



ON DOUBLE 3-TERM ARITHMETIC PROGRESSIONS

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asp11@sfu.ca*Received: 3/26/13, Revised: 4/23/14, Accepted: 5/27/14, Published: 8/27/14***Abstract**

In this note we are interested in the problem of whether or not every increasing sequence of positive integers $x_1x_2x_3\cdots$ with bounded gaps must contain a *double* 3-term arithmetic progression, i.e., three terms x_i , x_j , and x_k such that $i+k=2j$ and $x_i+x_k=2x_j$. We consider a few variations of the problem, discuss some related properties of double arithmetic progressions, and present several results obtained by using `RamseyScript`, a high-level scripting language.

1. Introduction

In 1987, Tom Brown and Allen Freedman ended their paper titled *Arithmetic progressions in lacunary sets* [3] with the following conjecture.

Conjecture 1. Let $(x_i)_{i\geq 1}$ be a sequence of positive integers with $1 \leq x_i \leq K$. Then there are two consecutive intervals of positive integers I, J of the same length, with $\sum_{i\in I} x_i = \sum_{j\in J} x_j$. Equivalently, if $a_1 < a_2 < \cdots$ satisfy $a_{n+1} - a_n \leq K$, for all n , then there exist $i < j < k$ such that $i+k=2j$ and $a_i + a_k = 2a_j$.

If true, Conjecture 1 would imply that if the sum of the reciprocals of a set $A = \{a_1 < a_2 < a_3 < \cdots\}$ of positive integers diverges, and $a_{n+1} - a_n \rightarrow \infty$ as $n \rightarrow \infty$, and there exists K such that $a_{i+1} - a_i \leq a_{j+1} - a_j + K$ for all $1 \leq i \leq j$, then A contains a 3-term arithmetic progression. This is a special case of the famous Erdős conjecture that if the sum of the reciprocals of a set A of positive integers diverges, then A contains arbitrarily long arithmetic progressions.

Conjecture 1 is a well-known open question in combinatorics of words and it is usually stated in the following form:

Must every infinite word over a finite alphabet consisting of positive integers contain an additive square, i.e., two adjacent blocks of the same length and the same sum?

The answer is trivially yes in case the alphabet has size at most 3. For more on this question see, for example, [2, 5, 7]. Also see [8, 10, 11].

We mention two relatively recent positive results. Freedman [5] has shown that if $a < b < c < d$ satisfy the Sidon equation $a + d = b + c$, then every word on $\{a, b, c, d\}$ of length 61 contains an additive square. His proof is a clever reduction of the general problem to several cases that are then checked by computer.

Ardal, Brown, Jungić, and Sahasrabudhe [1] proved that if an infinite word $\omega = a_1a_2a_3\cdots$ has the property that there is a constant M , such that for any positive integer n the number of possible sums of n consecutive terms in ω does not exceed M , then for any positive integer k there is a k -term arithmetic progression $\{m + id : i = 0, \dots, k - 1\}$ such that

$$\sum_{i=m+1}^{m+d} a_i = \sum_{i=m+d+1}^{m+2d} a_i = \cdots = \sum_{i=m+(k-2)d+1}^{m+(k-1)d} a_i.$$

The proof of this fact is based on van der Waerden’s theorem [12].

This note is inspired by the second statement in Conjecture 1. Before restating this part of the conjecture we introduce the following terms.

We say that a sequence of positive integers $a_1a_2a_3\cdots$ is with *bounded gaps* if there is a constant K such that

$$a_{n+1} - a_n \leq K$$

for all positive integers n .

We say that a sequence of positive integers $a_1a_2a_3\cdots$ contains a *double k -term arithmetic progression* if there are i , d , and δ such that

$$a_{i+jd} = a_i + j\delta$$

for all $j \in \{0, 1, \dots, k - 1\}$.

Problem 1. Does every increasing sequence of positive integers with bounded gaps contain a *double 3-term arithmetic progression*?

It is straightforward to check that Problem 1 is equivalent to the question above concerning additive squares: Given positive integers K and $a_1 < a_2 < a_3 < \cdots$, with $a_{i+1} - a_i \leq K$ for all $i \geq 1$, let $x_i = a_{i+1} - a_i$, $i \geq 1$. Then $x_1x_2x_3\cdots$ is an infinite word over a finite alphabet of positive integers. Given an infinite word $x_1x_2x_3\cdots$ over a finite alphabet of positive integers, define a_1, a_2, a_3, \dots recursively by $a_1 \in \mathbb{N}$, $a_{i+1} = x_i + a_i$, $i \geq 1$. Then $a_1 < a_2 < a_3 < \cdots$, and $a_{i+1} - a_i \leq K$ for

some K and all $i \geq 1$. In both cases, an additive square in $x_1x_2x_3 \cdots$ corresponds exactly to a double 3-term arithmetic progression in $a_1 < a_2 < a_3 < \cdots$.

The existence of an infinite word on four integers with no additive cubes, i.e., with no three consecutive blocks of the same length and the same sum, established by Cassaigne, Currie, Schaeffer, and Shallit [4], translates into the fact that there is an increasing sequence of positive integers with bounded gaps with no double 4-term arithmetic progression.

But what about a *double* variation on van der Waerden's theorem?

Problem 2. If the set of positive integers is finitely coloured, must there exist a colour class, say $A = \{a_1 < a_2 < a_3 < \cdots\}$ for which there exist $i < j < k$ with $a_i + a_k = 2a_j$ and $i + k = 2j$?

We have just seen that an affirmative answer to Problem 1 gives an affirmative answer to the question concerning additive squares. It is also true that an affirmative answer to Problem 1 implies an affirmative answer to Problem 2.

Proposition 1. *Assume that every increasing sequence of positive integers $x_1x_2x_3 \cdots$ with bounded gaps contains a double 3-term arithmetic progression. Then if the set of positive integers is finitely coloured, there must exist a colour class, say $A = \{a_1 < a_2 < a_3 < \cdots\}$, which contains a double 3-term arithmetic progression.*

Proof. We use induction on the number of colours, denoted by r . For $r = 1$ the conclusion trivially follows. Now assume that for every r -coloring of \mathbb{N} there exists a colour class which contains a double 3-term arithmetic progression. By the compactness principle [9, Theorem 2.4] there exists $M \in \mathbb{N}$ such that every r -coloring of $[1, M]$ (or of any translate of $[1, M]$) yields a monochromatic double 3-term arithmetic progression.

Assume now that there is an $(r + 1)$ -coloring of \mathbb{N} for which there does *not* exist a monochromatic double 3-term arithmetic progression. Let the $(r + 1)$ st colour class be $C(r + 1) = \{x_1 < x_2 < \cdots\}$. By the induction hypothesis on r colours, $C(r + 1)$ is infinite. By the assumption that every increasing sequence of positive integers $x_1x_2x_3 \cdots$ with bounded gaps contains a double 3-term arithmetic progression, $C(r + 1)$ does not have bounded gaps. In particular, there is $p \geq 1$ such that $x_{p+1} - x_p \geq M + 2$. But then the interval $[x_p + 1, x_{p+1} - 1]$ contains a translate of $[1, M]$ and is coloured with only r colours, so that $[x_p + 1, x_{p+1} - 1]$ does contain a monochromatic double 3-term arithmetic progression. This contradiction completes the proof. \square

More generally, if the set of positive integers is finitely coloured and if each colour class is regarded as an increasing sequence, must there be a monochromatic double k -term arithmetic progression, for a given positive integer k ? What if the gaps between consecutive elements coloured with same colour are pre-prescribed, say at

most 4 for the first colour, at most 6 for the second colour, and at most 8 for the third colour, and so on?

In the spirit of van der Waerden's numbers $w(r, k)$ [6] we define the following.

Definition 1. For given positive integers r and k greater than 1, let $w^*(r, k)$ be the least integer, if it exists, such that for any r -coloring of the interval $[1, w^*(r, k)]$ there is a monochromatic double k -term arithmetic progression.

For given positive numbers $r, k, a_1, a_2, \dots, a_r$ let $w^*(k; a_1, a_2, \dots, a_r)$ be the least integer, if it exists, such that for any r -coloring of the interval $[1, w^*(k; a_1, a_2, \dots, a_r)] = A_1 \cup A_2 \cup \dots \cup A_r$ such that for each i the gap between any two consecutive elements in A_i is not greater than a_i there is a monochromatic double k -term arithmetic progression.

We will show that $w^*(2, 3)$ is relatively simple to obtain. We will give lower bounds for $w^*(3, 3)$ and $w^*(4, 2)$ and a table with values of $w^*(3; a_1, a_2, a_3)$ for various triples (a_1, a_2, a_3) and propose a related conjecture.

We will share with the reader some insights related to the general question about the existence of double 3-term arithmetic progressions in increasing sequences with bounded gaps.

Finally, we will describe `RamseyScript`, a high-level scripting language developed by the third author that was used to obtain the colorings and bounds that we have established.

2. $w^*(r, 3)$

Now we look more closely at $w^*(r, 3)$, the least integer, if it exists, such that for every r -coloring of the interval $[1, w^*(r, 3)]$ there is a monochromatic double 3-term arithmetic progression.

Suppose that $w^*(r, 3)$ does not exist for some r , but $w^*(r - 1, 3)$ does exist. Then, by the Compactness Principle, there is a coloring of the positive integers with r colours, say with colour classes A_1, A_2, \dots, A_r , such that no colour class contains a double 3-term arithmetic progression. Then (a) A_1 contains no double 3-term arithmetic progression, (b) A_1 has bounded gaps because $w^*(r - 1, 3)$ exists, and (c) A_1 is infinite, because $w^*(r - 1, 3)$ exists.

Let d_1, d_2, \dots be the sequence of consecutive differences of the sequence A_1 . That is, if $A_1 = \{a_1, a_2, a_3, \dots\}$ then $d_n = a_n - a_{n-1}$, $n \geq 1$. Then the sequence d_1, d_2, \dots is a sequence over a finite set of integers which does not contain any additive square.

Thus if there exists r such that $w^*(r, 3)$ does not exist, then there exists a sequence over a finite set of integers which does not contain an additive square.

It is conceivable that proving that $w^*(r, 3)$ does not exist for all r (if this is true!) is easier than directly proving the existence of a sequence over a finite set of integers

with no additive square.

Theorem 2. $w^*(2, 3) = 17$.

Proof. Colour $[1, m]$ with two colours, with no monochromatic double 3-term arithmetic progressions. Then the first colour class must have gaps of either 1, 2, or 3. Thus the sequence of gaps of the first colour class is a sequence of 1s, 2s, and 3s, and this sequence must have length at most 7, otherwise there is an additive square, which would give a double 3-term arithmetic progression in the first colour class. Hence, the first colour class can contain at most 8 elements (only 7 consecutive differences) and similarly for the second colour class. This shows that $w^*(2, 3) \leq 8 + 8 + 1 = 17$. On the other hand, the following 2-coloring of $[1, 16]$ is with no monochromatic double 3-term arithmetic progressions:

0010110100101101.

Hence $w^*(2, 3) = 17$. □

Theorem 3. $w^*(3, 3) \geq 414$.

The following 3-coloring of $[1, 413]$ avoids monochromatic 3-term double arithmetic progressions:

```
0101102210100201200100221221010010220010112011211202210112122112202210
0110010220201122022002202001012212112122001001120121100110020022002110
2001101001121120210020011210201121122112122010110100110102201220201221
1210021122112122112200110011212200202202001212212112212200110010110012
0211212200220100112202200220200122102212211211002101220022001001100221
211010010110020022110010110010221211020220200220221001122011211.
```

This coloring is the result of about 8 trillion iterations of `RamseyScript`, using the Western Canada Research Grid¹. We started with a seed 3-coloring of the interval $[1, 61]$ and searched the entire space of extensions. Figure 1 gives the number of double 3-AP free extensions of the seed coloring versus their lengths.

To get more information about $w^*(3, 3)$ we define $w^*(3, 3; d)$ to be the smallest m such that whenever $[1, m]$ is 3-coloured so that each colour class has maximum gap at most d , then there is a monochromatic double 3-term arithmetic progression. Our goal was to compute $w^*(3, 3; d)$ for small values of d . (See Table 1.)

We note that $w^*(3, 3; d)$ is already difficult to compute when d is much smaller than $w^*(2, 3) = 17$. (In a 3-coloring containing no monochromatic double 3-term arithmetic progression the maximum gap size of any colour class is 17.)

Freedman [5] showed that there are 16 double 3-AP free 51-term sequences having the maximum gap of at most 4. The fact that $w^*(3, 3; 4) = 39$ is an interesting contrast, and shows that considering a single sequence instead of partitioning an interval of positive integers into three sequences is somewhat less restrictive.

¹<http://www.westgrid.ca>

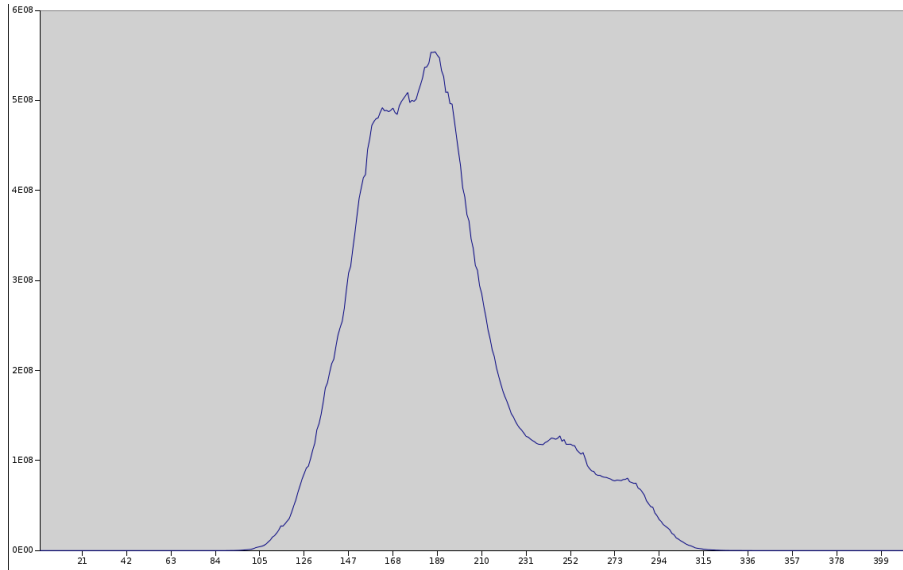


Figure 1: Number of double 3-AP free extensions versus length

	$w^*(3, 3; d)$
Max gap d	2
	3
	4
	5
	6
	7
	11
	22
	39
	100
	> 152
	?

Table 1: Known Values of $w^*(3, 3; d)$

Theorem 4. $w^*(4, 2) \geq 30830$.

Starting with the seed 2-coloring $[1, 10] = \{1, 4, 6, 7\} \cup \{2, 3, 5, 8, 9, 10\}$, after $2 \cdot 10^8$ iterations RamseyScript produced a double 4-AP free 2-coloring of the interval $[1, 30829]$, available at the web page <http://people.math.sfu.ca/~vjungic/Double/w-4-2.txt>.

3. $w^*(3; a, b, c)$ and $w^*(k; a, b)$

Recall that $w^*(3; a, b, c)$ is the least number such that every 3-coloring of $[1, w^*(3; a, b, c)]$, with gap sizes on the three colours restricted to a, b , and c , re-

spectively, has a monochromatic double 3-term arithmetic progression. Similarly, $w^*(k; a, b)$ is the least number such that every 2-coloring of $[1, w^*(k; a, b)]$, with gap sizes on the two colours restricted to a and b , respectively, has a monochromatic double k -term arithmetic progression.

Table 2 shows values of $w^*(3; a, b, c)$ for some small values of a, b , and c . Table 3 shows values of $w^*(k; a, b)$ for some small values of a, b , and k .

		Max Green Gaps				
		3	4	5	6	7+
Max Blue Gaps	3	22				
	4	31	31			
	5	33	38	43		
	6	33	41	44	45	
	7	33	41	46	46	46
	8+	33	41	46	46	47

Max Red Gap 3

		Max Green			
		5	6	7	8+
Max Blue	5	100			
	6	> 113	> 133		
	7	?	?	?	
	8+	?	?	?	?

Max Red Gap 5

		Max Green Gaps					
		4	5	6	7	8	9+
Max Blue Gaps	4	39					
	5	49	63				
	6	56	79	91			
	7	76	96	>105	>121		
	8	81	96	>114	>131	>131	
	9	81	96	>114	>133	>133	>133
	10	81	96	>114	>133	>135	>135
11+	81	97	>114	>133	>135	>135	

Max Red Gap 4

Table 2: Known Values and Bounds for $w^*(3; a, b, c)$

Based on this evidence, we propose the following conjecture.

Conjecture 2. The number $w^*(3, 3)$ exists. The numbers $w^*(4, 3)$ and $w^*(2, 4)$ do not exist.

Our guess would be that $w^*(3, 3) < 500$. Also we recall that $w^*(2, 3) = 17$ and $w^*(4, 2) \geq 30830$.

		Red	
		2	3
Blue	2	7	
	3	11	17

Double 3-AP's

		Red		
		2	3	4+
Blue	2	11		
	3	22	> 176	
	4+	22	> 2690	> 3573

Double 4-AP's

		Red			
		2	3	4	5+
Blue	2	15			
	3	37	> 131000		
	4	> 25503	?	?	
	5+	> 33366	?	?	?

Double 5-AP's

Table 3: Known Values and Bounds for $w^*(k; a, b)$

4. Double 3-term Arithmetic Progressions in Increasing Sequences of Positive Integers

In this section, we return to Problem 1: the existence of double 3-term arithmetic progressions in infinite sequences of positive integers with bounded gaps.

We remind the reader of the meaning of the following terms from combinatorics of words.

An infinite word over a finite subset S of \mathbb{Z} , called the *alphabet*, is defined as a map $\omega : \mathbb{N} \rightarrow S$ and is usually written as $\omega = x_1x_2\cdots$, with $x_i \in S, i \in \mathbb{N}$. For $n \in \mathbb{N}$, a *factor* B of the infinite word ω of length $n = |B|$ is the image of a set of n consecutive positive integers by ω , $B = \omega(\{i, i + 1, \dots, i + n - 1\}) = x_i x_{i+1} \cdots x_{i+n-1}$. The *sum* of the factor B is $\sum B = x_i + x_{i+1} + \cdots + x_{i+n-1}$. A factor $B = \omega(\{1, 2, \dots, n\}) = x_1 x_2 \cdots x_n$ is called a *prefix* of ω .

Theorem 5. *The following statements are equivalent:*

- (1) *For all $k > 1$, every infinite word on $\{1, 2, \dots, k\}$ has two adjacent factors with equal length and equal sum.*
- (1a) *For all $k > 1$, there exists $R = R(k)$ such that every word on $\{1, 2, \dots, k\}$ of length R has two adjacent factors with equal length and equal sum.*
- (2) *For all $n > 1$, if $x_1 < x_2 < x_3 < \cdots$ is an infinite sequence of positive integers such that $x_{i+1} - x_i \leq n$ for all $i > 1$, then there exist $1 \leq i < j < k$ such that $x_i + x_k = 2x_j$ and $i + k = 2j$.*

(2a) For all $n > 1$, there exists $S = S(n)$ such that if $x_1 < x_2 < x_3 < \dots < x_S$ are positive integers with $x_{i+1} - x_i \leq n$ whenever $1 \leq i \leq S - 1$, then there exist $1 \leq i < j < k \leq S$ such that $x_i + x_k = 2x_j$ and $i + k = 2j$.

(3) For all $t > 1$, if $\mathbb{N} = A_1 \cup A_2 \cup \dots \cup A_t$, then there exists q , $1 \leq q \leq t$, such that if $A_q = \{x_1 < x_2 < \dots\}$, then there are $1 \leq i < j < k$ such that $x_i + x_k = 2x_j$ and $i + k = 2j$.

(3a) For all $t > 1$, there exists $T = T(t)$ such that for all $a > 1$, if $\{a, a + 1, \dots, a + T - 1\} = A_1 \cup A_2 \cup \dots \cup A_t$, then there exists q , $1 \leq q \leq t$, such that if $A_q = \{x_1 < x_2 < \dots < x_p\}$, then there are $1 \leq i < j < k$ such that $x_1 + x_k = 2x_j$ and $i + k = 2j$.

Remark 1. Note that in (3) and (3a) the statements concern coverings and not partitions (colorings). This turns out to be essential, since if we used colorings in (3) and (3a) (call these new statements (3') and (3a')), then (3') would not imply (2), although (2) would still imply (3a'). This can be seen from the proofs below.

Remark 2. In each case $i = 1, 2, 3$, the statement (ia) is the finite form of the statement (i).

Proof. We start by proving that (2) implies (2a). (The proof that (1) implies (1a) follows the same form, and is a little more routine.)

Suppose that (2a) is false. Then there exists n such that for all $S > 1$ there are $x_1 < x_2 < x_3 < \dots < x_S$, with $x_{i+1} - x_i \leq n$ whenever $1 \leq i \leq S - 1$, such that there do not exist $1 \leq i < j < k \leq S$ such that $x_i + x_k = 2x_j$ and $i + k = 2j$. Replace $x_1 < x_2 < x_3 < \dots < x_S$ by its characteristic binary word (of length x_S)

$$B_S = b_1 b_2 b_3 \dots b_{x_S}$$

defined by $b_i = 1$ if i is in $\{x_1, x_2, x_3, \dots, x_S\}$, and $b_i = 0$ otherwise. Let H be the (infinite) collection of binary words obtained in this way. Note that if B_S is in H , then each pair of consecutive 1s in B_S is separated by at most $n - 1$ 0s.

Now construct, inductively, an infinite binary word w such that each prefix of w is a prefix of infinitely many words B_S in H in the following way. Let w_1 be a prefix of an infinite set H_1 of words in H . Let $w_1 w_2$ be a prefix of an infinite set H_2 of words in H_1 . And so on. Set $w = w_1 w_2 \dots$.

Define $x_1 < x_2 < x_3 < \dots$ so that w is the characteristic word of $x_1 < x_2 < x_3 < \dots$ and note that $x_{i+1} - x_i \leq n$ for all $i > 1$. Now it follows that there cannot exist $1 \leq i < j < k$ with $x_1 + x_k = 2x_j$ and $i + k = 2j$. (For these i, j, k would occur inside some prefix of w . But that prefix is itself a prefix of some word $B_S = b_1 b_2 b_3 \dots b_S$, where there do not exist such i, j, k .) Thus if (2a) is false, (2) is false.

Next we prove that (3) implies (3a). Suppose that (3a) is false. Then there exists t such that for all T there is, without loss of generality, a covering $\{1, 2, \dots, T\} = A_1 \cup A_2 \cup \dots \cup A_t$ such that there does not exist q with $A_q = \{x_1 < x_2 < \dots < x_p\}$ and $i < j < k$ with $x_1 + x_k = 2x_j$ and $i + k = 2j$. Represent the cover $\{1, 2, \dots, T\} = A_1 \cup A_2 \cup \dots \cup A_t$ by a word $B_T = b_1 b_2 b_3 \dots b_T$ on the alphabet consisting of the non-empty subsets of $\{1, 2, \dots, t\}$. Here for each i , $1 \leq i \leq T$, $b_i = \{\text{the set of } p, 1 \leq p \leq t, \text{ such that } i \text{ is in } A_p\}$. Let H be the set of all words B_T obtained in this way. Construct an infinite word w , on the alphabet consisting of the non-empty subsets of $\{1, 2, \dots, t\}$, such that each prefix of w is a prefix of infinitely many of the words B_T in H . Thus w represents a cover $\mathbb{N} = A_1 \cup A_2 \cup \dots \cup A_t$, where $A_i = \{j \geq 1 \text{ such that } i \text{ is in } w_j\}$, $1 \leq i \leq t$, for which there does not exist i , $A_i = \{x_1 < x_2 < \dots\}$, with $1 \leq i < j < k$ such that $x_1 + x_k = 2x_j$ and $i + k = 2j$, contradicting (3).

It is not difficult to show that (1) is equivalent to (2), that (1) is equivalent to (1a), that (2a) implies (2), and that (3a) implies (3). We have shown that (2) implies (2a) and that (3) implies (3a).

The final steps are:

Proof that (3) implies (2). If n and $A_0 = \{x_1 < x_2 < x_3 < \dots\}$ are given, with $x_{i+1} - x_i \leq n$ for all $i > 1$, let $A_i = A_0 + i$, $0 \leq i \leq n - 1$. Then $\mathbb{N} = A_0 \cup A_1 \cup \dots \cup A_{n-1}$, and now (3) implies (2).

Proof that (2) implies (3a). Assume (2), and use induction on t to show that if $\mathbb{N} = A_0 \cup A_1 \cup \dots \cup A_{t-1}$, then there exists q , $0 \leq q \leq t-1$, with $A_q = \{x_1 < x_2 < \dots\}$ for which there exist $i < j < k$ with $x_i + x_k = 2x_j$ and $i + k = 2j$. For $t = 1$ this is trivial. Fix $t > 1$, assume the statement 3a for this t , and let $\mathbb{N} = A_0 \cup A_1 \cup \dots \cup A_t$. If $A_t = \{x_1 < x_2 < \dots\}$ is either finite or there exists n with $x_{i+1} - x_i \leq n$ for all $i > 1$, then we are done by 2.

Otherwise, there are arbitrarily long intervals $[a, b] = B$ which are subsets of $A_0 \cup A_1 \cup \dots \cup A_{t-1}$, and we are done by the induction hypothesis. \square

Remark 3. If true, perhaps (3a) can be proved by a method such as van der Waerden’s proof that any finite coloring of \mathbb{N} has a monochromatic 3-AP.

Here is another remark on double 3-term arithmetic progressions.

Theorem 6. *The following two statements are equivalent:*

- (1) *For all $n \geq 1$, every infinite sequence of positive integers $x_1 < x_2 < \dots$ such that $x_{i+1} - x_i \leq n$ contains a double 3-term arithmetic progression.*
- (2) *For all $n \geq 1$, every infinite sequence of positive integers $x_1 < x_2 < \dots$ such that $x_{i+1} - x_i \leq n$ contains a double 3-term arithmetic progression x_i, x_j, x_k with the property that $j - i = k - j \geq m$ for any fixed $m \in \mathbb{N}$.*

Proof. Certainly (2) implies (1). We prove that (1) implies (2).

Let n and m be given positive integers. Let $X = \{x_1 < x_2 < \dots\}$ be an infinite sequence with gaps from $\{1, \dots, n\}$. For $j \in \mathbb{N}$ we define $y_j = x_{jm+1} - x_{(j-1)m+1}$. Note that $m \leq y_j \leq nm$. Next we define an increasing sequence $Z = \{z_1 < z_2 < \dots\}$ with gaps from $\{m, m+1, \dots, nm\}$ by

$$z_i = \sum_{j=1}^i y_j = \sum_{j=1}^i x_{jm+1} - \sum_{j=0}^{i-1} x_{jm+1}.$$

By (1) the sequence Z contains a double 3-term arithmetic progression z_p, z_q, z_r with

$$z_r - z_q = z_q - z_p \text{ and } p + r = 2q.$$

It follows that

$$\sum_{j=q+1}^r x_{jm+1} - \sum_{j=q}^{r-1} x_{jm+1} = \sum_{j=p+1}^q x_{jm+1} - \sum_{j=p}^{q-1} x_{jm+1}$$

and

$$x_{rm+1} - x_{qm+1} = x_{qm+1} - x_{pm+1}.$$

From

$$(pm + 1) + (rm + 1) = m(p + r) + 2 = 2mq + 2 = 2(mq + 1)$$

we conclude that $x_{pm+1}, x_{qm+1}, x_{rm+1}$ form a double 3-term arithmetic progression with gap

$$rm + 1 - (qm + 1) = (r - q)m \geq m.$$

Since m and X are arbitrary, we conclude that (2) holds. □

We wonder if one could get some intuitive “evidence” that it is easier to show that $w^*(3, 3)$ exists than it is to show that every increasing sequence with gaps from $\{1, 2, 3, \dots, 17\}$ has a double 3-term arithmetic progression. The “17” is chosen because in a 3-coloring of $[1, m]$ which has no monochromatic double 3-AP, the gaps between elements of this color class are colored with 2 colors, and $w^*(2, 3) = 17$.

The utility `RamseyScript` was used for search of an increasing sequence with gaps from $\{1, 2, 3, \dots, 17\}$ with no double 3-term arithmetic progressions. The first search produced a sequence of the length 2207. The histogram with the distribution of gaps in this sequence is given on Figure 2.

In another attempt we changed the order of gaps in the search, taking

$$[16, 12, 11, 17, 10, 14, 15, 8, 5, 3, 6, 4, 2, 1, 13, 7, 9]$$

instead of $[1, 2, \dots, 17]$. `RamseyScript` produced a 5234-term double 3-AP free sequence. The corresponding histogram of gaps is given on Figure 3.

Here are a few conclusion that one can make from this experiment.

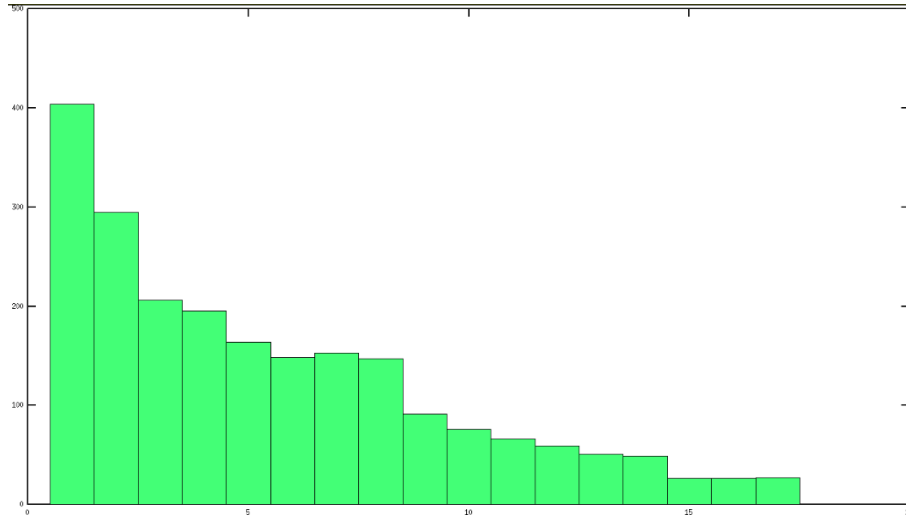


Figure 2: Histogram of Gaps in a 2207-term Double 3-AP Free Sequence

1. Initial choices of the order of gaps matter very much when constructing a double 3-AP free sequence, because we cannot backtrack in a reasonable (human) timespan at these lengths.
2. We do not really know anything about how long a sequence there will be.
3. The search space is very big. Table 4 gives the recursion tree size vs. maximum sequence length considered.

Max. Sequence	Size of Search
0	1
1	18
2	307
3	4931
4	78915
5	1216147
6	18695275
7	278661995
8	???

Table 4: Recursion Tree Size vs. Maximum Sequence Length

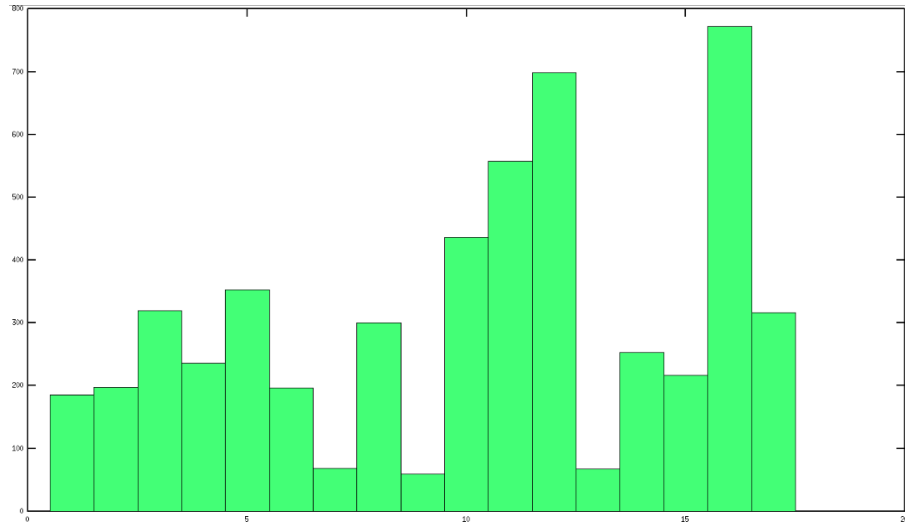


Figure 3: Histogram of Gaps in a 5234-term Double 3-AP Free Sequence

5. RamseyScript

To handle the volume and variety of computation required by this project and related ones, we use the utility `RamseyScript`, developed by the third author, which provides a high-level scripting language. In creating `RamseyScript`, we had two goals:

- To provide a uniform framework for Ramsey-type computational problems (which despite being minor variations of each other, are traditionally handled by *ad hoc* academic code).
- To provide a correct and efficient means to actually carry out these computations.

To achieve these goals, `RamseyScript` appears to the user as a declarative scripting language which is used to define a backtracking algorithm to be run. It exposes three main abstractions: search space, filters and targets.

The *search space* is a set of objects — typically r -colorings of the natural numbers or sequences of positive integers — which can be generated recursively and checked to satisfy certain conditions, such as being squarefree or containing no monochromatic progressions.

The conditions to be checked are specified as *filters*. Typically when extending `RamseyScript` to handle a new type of problem, only a new filter needs to be written. This saves development time and effort compared to writing a new program, while

also making available additional features, e.g., for splitting the problem across a computing cluster.

Finally, *targets* describe the information that should be shown to the user. The default target, `max-length`, informs the user of the largest object in the search space which passed the filters.

With these parameters set, `RamseyScript` then runs a standard backtracking algorithm, which essentially runs as follows:

1. Start with some element x in the search space. For example, x might be the trivial coloring of the empty interval.
2. Check that x passes each filter. If not, skip steps 3 and 4.
3. Check each target against x (e.g., is x the longest coloring obtained so far?).
4. For each possible extension \hat{x} of x , repeat step 2. For example, if x is the interval $[1, n]$ and the search space is the set of r -colorings, then the possible extensions of x are the r colorings of $[1, n + 1]$ which match x on the first n elements.
5. Output the current state of all targets.

Here is an example script to demonstrate these ideas and syntax:

```
# Output a brief description
echo Find the longest interval [1, n] that cannot be 4-colored
echo without a monochromatic 3-AP or a rainbow 4-AP.

# Set up environment
set n-colors 4
set ap-length 3

# Choose filters
filter no-n-aps
filter no-rainbow-aps

# Use the default target (max-length)

# Backtrack on the space of 4-colorings
search colorings
```

Its output is

```
find the longest interval [1, n] that cannot be 4_colored
without a monochromatic 3_ap or a rainbow 4_ap.
```

```

Added filter 'no-3-aps'.
Added filter 'no-rainbow-aps'.
#### Starting coloring search ####
  Targets: max-length
  Filters: no-rainbow-aps no-3-aps
  Dump data:
  Seed: [[] [] [] []]
Max. coloring (len 56): [[removed due to length]]
Time taken: 7s. Iterations: 4546107
#### Done. ####

```

RamseyScript has many options to control the backtracking algorithm and its output. For full details see the README, available alongside its source code at <https://www.github.com/apoelstra/RamseyScript>. It is licensed under the Creative Commons 0 public domain dedication license.

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