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**BUILDING A BETTER BIJECTION BETWEEN CLASSES OF COMPOSITIONS****James D. Diffenderfer***Dept. of Mathematical Sciences, Georgia Southern University, Statesboro, Georgia*  
jdifferfer@georgiasouthern.edu*Received: 1/19/14, Accepted: 5/27/14, Published: 8/27/14***Abstract**

A bijective proof is given for the following theorem: The number of compositions of  $n$  into parts congruent to  $a \pmod{b}$  equals the number of compositions of  $n + b - a$  into parts congruent to  $b \pmod{a}$  that are greater than or equal to  $b$ . The bijection is then shown to preserve palindromicity.

**1. Introduction**

A *composition* of an integer  $n$  is a representation of  $n$  as a sum of strictly positive integers called *parts*. A composition,  $\mu$ , with parts  $x_1, x_2, \dots, x_m$  is represented by  $\mu : x_1 + x_2 + \dots + x_m$ . The aim of this paper is to provide a bijective proof of the following theorem:

**Theorem 1.1.** *Suppose  $a \leq b$ . The number of compositions of  $n$  into parts congruent to  $a \pmod{b}$  equals the number of compositions of  $n + b - a$  into parts congruent to  $b \pmod{a}$  where each part is greater than or equal to  $b$ .*

The search for a bijective proof of this theorem was first inspired by Sills [4] who produced a bijective proof of the restricted case when  $a = 1$  and  $b = 2$ . This bijection was then generalized to establish a bijection for the stronger case when  $a = 1$  and  $b$  was any positive integer [1]. While Theorem 1.1 can be easily proven using generating functions, at the time of [1] a bijective proof seemed elusive. As we shall see, this is possibly because the bijection presented in this paper does not follow from the previous bijections and is, perhaps, even more elegant in its simplicity. In the spirit of [1], after establishing the bijective proof of Theorem 1.1 we will show that the bijective map preserves palindromicity and thereby prove that the two classes are also equal for palindromic compositions.

**2. Building a Better Bijection**

*Bijection Proof of Theorem 1.1.* Let  $\mu : x_1 + x_2 + x_3 + \dots + x_m$  be a composition of  $n$  with parts congruent to  $a \pmod{b}$ . Since each  $x_i \equiv a \pmod{b}$  let  $x_i = a + bk_i$ ,

where  $k_i \geq 0$  for  $1 \leq i \leq m$ . We can now think of the composition as

$$a + bk_1 + a + bk_2 + a + bk_3 + \cdots + a + bk_{m-1} + a + bk_m.$$

Let  $\{s_i\}_{i=1}^\ell$  be the subsequence of the sequence of positive integers such that  $k_{s_i} \neq 0$ , for  $\ell \leq m$ . Thus, our composition more accurately looks like

$$\underbrace{a + \cdots + a}_{s_1} + bk_{s_1} + \underbrace{a + \cdots + a}_{s_2 - s_1} + bk_{s_2} + \cdots + \underbrace{a + \cdots + a}_{s_\ell - s_{\ell-1}} + bk_{s_\ell} + \underbrace{a + \cdots + a}_{m - s_\ell}.$$

We can now combine these groups of  $a$ 's into one part which gives us

$$as_1 + bk_{s_1} + a(s_2 - s_1) + bk_{s_2} + \cdots + a(s_\ell - s_{\ell-1}) + bk_{s_\ell} + a(m - s_\ell).$$

Since we are mapping this to a composition of  $n + b - a$  we now place  $b$  in the left most position and add  $(-a)$  to  $as_1$  resulting in the composition

$$b + a(s_1 - 1) + bk_{s_1} + a(s_2 - s_1) + bk_{s_2} + \cdots + a(s_\ell - s_{\ell-1}) + bk_{s_\ell} + a(m - s_\ell).$$

At this point we take one  $b$  from each  $bk_{s_i}$  and adjoin it with the multiple of  $a$  to the right to form a new part. Letting  $y_0 = b + a(s_1 - 1)$ ,  $y_\ell = b + a(m - s_\ell)$ , and  $y_i = b + a(s_{i+1} - s_i)$  for  $1 \leq i < \ell$ , we have the composition

$$y_0 + b(k_{s_1} - 1) + y_1 + b(k_{s_2} - 1) + y_2 + \cdots + b(k_{s_\ell} - 1) + y_\ell.$$

We now split the remaining multiples of  $b$  up into sums yielding

$$\mu' : y_0 + \underbrace{b + \cdots + b}_{k_{s_1} - 1} + y_1 + \underbrace{b + \cdots + b}_{k_{s_2} - 1} + y_2 + \cdots + \underbrace{b + \cdots + b}_{k_{s_\ell} - 1} + y_\ell. \tag{1.1.1}$$

which is a composition of  $n + b - a$  in which every part is greater than or equal to  $b$  and each part is congruent to  $b \pmod{a}$ .

To map this composition back to the original one, split each  $y_i$  up into  $b + ar_i$  and combine adjacent  $b$ 's into the part  $bk_{s_i}$ , for  $0 \leq i \leq \ell$ . Add  $(-b)$  to the left most  $b$  in the composition and then add  $a$  to  $ar_0$ . Finally, separate the multiples of  $a$  into sums of  $a$ 's and combine one  $a$  that is left adjacent with  $bk_{s_i}$  to form the part  $x_i = a + bk_{s_i}$ . □

### 3. Preserving Palindromicity

A *palindromic composition* of  $n$  is a composition in which the parts are ordered such that they are read the same forward and backwards. Thus,  $\mu : x_1 + x_2 + \cdots + x_m$  is a palindromic composition iff  $x_{m-j+1} = x_j$ , for  $1 \leq j \leq m$ .

**Corollary 3.1.** *The number of palindromic compositions of  $n$  into parts congruent to  $a \pmod{b}$  equals the number of palindromic compositions of  $n + b - a$  into parts congruent to  $b \pmod{a}$  that are greater than or equal to  $b$ .*

*Proof.* Let  $\mu : x_1 + x_2 + \dots + x_m$  be a palindromic composition of  $n$  with parts congruent to  $a \pmod{b}$ . In the bijection, we let  $x_i = a + bk_i$ . Since  $\mu$  is palindromic,

$$x_{m-j+1} = x_j \iff a + bk_{m-j+1} = a + bk_j \iff k_{m-j+1} = k_j.$$

Thus, when we first rewrite the composition as

$$a + bk_1 + a + bk_2 + a + bk_3 + \dots + a + bk_{m-1} + a + bk_m$$

it is not palindromic because of the leftmost  $a$ . However, later in the bijection we add  $(-a)$  to this  $a$  which would cause this to be a palindromic composition. Say we have added  $(-a)$  to the first  $a$  to make the composition

$$bk_1 + a + bk_2 + a + bk_3 + \dots + a + bk_{m-1} + a + bk_m.$$

Once again, letting  $\{s_i\}_{i=1}^\ell$  be the subsequence of integers such that  $k_{s_i} \neq 0$ , we see that the previous composition is, more accurately,

$$\underbrace{a + \dots + a}_{s_1-1} + bk_{s_1} + \underbrace{a + \dots + a}_{s_2-s_1} + bk_{s_2} + \dots + \underbrace{a + \dots + a}_{s_\ell-s_{\ell-1}} + bk_{s_\ell} + \underbrace{a + \dots + a}_{m-s_\ell}.$$

Since we know this to be palindromic, we find that  $s_1 - 1 = m - s_\ell$  and  $s_{\ell-j+1} - s_{\ell-j} = s_{j+1} - s_j$ , for  $1 \leq j < \ell$ , and that  $k_{s_{\ell-i+1}} = k_{s_i}$ , for  $1 \leq i \leq \ell$ . This implies that  $y_{\ell-j+1} = y_j$  in the composition  $\mu'$  given in (1.1.1), thereby proving the map preserves palindromicity.  $\square$

#### 4. Conclusion

Clearly the bijection presented here is much more simplistic as it requires no outside machinery, such as binary representation [3, Sec. IV, Ch. 1, p. 151] which was employed in the less general bijections. Corollary 3.1 can also be established using generating functions for palindromic compositions found in [2, p. 350-351].

#### References

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