_b(t,b^{k_j})| = (b-1)u \leq (b-1)(\log_b N+1).$$

Hence, $S_b(t,N)=O_b(\log N)$ for any $N>1$, as claimed.

Finally, suppose we are in case (iv). So by Proposition 2.5, $f_b(t,b,h)\neq 0$ for all $h \geq 0$. Thus, Lemma 2.3 yields

$$\begin{aligned}\lim_{k\to\infty}\frac{\log_b|S_b(t,b^k)|}{k} &= \lim_{k\to\infty}\frac{1}{k}\sum_{h=0}^{k-1}\log_b|f_b(t,b,h)| \\ &= \lim_{k\to\infty}\frac{1}{k}\sum_{h=0}^{k-1}\log_b|F_b(t,b,w_b(h))| = \frac{1}{m}\sum_{r=1}^{m}\log_b|F_b(t,b,r)|,\end{aligned}$$

where the last equality follows since, by Corollary 1.1, $\{w_b(n)\}_{n=1}^{\infty}$ is uniformly distributed modulo m and $\log|F_b(t,b,\cdot)|$ is a function of period m. Observe that

$$\prod_{r=1}^{m}|1+(rt)z| = |1-(-z)^m|$$

for any complex number z, thus putting $z=(\frac{1}{2}bt)\sin(\pi(b-1)t)/\sin(\pi t)$ we have

$$\frac{1}{m}\sum_{r=1}^{m}\log_b|F_b(t,b,r)| = \frac{1}{m}\log_b\left|1-(-1)^{m+b\ell}\left(\frac{\sin(\pi(b-1)t)}{\sin(\pi t)}\right)^m\right| = C_{b,t}.$$

From here, we get

$$|S_b(t,b^k)| = (b^k)^{C_{b,t}+o(1)}, \tag{4}$$

as $k\to\infty$. Therefore, if $\varepsilon>0$ then $S_b(t,b^k) \ll_{b,t,\varepsilon} (b^k)^{C_{b,t}+\varepsilon}$ for any $k\geq 0$, and taking N as in Lemma 2.3 yields that

$$\begin{aligned}|S_b(t,N)| &\leq \sum_{j=1}^{u}|S_b(t,a_jb^{k_j})| \leq (b-1)\sum_{j=1}^{u}|S_b(t,b^{k_j})| \ll_{b,t,\varepsilon} \sum_{j=1}^{u}(b^{k_j})^{C_{b,t}+\varepsilon} \\ &\ll_{b,t,\varepsilon} \sum_{k=0}^{k_u}(b^k)^{C_{b,t}+\varepsilon} \ll_{b,t,\varepsilon} (b^{k_u})^{C_{b,t}+\varepsilon} \leq N^{C_{b,t}+\varepsilon}\end{aligned}$$

for any $N\geq 1$, as claimed.

Now we prove that $C_{b,t} \in]0,1[$. On the one hand, it must be $C_{b,t} \geq 0$, otherwise $S_b(t,N) \to 0$ as $N \to \infty$, which is absurd. If $C_{b,t} = 0$ then, since $\sin(\pi(b-1)t) \neq 0$, it results

$$2 = (-1)^{m+b\ell} \left(\frac{\sin(\pi(b-1)t)}{\sin(\pi t)}\right)^m = \left(-\frac{((b-1)t)-1}{(t)-1}\right)^m,$$

so that $\mathbb{Q}(\sqrt[m]{2})$ is a subfield of $\mathbb{Q}((t))$, but

$$\left[\mathbb{Q}(\sqrt[m]{2}) : \mathbb{Q}\right] = m > \phi(m) = [\mathbb{Q}((t)) : \mathbb{Q}],$$

where $\phi(\cdot)$ is the totient function, which is a contradiction, hence $C_{b,t} > 0$. On the other hand, since $m \geq 2$, we see that

$$C_{b,t} \leq \frac{1}{m} \log_b \left(1 + \left|\frac{\sin(\pi(b-1)t)}{\sin(\pi t)}\right|^m\right) \leq \frac{1}{m} \log_b \left(1 + (b-1)^m\right) < 1.$$

It remains only to prove that $C_{b,t}$ is the best constant possible for the bound obtained, i.e., that for any $C < C_{b,t}$ there exists $\varepsilon > 0$ such that $S_b(t,N) \neq O_{b,t,\varepsilon}(N^{C+\varepsilon})$, as $N \to +\infty$. This is straightforward since if $\varepsilon \in]0, C_{b,t} - C[$ then from (4) we get

$$|S_b(t,b^k)| = (b^k)^{C_{b,t}-C-\varepsilon+o(1)} \cdot (b^k)^{C+\varepsilon} \neq O_{b,t,\varepsilon}((b^k)^{C+\varepsilon}),$$

as $k \to \infty$, so a fortiori $S_b(t,N) \neq O_{b,t,\varepsilon}(N^{C+\varepsilon})$. This completes the proof.

4. Proof of Theorem 1.2

Observe that, since $b \geq 3$ and t is irrational, by Proposition 2.6 it follows that $\log_b |E_b(t,b,\cdot)|$ is a well-defined function. On the other hand, since t is irrational, Proposition 2.5 shows that $f_b(t,b,h) \neq 0$ for all $h \geq 0$. Thus, by Lemma 2.3,

$$\begin{aligned} \lim_{k\to\infty} \frac{\log_b |S_b(t,b^k)|}{k} &= \lim_{k\to\infty} \frac{1}{k} \sum_{h=0}^{k-1} \log_b |f_b(t,b,h)| \qquad (5) \\ &= \lim_{k\to\infty} \frac{1}{k} \sum_{h=0}^{k-1} \log_b |E_b(t,b,w_b(h)t)| = \int_0^1 \log_b |E_b(t,b,\theta)| \,\mathrm{d}\theta, \end{aligned}$$

since $\log_b |E_b(t,b,\cdot)|$ is a continuous function of period 1 and, by Corollary 1.2, the sequence $\{w_b(h)t\}_{h=0}^{\infty}$ is uniformly distributed modulo 1 (see [8, Chap. 1, Corollary 1.2]). Now, from Jensen's formula [13, Theorem 15.18], we know that

$$\int_0^1 \log|1 + (\theta)A| \,\mathrm{d}\theta = \begin{cases} \log|A| & \text{if } |A| > 1 \\ 0 & \text{if } |A| \leq 1 \end{cases}$$

for all complex number A. Putting $A = (\frac{1}{2}b\,t)\sin(\pi(b-1)t)/\sin(\pi t)$ we can evaluate the integral in (5) and get

$$\int_0^1 \log_b |E_b(t,b,\theta)|\,\mathrm{d}\theta = \max\left(0, \log_b\left|\frac{\sin(\pi(b-1)t)}{\sin(\pi t)}\right|\right) = C_{b,t}.$$

Hence, $S_b(t,b^k) = (b^k)^{C_{b,t}+o(1)}$ as $k \to \infty$. At this point, the claim follows by the same arguments used at the end of the proof of Theorem 1.1.

5. Concluding Remarks

It is perhaps interesting that, in the proof of Theorem 1.1 (respectively Theorem 1.2), we used a coarse upper bound on $S_b(t,N)$, namely Lemma 2.4, to get that $\{w_b(n)\}_{n=1}^\infty$ is uniformly distributed modulo m (respectively $\{w_b(n)\alpha\}_{n=1}^\infty$ is uniformly distributed modulo 1), and then we used this result to get an improved upper bound on $S_b(t,N)$.

An open question is whether it is possible to get an upper bound, similar to that of Theorem 1.2, for $S_2(t,N)$ when t is irrational. Note that we cannot use the same arguments as in the proof of Theorem 1.2, because Proposition 2.6 breaks down for $b = 2$. Precisely, the function $\log_2|E_2(t,2,\cdot)| = \log_2|1 + (\,\cdot + t)|$ has singularities on $\mathbb{Z} + (\frac{1}{2} - t)$ and although the integral

$$\int_0^1 \log_2 |E_2(t,2,\theta)|\,\mathrm{d}\theta$$

is finite (in fact, it is equal to zero), it is no longer true that

$$\int_0^1 \log_2 |E_2(t,2,\theta)|\,\mathrm{d}\theta = \lim_{k\to\infty} \frac{1}{k}\sum_{h=0}^{k-1} \log_2|E_2(t,2,\theta_h)|$$

for any sequence $\{\theta_h\}_{h=0}^\infty$ which is uniformly distributed modulo 1. For example, for all $h \geq 0$ not a power of 2 take $\theta_h := \theta'_h$, where $\{\theta'_h\}_{h=0}^\infty$ is some sequence uniformly distributed modulo 1; and for all $n \geq 0$ take θ_{2^n} such that $\sum_{h=0}^{2^n} \log_2|E_2(t,2,\theta_h)| < -(2^n+1)$. We leave it to the reader to prove that then $\{\theta_h\}_{h=0}^\infty$ is uniformly distributed modulo 1 and

$$\liminf_{k\to\infty} \frac{1}{k}\sum_{h=0}^{k-1} \log_2|E_2(t,2,\theta_h)| \leq -1.$$

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References

[1] M. Drmota and G. Larcher. The sum-of-digits-function and uniform distribution modulo 1. *J. Number Theory*, 89(1):65–96, 2001.

[2] M. Drmota, J. Rivat, and T. Stoll. The sum of digits of primes in $\mathbb{Z}[i]$. *Monatsh. Math.*, 155:317–348, 2008.

[3] A. Gelfond. Sur les nombres qui ont des propriétés additives et multiplicatives données. *Acta Arith.*, 13(3):259–265, 1968.

[4] R. L. Goodstein. On the restricted ordinal theorem. *J. Symbolic Logic*, 9(2):33–41, 1944.

[5] P. J. Grabner and R. F. Tichy. Contributions to digit expansions with respect to linear recurrences. *J. Number Theory*, 36(2):160–169, 1990.

[6] D.-H. Kim. On the joint distribution of q-additive functions in residue classes. *J. Number Theory*, 74(2):307–336, 1999.

[7] L. Kirby and J. Paris. Accessible independence results for Peano arithmetic. *Bull. Lond. Math. Soc.*, 14:285–293, 1982.

[8] L. Kuipers and H. Niederreiter. *Uniform Distribution of Sequences*. Dover Publications, Incorporated, 2012.

[9] M. Lamberger and J. M. Thuswaldner. Distribution properties of digital expansions arising from linear recurrences. *Math. Slovaca*, 53:1–20, 2003.

[10] P. Liardet and P. Grabner. Harmonic properties of the sum-of-digits function for complex bases. *Acta Arith.*, 91(4):329–349, 1999.

[11] M. G. Madritsch. The sum-of-digits function of canonical number systems: Distribution in residue classes. *J. Number Theory*, 132(12):2756–2772, 2012.

[12] M. Olivier. Répartition des valeurs de la fonction “somme des chiffres”. *Séminaire Delange-Pisot-Poitou. Théorie des nombres*, 12:1–5, 1970-1971.

[13] W. Rudin. *Real and Complex Analysis, 3rd Ed.* Mathematics series. McGraw-Hill, Inc., New York, NY, USA, 1987.

[14] J. M. Thuswaldner. The sum of digits function in number fields: Distribution in residue classes. *J. Number Theory*, 74(1):111–125, 1999.