NOTE ON ODD/ODD VERTEX REMOVAL GAMES ON BIPARTITE GRAPHS

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Abstract
We analyze the Odd/odd vertex removal game introduced by P. Ottaway. We prove that every bipartite graph has Grundy value 0 or 1 only depending on the parity of the number of edges in the graph, which is a generalization of a conjecture of K. Shelton.

1. Introduction

In this paper we will analyze some properties of the Odd/odd vertex removal game introduced by Ottaway [6, 5]. In particular we will prove that the Odd/odd vertex deletion games on bipartite graphs have Grundy values 0 or 1 depending on the parity of the number of edges, which as a consequence, proves a conjecture of Shelton [7].

We will assume that the reader is familiar with the Sprague-Grundy theory of impartial combinatorial games (see, for example [2] or [1]). We will denote the Sprague-Grundy value (or simply Grundy value) of a game G by $\mathcal{G}(G)$, the disjunctive sum of games G_1 and G_2 by $G_1 + G_2$. The nim-sum of two integers a and b is denoted by $a \oplus b$. The set of games where the first player to move will win (when played optimally), i.e. where $\mathcal{G}(G) > 0$, will be denoted $\mathcal{N}$. The complement of that in the set of all impartial games will be denoted $\mathcal{P}$.

Of importance is the main theorem of the Sprague-Grundy theory which states

Theorem 1. *[2] Let G and H be two impartial games. The Grundy value of the disjunctive sum $G + H$ is*

$$\mathcal{G}(G + H) = \mathcal{G}(G) \oplus \mathcal{G}(H)$$

1.1. Impartial Vertex Removal Games

The vertex removal games are played on graphs and each move consists of removing a vertex and all the edges incident to that vertex. Following Ottaway, [6, 5], we decide what vertices we are allowed to remove by some rule concerning the parity of the degree of the vertices. For these games, every position may be denoted by a graph and we will make no distinction between a graph and a position in such games.

For example, one of the two players may only be allowed to remove vertices of even degree and the other only vertices of odd degree. This example is obviously a partizan game since each player has a different set of options. Games of the same type but played on digraphs are also possible, but will not be considered here.

Shelton studies a more general problem than this [6], where one considers every vertex in a graph to be a coin, i.e. a binary value of either heads or tails. A player can remove any coin with heads up, and then flip any adjacent coins (adjacent vertices in the graph). This version is more general than a vertex removal game. We get a vertex removal game if one starts with heads up on every vertex with the parity we want to be the "removable parity". Furthermore, Shelton proves some results for the Grundy values of these types of games on particular graphs and that the problem of determining if $G \in \mathcal{N}$ for directed graphs G is PSPACE-Complete. He also makes two conjectures concerning the Odd/odd vertex removal game, one of which we will study.

More recently Harding and Ottaway [4] proved that there are Odd/odd vertex deletion games with Grundy value n for all non-negative integers n.

1.2. Odd/Odd Vertex Removal

We will be concerned with *impartial* vertex removal games in undirected graphs and, in particular, we will analyze the game where both players only remove vertices of odd degree.

The Grundy values for families of graphs such as complete graphs, complete bipartite graphs, stars and paths were exhibited by Ottaway [6]. Harding and Ottaway [4] show that given a specific Grundy value, there exists a graph with that Grundy value. The construction is explicit and obtained by forming the line graphs of specific trees.

In this paper we give the Grundy value for any bipartite graph. The family of bipartite graphs include such graph classes as paths, grids, stars, and trees. In particular this settles a conjecture of Shelton [7] concerning the Grundy value for grid graphs.

The main problem of constructing a function for computing the Grundy value of *any* graph is still an open problem. The result about Grundy values of bipartite graphs does not help since we use specific properties of the cycles in bipartite graphs

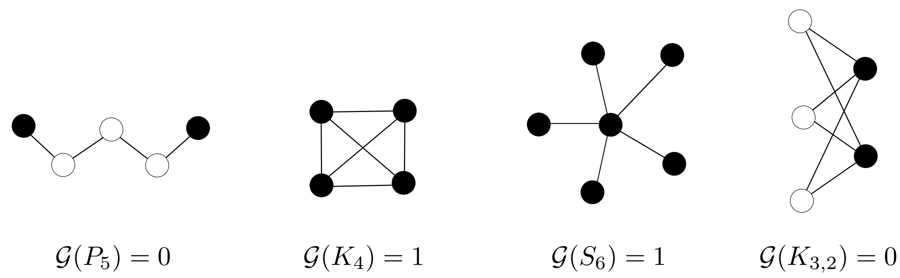

Figure 1: Examples of positions with known Grundy values. The vertices which are removable in the positions are filled in.

and prove that these games are much simpler than the ones played on arbitrary graphs (it does not matter what move we make). Hence, when solving this problem one will have to use some method other than the one for bipartite graphs.

Even though the problem of determining the Grundy value in a directed graph has been shown to be PSPACE-complete [7], there is no reason to despair. The problem in a directed graph might be much more complicated than in the undirected case (as illustrated by the Even/even vertex removal game). There is still hope for some simple graph invariant that characterizes the games in terms of which player wins or the Grundy value.

1.3. Bipartite Positions

Recall that one way of defining a *bipartite* graph is as a graph where every circuit has even length [3, Proposition 1.6.1].

Theorem 2. *[3, Theorem 1.8.1] A connected graph G has an Eulerian circuit if and only if every vertex has even degree.*

Corollary 3. *A position H, in an Odd/odd vertex deletion game, is terminal if and only if H is a graph where every connected component has an Eulerian circuit (with the convention that the empty circuit, in the single vertex component, is an Eulerian circuit).*

Lemma 4. *Every terminal position, H, in an Odd/odd vertex removal game played on a bipartite graph, G, has an even number of edges.*

Proof. By Corollary 3 each edge in H is part of some Eulerian circuit. Since every subgraph of a bipartite graph is bipartite (and by the definition of a bipartite graph as only having circuits of even length) we know that each circuit must have even length. Also note that each terminal position must be a subgraph of the initial graph G.

Thus each connected component of H has an even number of edges. The total number of edges is the sum of these. $\square$

Theorem 5. *Let $G = (V, E)$ be a bipartite graph, then*

$$\mathcal{G}(G) = \begin{cases} 1 & \textit{if } |E| \textit{ odd} \\ 0 & \textit{if } |E| \textit{ even} \end{cases}$$

Proof. We only need to use Lemma 4 to see that all terminal positions have an even number of edges. So, any position with an odd number of edges cannot be terminal.

In each move an odd number of edges is removed so the number of edges remaining in the graph alternate between even and odd with every move. One player always moves to a position with an even number of edges and the other always to a position with an odd number of edges. The player who moves to a position with an even number of edges must necessarily win. This means that for $G = (V, E)$ we have $G \in \mathcal{P}$ if $|E|$ is even and $G \in \mathcal{N}$ is $|E|$ is odd.

Since *every* move from any $\mathcal{N}$-position takes us to a $\mathcal{P}$-position we cannot get any Grundy value larger than one. The assertion above follows. $\square$

Corollary 6. *[7, Conjecture 17] The Grundy value of an Odd/odd vertex removal game played on a grid graph is either 0 or 1.*

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