FIBONACCI NUMBERS WITH PRIME SUMS OF COMPLEMENTARY DIVISORS

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Abstract

Here, we study the set of positive integers n such that with F_n being the nth Fibonacci number, the number $F_n/d + d$ is prime for all proper divisors d of F_n.

1. Introduction

In [2], Becheanu, Luca and Shparlinski studied the set of positive integers

$$\mathcal{N} = \{n : n/d + d \text{ is prime for all divisors } d \text{ of } n\}.$$

They noted that if p is a prime such that $2p + 1$ and $p + 2$ are both primes, then $n = 2p \in \mathcal{N}$, since in that case, the sums $n/d+d$ for $d \mid n$ are in the set $\{2p+1, p+2\}$. The main result in [2] is that the subset $\mathcal{N}$ is of asymptotic density zero, and in fact

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the estimate $\#\mathcal{N}(x) = O(x/(\log x)^3)$ holds. Here and in what follows, for a subset $\mathcal{A}$ of the positive integers and a real number $x \geq 1$, we write $\mathcal{A}(x) = \mathcal{A} \cap [1, x]$. Here, we study the same problem for the Fibonacci numbers F_n given by $F_0 = 0,\ F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. We start by remarking that for the Fibonacci numbers we have to slightly modify our requirement that $F_n/d + d$ is always a prime for all divisors d of F_n, because when $d = 1$ (or $d = F_n$) and $n = 4k + r$, $r \in \{0, 1, 2, 3\}$, we have the identities

$$\begin{array}{rcl} F_{4k} + 1 & = & F_{2k-1}L_{2k+1} \qquad F_{4k+1} + 1 = F_{2k+1}L_{2k} \\ F_{4k+2} + 1 & = & F_{2k+2}L_{2k} \qquad F_{4k+3} + 1 = F_{2k+1}L_{2k+2} \end{array}$$

for all positive integers k (see [3]), where $\{L_n\}_{n \geq 0}$ is the companion Lucas sequence of the Fibonacci sequence given by $L_0 = 2,\ L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ for all $n \geq 0$. In particular, $F_n + 1$ is never prime for $n \geq 4$. So, we put

$$\mathcal{M} = \{n : F_n/d + d \text{ is prime for all divisors } d \text{ with } 1 < d < F_n \text{ of } F_n\}.$$

Note that 9 and 15 are members of $\mathcal{M}$. Indeed, the divisors d of $N = F_9 = 34$ are $1, 2, 17, 34$, and the only sum $d + N/d$ with $1 < d < N$ is $2 + 17 = 19$, which is a prime. Similarly, the only divisors d of $N = F_{15} = 610$ and $1, 2, 5, 10, 61, 122, 305, 610$ and the only sums of the form $d + N/d$ for $1 < d < N$ are $2 + 305 = 307,\ 5 + 122 = 127$ and $10 + 61 = 71$, which are all prime.

Our main result is the following.

Theorem 1. *The set $\mathcal{M}$ has asymptotic density zero. Additionally, the estimate $\#\mathcal{M}(x) = O(x/(\log\log x)^{3/16})$ holds for all $x > 10$.*

The proof uses an assortment of tools from elementary/analytic number theory, the most important one being a result of Heath–Brown [6] to the effect that if $q,\ r,\ s$ are three different primes, then there exist infinitely many primes p such that one of $q,\ r,\ s$ is a primitive root modulo p.

2. Preliminary Results

The first result, is a sieve result due to Heath-Brown. In order to state it, we need some definitions. Let $\alpha,\ \delta \in (0, 1/2)$ with $\alpha + \delta < 1/2$. We let $P_2(\alpha, \delta)$ be the set of numbers n which are either primes, or $n = p_1 p_2$, where $p_1 < p_2$ are primes and $n^\alpha < p_1 < n^{1/2-\delta}$. The following result is Lemma 3 in [6]. For an odd prime p and an integer a we write $\left(\dfrac{a}{p}\right)$ for the Legendre symbol of a with respect to p.

Lemma 1. *Let $q,\ r,\ s$ be three primes, $k \in \{1, 2, 3\}$, u and v be positive integers such that $16 \mid v$, $K = 2^k \mid (u - 1)$, $\gcd((u - 1)/K, v) = 1$ and if $p \equiv u \pmod{v}$,* *then*

$$\left(\frac{-3}{p}\right) = \left(\frac{q}{p}\right) = \left(\frac{r}{p}\right) = \left(\frac{s}{p}\right) = -1.$$

Then there exists $\alpha \in (1/4, 1/2)$ *and* $\delta \in (0, 1/2 - \alpha)$ *such that for large* x*, the set of primes*

$$\begin{aligned} \mathcal{P}(x; u, v) &= \{p \le x : p \equiv u \pmod v, (p-1)/K \in P_2(\alpha, \delta), \textit{and one of} \\ &\quad q,\ r,\ s \textit{ is a primitive root modulo } p\} \end{aligned}$$

has cardinality satisfying

$$\#\mathcal{P}(x; u, v) \gg \frac{x}{(\log x)^2}.$$

The following result is a theorem of Turán [11] of 1934 (see also inequality (1.2) in Norton's paper [8]) and is an upper bound for the number of positive integers $n \le x$ having the number of prime factors from a given set E of prime numbers away from the expected mean $E(x) = \sum_{\substack{p \le x \\ p \in E}} 1/p$.

Lemma 2. *Let* E *be an arbitrary set of primes, and define*

$$E(x) = \sum_{\substack{p \le x \\ p \in E}} \frac{1}{p}, \qquad \omega(n, E) = \sum_{\substack{p \mid n \\ p \in E}} 1.$$

Then, given $\varepsilon > 0$*, we have*

$$\#\{n \le x : |\omega(n; E) - E(x)| > \varepsilon E(x)\} \ll \frac{\varepsilon^{-2} x}{E(x)}.$$

For every positive integer k we use $z(k)$ for the least positive integer m such that $k \mid F_m$. The number $z(k)$ is sometimes called the *index* (or *order*) of appearance of k in the Fibonacci sequence. It is well-known that this exists for all $k \ge 1$. Furthermore, for positive integers k and n the divisibility relation $k \mid F_n$ holds if and only if $z(k) \mid n$. Additionally, if p is a prime, then $z(p)$ divides $p - \left(\frac{p}{5}\right)$. Furthermore, for a prime number p, let $t(p)$ be the period of the Fibonacci sequence modulo p. It is well-known that $t(p) \in \{z(p), 2z(p), 4z(p)\}$. All these properties of the index of appearance are useful in the proof of our main result.

3. Proof of the Theorem

We start with $q = 2 = F_3,\ r = 13 = F_7$ and $s = 89 = F_{11}$. We find suitable u and v such that the hypotheses of Lemma 1 are satisfied with $k = 2$ and additionally

$\left(\frac{5}{p}\right) = 1$. We note that if we take

$$\begin{aligned} &p \equiv 5 \pmod{16}, \qquad p \equiv 4 \pmod{5}, \quad p \equiv 2 \pmod{3}, \\ &p \equiv 2 \pmod{13}, \qquad p \equiv 7 \pmod{89}, \end{aligned} \tag{1}$$

then indeed

$$\left(\frac{2}{p}\right) = \left(\frac{-3}{p}\right) = \left(\frac{13}{p}\right) = \left(\frac{89}{p}\right) = -1 \quad \text{and} \quad \left(\frac{5}{p}\right) = 1.$$

Using the Chinese Remainder Theorem to solve congruences (1), we get $u = 270389$ and $v = 277680$, for which $u - 1 = 2^2 \times 23 \times 2939$ and $v = 2^4 \times 3 \times 5 \times 13 \times 89$, so indeed we may take $k = 2,\ K = 2^2$ and then $16 \mid v$ and $\gcd((u-1)/K, v) = 1$.

We next take a large real number x and put $y = 0.4(\log\log x)^{1/2}$. Consider the set $\mathcal{P}(y; u, v)$ defined in Lemma 1 and let $c_1 > 0$ be that constant such that

$$\#\mathcal{P}(y; u, v) > \frac{c_1 y}{(\log y)^2}. \tag{2}$$

We write $P(n)$ for the largest prime factor of n with the convention that $P(1) = 1$. We eliminate from the set $\mathcal{P}(y; u, v)$ the primes p such that $P(p-1) \nmid z(p)$. Let p be such a prime. Since $p \equiv 4 \pmod{5}$, it follows that $z(p) \mid p - 1$, and since $(p-1)/4 \in P_2(\alpha, \delta)$, we conclude that $z(p)$ is at most $4y^{1/2-\delta}$. By an argument of Erdős and Murty [5], the number M of such primes p satisfies

$$2^M \le \prod_{z(p) \le 4y^{1/2-\delta}} p \le \prod_{t \le 4y^{1/2-\delta}} F_t < \gamma^{\sum_{t \le 4y^{1/2-\delta}} t} = \exp(O(y^{1-2\delta})),$$

where $\gamma = (1 + \sqrt{5})/2$. Here and in what follows, we use the fact that

$$\gamma^{k-2} \le F_k \le \gamma^{k-1} \qquad \text{holds for all} \qquad k \ge 1.$$

The above argument shows that $M \ll y^{1-2\delta}$. Thus, in the definition of $\mathcal{P}(y; u, v)$, we additionally assume that $P(p-1) \mid z(p)$, and then inequality (2) still holds for all $x > x_0$, maybe with a slightly smaller c_1. Observe next that if $p \in \mathcal{P}(t; u, v)$, where t is sufficiently large, then $p \equiv 1 \pmod{K}$ and either $(p-1)/K$ is prime, or $(p-1)/K = p_1 p_2$, with $p_1 \le p_2$ and $p_1 > p^{1/4}$. By the sieve, it follows that

$$\#\mathcal{P}(t; u, v) \ll \frac{t}{(\log t)^2} \qquad (t \ge 10).$$

Let c_2 be the constant implied by the symbol $\ll$ above and put $c_3 = c_1/(2c_2)$. We let

$$\mathcal{Q} = \mathcal{P}(y; u, v) \backslash \mathcal{P}(c_3 y; u, v),$$

and note that the inequality

$$\#\mathcal{Q} \geq \#\mathcal{P}(y; u, v) - \#\mathcal{P}(c_3 y; u, v) \geq \frac{c_1 y}{2(\log y)^2}$$

holds for all sufficiently large x. We now let $T = \lfloor y^{1/8} \rfloor$ and select $\mathcal{Q}' \subseteq \mathcal{Q}$ with T elements such that $\gcd((p-1)/4, (p'-1)/4) = 1$ for all $p \neq p'$ in $\mathcal{Q}'$. To do that, start with the first (minimal) prime $p_1 \in \mathcal{Q}$. Then $(p_1 - 1)/4$ is either prime, or a product of two primes $p_{1,1}p_{1,2}$ each exceeding $y^{1/4}$ for x sufficiently large. Assume that p' is another member of $\mathcal{Q}$ such that $(p'-1)/4$ is not coprime to $(p-1)/4$. If $(p-1)/4$ is prime, then $p' - 1 \leq y$ is a multiple of $(p-1)/4$, and the number of such multiples is $O(1)$. If $(p-1)/4 = p_{1,1}p_{1,2}$, then $p' - 1 \leq y$ is divisible either by $p_{1,1}$ or by $p_{1,2}$, and the number of such numbers is $O(y/p_{1,1} + y/p_{1,2}) = O(y^{3/4})$. We eliminate all such potential values of p' and let p_2 be the smallest remaining prime in $\mathcal{Q}$. We next repeat the argument for p_2. Proceeding in this way, we create a sequence of primes $p_1, p_2, \ldots, p_t$, such that $(p_i - 1)/4$ and $(p_j - 1)/4$ are coprime for all $i \neq j$ in $\{1, \ldots, t\}$ and such that furthermore, at step t, the number of primes p' which have been eliminated from $\mathcal{Q}$ because $(p'-1)/4$ is not coprime with one of $(p_1 - 1)/4, \ldots, (p_t - 1)/4$ is $O(ty^{3/4})$. In particular, if $t \leq T$, then the number of such eliminated primes is

$$O(y^{1/8+3/4}) = O(y^{7/8}) = o\left(\frac{y}{(\log y)^2}\right) \qquad \text{as} \qquad x \to \infty,$$

which validates the above argument.

We write $\mathcal{R} = \{p_1, \ldots, p_T\}$.

Now we start working on the set $\mathcal{M}$. We assume that x is large and that $n \leq x$. Since there are $O(x/\log x)$ numbers $n \leq x/\log x$, we assume additionally that $n > x/\log x$. We put

$$\mathcal{M}_1(x) = \{n \leq x : p_i \mid F_n \text{ for some } i = 1, \ldots, T\}. \tag{3}$$

Fix $i \in \{1, \ldots, T\}$. We count the number of $n \leq x$ such that $p_i \mid F_n$. This is equivalent to $z(p_i) \mid n$, and since $P(p_i - 1) \mid z(p_i)$, we conclude that either $(p_i - 1)/4$ is prime and $(p_i - 1)/4 \mid n$, or $(p_i - 1)/4 = p_{i,1}p_{i,2}$, with $p_{i,1} < p_{i,2}$, and $p_{i,2} \mid n$. Since $p_{i,2} > y^{1/2}$ for large x, it follows that the number of such numbers $n \leq x$ is $O(x/P(p_i - 1)) = O(x/y^{1/2})$. Summing up over all $i = 1, \ldots, T$, we get that

$$\#\mathcal{M}_1(x) \ll \frac{xT}{y^{1/2}} = \frac{x}{y^{3/8}}. \tag{4}$$

From now on, we assume that $n \in \mathcal{M}(x) \backslash \mathcal{M}_1(x)$.

We now let $i \in \{1, \ldots, T\}$, let $p_0 \in \{3, 7, 11\}$ and put

$$E_{p_0,i} := E = \{p \equiv p_0 \pmod{4z(p_i)}\}.$$

Theorem 1 in [9] shows that

$$E(x) = \sum_{\substack{p \le x \\ p \in E}} \frac{1}{p} = \frac{\log\log x}{\phi(4z(p_i))} + O(1).$$

Since $z(p_i) \le p_i - 1 < y$, and $\phi(4z(p_i)) \le 2z(p_i) < 2y$, it follows that

$$E(x) > \frac{\log\log x}{2z(p_i)} + O(1) > \frac{\log\log x}{2y} + O(1) > \frac{\log\log x}{3y} > 2y \qquad (x > x_0). \tag{5}$$

Apply Lemma 2 with $\varepsilon = 1/2$ to conclude that

$$\#\{n \le x : |\omega(n, E) - E(x)| > \varepsilon E(x)\} \ll \frac{\varepsilon^{-2} x}{E(x)} \ll \frac{x}{y}. \tag{6}$$

Summing up the above inequality over all $p_0 \in \{3, 7, 11\}$ and $1 \le i \le T$, it follows that if we put

$$\mathcal{M}_2(x) = \bigcup_{\substack{p_0 \in \{3,7,11\} \\ 1 \le i \le T}} \{n \le x : \omega(E_{p_0,i}, n) < E_{p_0,i}(x)/2\}, \tag{7}$$

then

$$\#\mathcal{M}_2(x) \ll \sum_{\substack{p_0 \in \{3,7,11\} \\ 1 \le i \le T}} \frac{x}{y} \ll \frac{x}{y^{7/8}}. \tag{8}$$

From now on, we work with $n \in \mathcal{M}_3(x) = \mathcal{M}(x) \backslash \left(\mathcal{M}_1(x) \bigcup \mathcal{M}_2(x)\right)$. Observe that for all $p_0 \in \{3, 7, 11\}$ and all $1 \le i \le T$, we have that

$$\omega(n, E_{p_0,i}) \ge E_{p_0,i}(x)/2 > y$$

for all $x > x_0$. In particular, n has at least y distinct primes p in the progression $p \equiv p_0 \pmod{4z(p_i)}$. Since the formula

$$F_a - F_b = F_{(a-b)/2} L_{(a+b)/2}$$

holds for all integers $a,\ b$ which are congruent modulo 4 (see Lemma 2 in [7]), it follows that n has at least y distinct primes p such that $F_p \equiv F_{p_0} \pmod{p_i}$. In particular, each one of the sets

$$\begin{aligned} S_q &= \{p \mid n : F_p \equiv 2 \pmod{p_i}\}, \\ S_r &= \{p \mid n : F_p \equiv 13 \pmod{p_i}\}, \\ S_s &= \{p \mid n : F_p \equiv 89 \pmod{p_i}\} \end{aligned}$$

has at least $y > p_i$ elements. In particular, each of

$$q,\ q^2,\ \ldots,\ q^{p_i-1},\ r,\ r^2,\ \ldots,\ r^{p_i-1},\ s,\ s^2,\ \ldots,\ s^{p_i-1}$$

modulo p_i is representable as $d = \prod_{p \in S} F_p$ for some subset S of prime factors of n, which in turn is a proper divisor of F_n. Since one of the primes $q,\ r,\ s$ is a primitive root modulo p_i, it follows that the d's obtained in this way cover all the nonzero residue classes modulo p_i. Note that it is not possible that $p_i \mid F_n/d + d$ for some divisor d of F_n, since then

$$F_n/d + d \geq 2\sqrt{F_n} > 2\gamma^{n/2-1} > \gamma^{x/(2\log x)} > y > p_i$$

for $x > x_0$ sufficiently large, so that p_i is a proper divisor of $F_n/d+d$, contradicting the primality of this last number. Imposing that $p_i \nmid F_n/d+d$ for all such d, we get that $F_n \not\equiv -d^2 \pmod{p_i}$, and since $p_i \equiv 1 \pmod 4$, so, in particular, $\left(\frac{-1}{p_i}\right) = -1$, we conclude that $\left(\frac{F_n}{p_i}\right) = -1$ for all $1 \leq i \leq T$. Let $z'(p_i)$ be the largest odd divisor of $z(p_i)$ and let $t(p_i)$ be the period of the Fibonacci sequence $\{F_n\}_{n\geq 0}$ modulo p_i. Since $t(p_i) \in \{z(p_i), 2z(p_i), 4z(p_i)\}$ and $2^2 \| p_i - 1$, we conclude that $t(p_i)/z'(p_i) \in \{1, 2, 4, 8, 16\}$. Fix the residue class of n modulo 16. Note that n is odd. Indeed, to justify this, observe first that F_n is even, for if not, $F_n/d + d$ will always be even. Hence, $3 \mid n$. If also $2 \mid n$, it follows that $6 \mid n$, so $8 \mid F_n$. Taking $d = 2$, we get that $F_n/d + d$ is an even number, a contradiction.

Let $n_0 \in \{1, 3, 5, 7, 9, 11, 13, 15\}$ and let us count the number of $n \leq x$ in $\mathcal{M}_3(x)$ with $n \equiv n_0 \pmod{16}$. For large x, the period of the sequence $\{F_{n_0+16n}\}_{n\geq 0}$ is $z'(p_i)$ (see Lemma 2.6 in [1]). By a result of Shparlinsky [10],

$$\sum_{n=0}^{z'(p_i)-1} \left(\frac{F_{n_0+16n}}{p_i}\right) = O(\sqrt{p_i}) = O(y^{1/2}).$$

It thus follows that the set

$$A_i = \left\{0 \leq n < z'(p_i) : \left(\frac{F_{n_0+16n}}{p_i}\right) = -1\right\}$$

satisfies

$$\begin{aligned} \#\mathcal{A}_i &= z'(p_i)/2 + O(y^{1/2}) = z(p_i)'/2\left(1 + O\left(\frac{y^{1/2}}{z(p_i)}\right)\right) \\ &= z(p_i)'/2\left(1 + O\left(\frac{1}{y^{\delta}}\right)\right). \end{aligned}$$

We now loop over all $i = 1, \ldots, T$ and use the Chinese Remainder Theorem noting that $z'(p_i)$ and $z'(p_j)$ are coprime for $i \neq j$ in $\{1, \ldots, T\}$ because they are divisors of $(p_i - 1)/4$ and $(p_j - 1)/4$, respectively, which are coprime. We get that the number $n \equiv n_0 \pmod{16}$ must be in $\prod_{i=1}^{T} \#A_i$ residue classes modulo $N = \prod_{i=1}^{T} z'(p_i)$, and the number of such is at most

$$\begin{aligned} &\leq \prod_{i=1}^{T}\left(\frac{\#A_i}{z'(p_i)}\right)x+N=\frac{x}{2^T}\left(1+O\left(\frac{1}{y^\delta}\right)\right)^T+O(y^T) \\ &\ll x\exp\left(-T\ln 2+O\left(\frac{T}{y^\delta}\right)\right)\ll\frac{x}{1.5^T}=O\left(\frac{x}{y}\right). \end{aligned}$$

Summing up over all possibilities for odd n_0 in $[1,16]$, we get that

$$\#\mathcal{M}_3(x)=O\left(\frac{x}{y}\right). \tag{9}$$

From equations (4), (8), and (9), we get that

$$\#\mathcal{M}(x)\ll\frac{x}{y^{3/8}}\ll\frac{x}{(\log\log x)^{3/16}},$$

which is what we wanted to prove.

4. An Open Problem

The conclusion of our theorem is too weak to deduce that

$$\sum_{n\in\mathcal{M}}\frac{1}{n}$$

is finite, a problem which we leave for the reader. In fact, we believe that $\mathcal{M}$ is a finite set. To see why, we first note that if $n\in\mathcal{M}$ and $n>9$, then F_n has at least three prime factors. Indeed, from what we have seen, $2\|F_n$ so $n=3m$ for some odd m. If $m>12$, then by Carmachael's primitive divisor theorem (see [4]), we deduce that each of F_m and F_{3m} has a primitive prime factor, that is a prime factor p that did not divide any previous Fibonacci number. This shows that F_{3m} has at least three prime factors for all $m>12$, and the fact that this is so for $m\in[5,11]$ can be checked on a case by case basis. Now let $p_1,\ p_2,\ p_3$ be three distinct prime factors of F_n. Then each of $F_n/p_i+p_i\geq 2\sqrt{F_n}\geq\gamma^{n/2}$ is a prime for $i=1,2,3$. By the Prime Number Theorem, the expectation that F_n/p_i+p_i is a prime should be about $1/\log(F_n/p_i+p_i)=O(1/n)$. Since this is true for $i=1,2,3$, and assuming that the above three events are independent, it follows that it is natural to expect that the probability that a random $n\in\mathcal{M}$ is of order $O(1/n^3)$. Since

$$\sum_{n\geq 1}\frac{1}{n^3}=\zeta(3)=O(1),$$

it would seem reasonable to conjecture that $\mathcal{M}$ is in fact a finite set.

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