# ON THE LAMBEK-MOSER THEOREM

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**Abstract**
We suggest an alternative proof of a partitioning theorem due to Lambek and Moser using a perceptible model.

## 1. Introduction

The notion of invertibility of sequences whose values are either non-negative integers or $\infty$ was introduced by J. Lambek and L. Moser. Adopting their terminology [4], such sequences are called sequences of *numbers*.

**Definition 1.** Two sequences $\bar{f} = (f(n))_{n=1}^{\infty}, \bar{g} = (g(n))_{n=1}^{\infty}$ of numbers are *mutually inverse* if for every pair of positive integers $m, n$ either $f(m) < n$ or $g(n) < m$, but not both.

It is shown [4, Theorem 1] that a sequence of numbers $(f(n))_{n=1}^{\infty}$ has an inverse if and only if it is non-decreasing. In this case, the unique inverse $(g(n))_{n=1}^{\infty}$ is given by

$$g(n) = |\{m|f(m) < n\}|. \tag{1}$$

It can be verified that the inverse of $(g(n))_{n=1}^{\infty}$ is again $(f(n))_{n=1}^{\infty}$.

Any non-decreasing sequence of numbers $\bar{f} = (f(n))_{n=1}^{\infty}$ also determines a set of positive integers

$$\hat{f} := \{n + f(n)\}_{f(n)<\infty}.$$

The correspondence $\bar{f} \mapsto \hat{f}$ between the non-decreasing sequences of numbers and the sets of positive integers is one-to-one [4, §3]. The following partitioning theorem is established.

**Lambek-Moser Theorem.** *[4, Theorem 2] Two non-decreasing sequences of numbers $\bar{f} = (f(n))_{n=1}^{\infty}, \bar{g} = (g(n))_{n=1}^{\infty}$ are mutually inverse if and only if the sets $\hat{f}$ and $\hat{g}$ are complementary, that is they disjointly cover the set of positive integers.*

The Lambek-Moser Theorem yields nice examples of complementary sets which are somehow surprising [4, §2].

In this note we suggest an alternative proof of the Lambek-Moser theorem, by applying the running model which was introduced in [3]. Another visual proof was given by E.W. Dijkstra [2]. The reader is referred to [5] for a detailed bibliography on complementary sequences and related topics.

## 2. The Model

Let $X$ and $Y$ be two athletes running around a circular track in opposite directions, starting at time $t = 0$ from the same starting point $\mathcal{O}$. Each time one of these athletes crosses the point $\mathcal{O}$, the number of their meetings (not including the meeting at time $t = 0$) is recorded for this athlete. Now, assume that they never meet exactly in $\mathcal{O}$. Then it is clear that between two consecutive meetings, exactly one of the two of them crosses $\mathcal{O}$. As a result, the set $\mathcal{S}_X$ recorded for $X$ and the set $\mathcal{S}_Y$ recorded for $Y$ are disjoint. Assume further that the athletes are immortal and never stop running, and that at least one of them crosses $\mathcal{O}$ infinitely many times. Under these assumptions, the sets $\mathcal{S}_X$ and $\mathcal{S}_Y$ partition the set of positive integers. Note that in order that $\mathcal{S}_X$ and $\mathcal{S}_Y$ partition the set of positive integers it is also necessary that none of the meetings occur at $\mathcal{O}$.

## 3. Preliminary Results

Normalize the circumference of the track to be 1, and let

$$\begin{array}{rcl} \varphi : [0,\infty) & \to & [0,\infty) \\ t & \mapsto & \varphi(t) \end{array}$$

be a strictly increasing continuous time function with $\varphi(0) = 0$ describing the motion of $X$. That is $\varphi(t)$ is the distance traveled by $X$ during the time interval $[0, t]$. Let $\psi(t) = t$ be the motion function of $Y$, who is running in the opposite direction. Then $Y$ crosses $\mathcal{O}$ exactly in integer time units. Since the relative motion function of $X$ and $Y$ is $\varphi(t) + t$, and since together they travel a unit between two consecutive meetings, the number of times $X$ and $Y$ meet until time $t$ is $\lfloor \varphi(t) + t \rfloor$, where $\lfloor \cdot \rfloor$ is the floor integer part function. Therefore, the set $\mathcal{S}_Y$ of positive integers recorded for $Y$ is just

$$\mathcal{S}_Y = \{\lfloor \varphi(n) + n \rfloor\}_{n=1}^{\infty}.$$

Next, $X$ crosses the point $\mathcal{O}$ each and every time $t$ such that $\varphi(t) \in \mathbb{Z}^+$ (where $\mathbb{Z}^+$ denotes the set of positive integers). Thus, the set $\mathcal{S}_X$ recorded for $X$ is exactly

$$\mathcal{S}_X = \{\lfloor \varphi(t) + t \rfloor\}_{\varphi(t) \in \mathbb{Z}^+}.$$

We can describe $\mathcal{S}_X$ in another way. Since $\varphi$ is continuous and strictly increasing, it maps $(0,\infty)$ onto an open segment $I := (0, M)$ (where $0 < M \leq \infty$), and admits an increasing, continuous inverse $\varphi^{-1} : I \to \mathbb{R}^+$. Then

$$\mathcal{S}_X = \{\lfloor n + \varphi^{-1}(n) \rfloor\}_{n \in \mathbb{Z}^+ \cap I}.$$

By the argument in §2, the sets $\mathcal{S}_X$ and $\mathcal{S}_Y$ partition the positive integers if and only if $X$ and $Y$ never meet at $\mathcal{O}$ after $t = 0$. But $X$ and $Y$ do meet at $\mathcal{O}$ at time $t > 0$ exactly when both $t$ and $\varphi(t)$ are positive integers. We obtain

**Corollary 1.** *Let $\varphi : [0,\infty) \to [0,\infty)$ be a strictly increasing continuous function with $\varphi(0) = 0$ and let $\varphi^{-1} : \mathrm{Im}(\varphi) \to [0,\infty)$ be its inverse. Then the sets $\{\lfloor \varphi(n) + n \rfloor\}_{n \in \mathbb{Z}^+}$ and $\{\lfloor n + \varphi^{-1}(n) \rfloor\}_{n \in \mathbb{Z}^+ \cap \mathrm{Im}(\varphi)}$ partition the set of positive integers if and only if $\varphi(\mathbb{Z}^+) \cap \mathbb{Z}^+ = \emptyset$.*

In order to exploit Corollary 1 to prove the Lambek-Moser Theorem, we need two lemmas. The first observation is easily verified by distinguishing between three types of sequences (see [4, §2]).

**Lemma 1.** *Let $(f(n))_{n=1}^{\infty}$ and $(g(n))_{n=1}^{\infty}$ be mutually inverse sequences of numbers. Then at least one of these sequences does not admit $\infty$ as a value, in other words, it is a sequence of non-negative integers.*

The second lemma is straightforward:

**Lemma 2.** *Let $(f(n))_{n=1}^{\infty}$ be a non-decreasing sequence of non-negative integers. Then there exists a strictly increasing continuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ such that $\lfloor \varphi(n) \rfloor = f(n)$, for every $n \in \mathbb{Z}^+$. Moreover, $\varphi$ can be chosen such that*

$$\varphi(\mathbb{Z}^+) \cap \mathbb{Z}^+ = \emptyset. \tag{2}$$

## 4. Proof of the Lambek-Moser Theorem

Since the correspondence $\bar{f} \mapsto \hat{f}$ is one-to-one, and since an inverse sequence and a complementary set are unique, it is enough to show the "only if" direction of the theorem. Indeed, let $\bar{f}$ and $\bar{g}$ be mutually inverse sequences of numbers. By Lemma 1 we may assume that $(f(n))_{n=1}^{\infty}$ is sequence of non-negative integers (else $(g(n))_{n=1}^{\infty}$ is). Next, by Lemma 2, there exists a strictly increasing continuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ such that for every $n \in \mathbb{Z}^+$, both
(a) $\lfloor \varphi(n) \rfloor = f(n)$, and
(b) $\varphi(n) \notin \mathbb{Z}^+$.

Let $\varphi^{-1} : \mathrm{Im}(\varphi) \to [0,\infty)$ be the increasing continuous inverse of $\varphi$. By the conditions on $\varphi$, using the alternative characterization (1), the inverse sequence

$\bar{g} = (g(n))_{n=1}^{\infty}$ of $\bar{f} = (f(n))_{n=1}^{\infty} = (\lfloor \varphi(n) \rfloor)_{n=1}^{\infty}$ is given by

$$g(n) = |\{m | \lfloor \varphi(m) \rfloor < n\}| = \begin{cases} \lfloor \varphi^{-1}(n) \rfloor & \text{if } n \in \mathrm{Im}(\varphi) \\ \infty & \text{otherwise.} \end{cases} \quad (3)$$

Consequently,

$$\hat{g} = \{\lfloor \varphi^{-1}(n) + n \rfloor\}_{n \in \mathbb{Z}^+ \cap \mathrm{Im}(\varphi)}. \quad (4)$$

By Corollary 1, together with (2) and (4), we deduce that $\hat{g}$ is the complement of

$$\{\lfloor \varphi(n) + n \rfloor\}_{n \in \mathbb{Z}^+} = \{f(n) + n\}_{n=1}^{\infty} = \hat{f}.$$

The proof of the theorem is complete. □

**Remark.** Note that S. Beatty's celebrated theorem [1] follows from Corollary 1 by taking $\varphi(t) := \lambda \cdot t$, where $\lambda > 0$ (and then $\varphi^{-1}(t) = \frac{1}{\lambda} \cdot t$), that is the case where the speeds of both athletes are constant.

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## References

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