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A COMBINATORIAL PROOF ON PARTITION FUNCTION PARITY

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Abstract

One of the most basic results on the number-theoretic properties of the partition function $p(n)$ is that $p(n)$ takes each value of parity infinitely often. First proved by Kolberg in 1959, this statement was strengthened by Kolberg and Subbarao in 1966 to say that both $p(2n)$ and $p(2n+1)$ take each value of parity infinitely often. These results have received several other proofs, each relying to a certain extent on manipulating generating functions. We give a new, self-contained proof of Subbarao's result by constructing a series of bijections and involutions, along the way getting a more general theorem concerning the enumeration of a special subset of integer partitions.

1. Introduction

A *partition* λ of a positive integer n is a nonincreasing list of positive integer *parts* $\lambda_1, \ldots, \lambda_k$ that sum to n. The *partition function* $p(n)$ counts the partitions of n.

The number-theoretic properties of $p(n)$ have been studied quite extensively. For example, Kolberg [3] proved in 1959 and Newman [4] proved independently in 1962 that $p(n)$ takes each value of parity infinitely often, with Fabrykowski and Subbarao [1] and Robbins [6] giving new proofs of this result in 1990 and 2004, respectively. Subbarao [7] strengthened the result in 1966 by proving that $p(2n+1)$ takes each value of parity infinitely often, though he was unable to prove the analogous result for $p(2n)$; this was later proved by Kolberg in private correspondence to Subbarao. Subbarao conjectured that $p(tn+r)$ takes each value of parity infinitely often for every pair r and t of integers satisfying $0 \leq r < t$. Over the years several authors confirmed this conjecture for various values of t, including the case $t = 16$ by Hirschhorn and Subbarao [2] in 1988. In 1995, Ono [5] proved that $p(tn+r)$ is either always even or takes each value of parity infinitely often. All of these proofs rely to some extent on manipulating generating functions.

We give a new self-contained proof that both $p(2n)$ and $p(2n+1)$ take each value of parity infinitely often. We show these results follow from a more general theorem on the enumeration of certain partitions of integers along arithmetic progressions, whose proof relies on a series of bijections rather than generating functions. We wonder if similar techniques could be applied to plane partitions.

2. Results

We let $\lambda_1, \ldots, \lambda_k$ denote the nonincreasing list of parts of a partition λ. For positive integers a and b, define $D_{a,b}(n)$ as the set of partitions of n into distinct parts each congruent to b modulo a.

Theorem 2.1. *Let integers a, b, c, and d satisfy $a \geq b \geq 1$, $c \geq 0$, and $d \geq 2$, and set $A = \{n : n \equiv bc \mod a\}$. Then there exist integers r and s satisfying $0 \leq r < s < d$ such that $|D_{a,b}(n)| \equiv r \mod d$ for infinitely many $n \in A$ and $|D_{a,b}(n')| \equiv s \mod d$ for infinitely many $n' \in A$.*

Proof. To show at least two congruence classes modulo d are hit by $|D_{a,b}(n)|$ for infinitely many $n \in A$, it suffices to show that for every m there exists $n \in A$ satisfying $n \geq m$ and $|D_{a,b}(n-a)| \not\equiv |D_{a,b}(n)| \mod d$. To this end, for $j \geq 1$ we define a set $D^j_{a,b}(n)$ containing certain partitions in $D_{a,b}(n)$ having j parts or more.

$$D^j_{a,b}(n) = \begin{cases} D_{a,b}(n) & j = 1 \\ \{\lambda \in D_{a,b}(n) : \lambda_1 - \lambda_2 = \cdots = \lambda_{j-1} - \lambda_j = a\} & j > 1 \end{cases}$$

Note that $D^{j+1}_{a,b}(n) \subseteq D^j_{a,b}(n)$ for $j \geq 1$. Since all parts of partitions in $D^j_{a,b}(n)$ lie in the same congruence class modulo a, a partition $\lambda \in D^j_{a,b}$ fails to be in $D^{j+1}_{a,b}(n)$ when either λ_{j+1} does not exist or $\lambda_j - \lambda_{j+1} = ta$ with $t > 1$. If $D^{j+1}_{a,b}(n) \neq \emptyset$, then $n \geq \sum_{i=0}^{j}(ai + b)$, so every partition $\lambda \in D^j_{a,b}(n)$ satisfies $\lambda_j \geq a + b$ (since $n = \sum_{i=0}^{j-1}(ai + b)$ otherwise).

If $j \geq 1$ and $|D^{j+1}_{a,b}(n)| \not\equiv 0 \mod d$, then $|D^j_{a,b}(n - aj)| \not\equiv |D^j_{a,b}(n)| \mod d$: we show this by constructing a bijection $\phi^j_n : (D^j_{a,b}(n) - D^{j+1}_{a,b}(n)) \to D^j_{a,b}(n - aj)$, which trims parts of partitions $\lambda \in D^j_{a,b}(n) - D^{j+1}_{a,b}(n)$ using the following rule.

$$(\phi^j_n(\lambda))_i = \begin{cases} \lambda_i - a & 1 \leq i \leq j \\ \lambda_i & i > j \end{cases}$$

Let $k = am+c$ and $n_1 = \sum_{i=0}^{k-1}(ai+b)$, so $n_1 \equiv bc \mod a$. Consider the partition λ of n_1 with k parts given by $\lambda_i = a(k - i) + b$; clearly λ is the only partition in $D^k_{a,b}(n_1)$, so $|D^k_{a,b}(n_1)| = 1 \not\equiv 0 \mod d$. This yields

$$|D^{k-1}_{a,b}(n_1 - a(k-1))| \not\equiv |D^{k-1}_{a,b}(n_1)| \mod d$$

so we can pick $n_2 \in \{n_1 - a(k-1), n_1\}$ to satisfy $|D^{k-1}_{a,b}(n_2)| \not\equiv 0 \mod d$. Similarly,

$$|D^{k-2}_{a,b}(n_2 - a(k-2))| \not\equiv |D^{k-2}_{a,b}(n_2)| \mod d$$

so we can pick $n_3 \in \{n_2 - a(k-2), n_2\}$ to satisfy $|D^{k-2}_{a,b}(n_3)| \not\equiv 0 \mod d$. Iterate this process to compute the sequence $n_1, n_2, \ldots, n_{k-1}$.

Putting everything together, we have the following.

- $n_i \equiv n_1 \equiv bc \mod a$ for any $i < k$
- $|D^{k-i}_{a,b}(n_i - a(k-i))| \not\equiv |D^{k-i}_{a,b}(n_i)| \mod d$ for any $i < k$
- $n_{k-1} \geq n_1 - \sum_{i=2}^{k-1} ai > \sum_{i=0}^{k-1}(ai + b - ai) = kb \geq m$

Since $D_{a,b}(n) = D^1_{a,b}(n)$, setting $n = n_{k-1}$ completes the proof. □

The *Ferrers diagram* of a partition λ is a pattern of upper left-justified dots, with λ_i dots in the ith row from the top. The *conjugate partition* of λ is the partition whose Ferrers diagram has λ_i dots in the ith column from the left.

Corollary 2.2. *Both $p(2n)$ and $p(2n+1)$ take each value of parity infinitely often.*

Proof. Partition conjugation is an involution on the set of partitions of n that fixes only the partitions whose Ferrers diagrams are symmetric about the diagonal from the upper left to lower right. Thus $p(n)$ has the same parity as the number of self-conjugate partitions of n, and the set of such partitions is in one-to-one correspondence with the set of partitions of n into distinct odd parts through the bijection that unfolds the Ferrers diagram of any self-conjugate partition about its axis of symmetry. Thus $p(n) \equiv |D_{2,1}(n)| \mod 2$. Applying Theorem 2.1 once with $(a, b, c, d) = (2, 1, 0, 2)$ and once with $(a, b, c, d) = (2, 1, 1, 2)$ yields both claims. □

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