



**PROPERTIES OF A RESTRICTED BINARY PARTITION
FUNCTION A LA ANDREWS AND LEWIS**

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Received: 1/20/15, Accepted: 4/29/15, Published: 5/8/15

Abstract

In 2001, Andrews and Lewis utilized an identity of F. H. Jackson to derive some new partition generating functions as well as identities involving some of the corresponding partition functions. At the end of their paper, they define a family of functions $W_1(S_1, S_2; n)$ to be the number of partitions of n into parts from $S_1 \cup S_2$ that do not contain both a_j and b_j as parts (where $S_1 = \{a_1, a_2, a_3, \dots\}$ and $S_2 = \{b_1, b_2, b_3, \dots\}$ and $S_1 \cap S_2 = \phi$). This definition is motivated by the main results of their paper; in that case, S_1 and S_2 contain elements in arithmetic progression with the same “skip value” k . Our goal in this note is to consider more general examples of such partition functions where S_1 and S_2 satisfy the requirements mentioned above but do not simply contain elements in an arithmetic progression. In particular, we consider the situation where S_1 and S_2 contain specific powers of 2. We then prove a number of arithmetic properties satisfied by this function using elementary generating function manipulations and classic results from the theory of partitions.

1. Introduction

In 2001, Andrews and Lewis [1] utilized an identity of F. H. Jackson to derive some new partition generating functions as well as identities involving some of the corresponding partition functions. At the end of their paper, they define a family of functions $W_1(S_1, S_2; n)$ to be the number of partitions of n into parts from $S_1 \cup S_2$ that do not contain both a_j and b_j as parts (where $S_1 = \{a_1, a_2, a_3, \dots\}$ and $S_2 = \{b_1, b_2, b_3, \dots\}$ and $S_1 \cap S_2 = \phi$). This definition is motivated by the

main results of their paper; in that case, S_1 and S_2 contain elements in arithmetic progression with the same “skip value” k . (An example of the kind of partition functions that arise in the work of Andrews and Lewis can be found in [6, Sequence A070047]. In that case, $S_1 = \{1, 4, 7, 10, 13, \dots\}$ and $S_2 = \{2, 5, 8, 11, 14, \dots\}$.)

In this note, we consider other sets S_1 and S_2 that satisfy the conditions above but do not simply consist of the values of a given arithmetic progression. In particular, let $S_1 = \{1, 4, 16, 64, \dots\}$ and $S_2 = \{2, 8, 32, 128, \dots\}$. That is, $S_1 = \{2^{2n-2}\}_{n \geq 1}$ and $S_2 = \{2^{2n-1}\}_{n \geq 1}$. In this context, $W_1(S_1, S_2; n)$ counts the number of partitions of n into powers of 2 such that 1 and 2 cannot both be parts of a particular partition, and 4 and 8 cannot both be parts of a particular partition, and 16 and 32, and so on. Thus, for example, the number of such partitions of $n = 10$ is given by $W_1(S_1, S_2; 10) = 8$ where the partitions in question are the following:

$$8 + 2, \quad 8 + 1 + 1, \quad 4 + 4 + 2, \quad 4 + 2 + 2 + 2, \\ 4 + 4 + 1 + 1, \quad 4 + 1 + 1 + 1 + 1 + 1 + 1, \quad 2 + 2 + 2 + 2 + 2, \quad 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$$

Throughout this note, we will let $H_1(q)$ denote the generating function for $W_1(S_1, S_2; n)$. Moreover, with the goal of simplifying notation, we will write $W(n)$ in place of $W_1(S_1, S_2; n)$ throughout the rest of this paper.

Thanks to the comments made in the concluding remarks of Andrews and Lewis, we have the following generating function identity:

Theorem 1. *We have*

$$H_1(q) := \sum_{n \geq 0} W(n)q^n = \prod_{j \geq 1} \frac{(1 - q^{2^{2j-2} + 2^{2j-1}})}{(1 - q^{2^{2j-2}})(1 - q^{2^{2j-1}})} = \prod_{j \geq 1} \frac{(1 - q^{3 \times 2^{2j-2}})}{(1 - q^{2^{2j-1}})}$$

We now wish to study $W(n)$ from two different perspectives. In Section 2, we prove a number of elementary recurrence properties satisfied by $W(n)$. In Section 3, we then prove a number of divisibility properties satisfied by $W(n)$. Our primary tool in proving these results is elementary generating function manipulations.

2. Recurrences

Our main goal in this section is to prove the following two recurrence relations satisfied by $W(n)$.

Theorem 2. *For all $n \geq 1$, we have $W(2n) = W(2n - 2) + W(\lfloor \frac{n}{2} \rfloor)$.*

Theorem 3. *For all $n \geq 1$, we have $W(2n + 1) = W(2n) - W(2n - 1)$.*

Prior to proving these results, it is instructive to note that recurrences of a similar form also hold for a number of other binary partition functions. See, for example, Churchhouse’s original work [2] on the unrestricted binary partition function as well as the work of Sloane and Sellers [10] on non-squashing partitions. Of course this is not surprising as such recurrences follow from the corresponding generating functions, and each of the generating functions mentioned above are, in essence, generating functions for binary partition functions.

Proof. (of Theorem 2) We begin this proof by noting that

$$\begin{aligned} H_1(-q) &= \frac{1+q^3}{1+q} \prod_{j \geq 2} \frac{(1-q^{3 \times 2^{2j-2}})}{(1-q^{2^{j-1}})} \\ &= \frac{(1+q^3)(1-q)}{(1+q)(1-q^3)} H_1(q). \end{aligned}$$

Thus, we know that

$$\begin{aligned} \sum_{n \geq 0} W(2n)q^{2n} &= \frac{1}{2} (H_1(q) + H_1(-q)) \\ &= \frac{1}{2} H_1(q) \left(1 + \frac{(1+q^3)(1-q)}{(1+q)(1-q^3)} \right) \\ &= \frac{1}{2} H_1(q) \left(\frac{2-2q^4}{(1+q)(1-q^3)} \right) \\ &= \frac{(1-q^4)}{(1+q)(1-q^3)} H_1(q) \\ &= \frac{(1+q^2)(1-q)}{1-q^3} H_1(q) \\ &= \frac{(1+q^2)(1-q^{12})(1-q^{48})(1-q^{192}) \dots}{(1-q^2)(1-q^4)(1-q^8) \dots} \end{aligned}$$

after simplification. Therefore,

$$\sum_{n \geq 0} W(2n)q^n = \frac{(1+q)(1-q^6)(1-q^{24})(1-q^{96}) \dots}{(1-q)(1-q^2)(1-q^4) \dots} := H_2(q).$$

Next, note that

$$\begin{aligned} H_2(-q) &= \frac{(1-q)(1-q^6)(1-q^{24})(1-q^{96}) \dots}{(1+q)(1-q^2)(1-q^4) \dots} \\ &= \frac{(1-q)^2}{(1+q)^2} H_2(q). \end{aligned}$$

Thus,

$$\sum_{n \geq 0} W(4n)q^{2n} = \frac{1}{2}(H_2(q) + H_2(-q)) = \frac{(1 + q^2)(1 - q^6)(1 - q^{24})(1 - q^{96}) \dots}{(1 - q^2)^2(1 - q^4)(1 - q^8) \dots}$$

after much simplification. Hence,

$$\sum_{n \geq 0} W(4n)q^n = \frac{(1 + q)(1 - q^3)(1 - q^{12})(1 - q^{48}) \dots}{(1 - q)^2(1 - q^2)(1 - q^4) \dots} = \frac{(1 + q)}{(1 - q)}H_1(q)$$

using Theorem 1.

In a similar vein, we know

$$\sum_{n \geq 0} W(4n + 2)q^{2n+1} = \frac{1}{2}(H_2(q) - H_2(-q)) = 2q \frac{(1 - q^6)(1 - q^{24})(1 - q^{96}) \dots}{(1 - q^2)^2(1 - q^4)(1 - q^8) \dots}$$

This means

$$\sum_{n \geq 0} W(4n + 2)q^n = 2 \frac{(1 - q^3)(1 - q^{12})(1 - q^{48}) \dots}{(1 - q)^2(1 - q^2)(1 - q^4) \dots} = \frac{2}{(1 - q)}H_1(q).$$

Now two important comments are in order. First, from a generating function perspective, we see that

$$\begin{aligned} \frac{(1 + q)}{(1 - q)}H_1(q) + H_1(q) &= \left(\frac{(1 + q)}{(1 - q)} + 1 \right) H_1(q) \\ &= \left(\frac{1 + q + 1 - q}{1 - q} \right) H_1(q) \\ &= \frac{2}{(1 - q)}H_1(q). \end{aligned}$$

From the above generating function work, this means that, for all $n \geq 1$,

$$W(4n + 2) = W(4n) + W(n). \tag{1}$$

Note also that the generating function for $W(4n - 2)$ is given by

$$\sum_{n \geq 0} W(4n - 2)q^n = \frac{2q}{(1 - q)}H_1(q)$$

by simply shifting the generating function for $W(4n + 2)$ by one factor of q . We then see that the sum of the generating function for $W(4n - 2)$ and the generating function for $W(n)$ is

$$\begin{aligned} \frac{2q}{(1 - q)}H_1(q) + H_1(q) &= \left(\frac{2q}{(1 - q)} + 1 \right) H_1(q) = \left(\frac{2q + 1 - q}{1 - q} \right) H_1(q) \\ &= \left(\frac{1 + q}{1 - q} \right) H_1(q) \end{aligned}$$

and this is the generating function for $W(4n)$. Thus, for all $n \geq 1$,

$$W(4n) = W(4n - 2) + W(n). \tag{2}$$

Combining (1) and (2) yields the result. □

Proof. (of Theorem 3) While proving Theorem 2, we noted that

$$\sum_{n \geq 0} W(2n)q^n = \frac{(1+q)(1-q^6)(1-q^{24})(1-q^{96}) \dots}{(1-q)(1-q^2)(1-q^4) \dots}.$$

It is worth noting that this is equivalent to saying that

$$\sum_{n \geq 0} W(2n)q^n = \frac{1+q}{1-q} H_1(q^2).$$

We now wish to find a similar generating function result for $W(2n + 1)$. Using techniques similar to those employed above, we see that

$$\begin{aligned} \sum_{n \geq 0} W(2n + 1)q^{2n+1} &= \frac{1}{2} (H_1(q) - H_1(-q)) = \frac{1}{2} H_1(q) \left(1 - \frac{(1+q^3)(1-q)}{(1+q)(1-q^3)} \right) \\ &= \frac{q(1-q^2)}{(1+q)(1-q^3)} H_1(q) \\ &= \frac{q(1-q)}{1-q^3} H_1(q) \\ &= \frac{q(1-q^{12})(1-q^{48})(1-q^{192}) \dots}{(1-q^2)(1-q^4)(1-q^8) \dots}. \end{aligned}$$

Thus,

$$\sum_{n \geq 0} W(2n + 1)q^n = \frac{(1-q^6)(1-q^{24})(1-q^{96}) \dots}{(1-q)(1-q^2)(1-q^4) \dots}.$$

This can be rewritten as

$$\sum_{n \geq 0} W(2n + 1)q^n = \frac{1}{1-q} H_1(q^2).$$

By shifting the above by one factor of q , we see then that

$$\sum_{n \geq 0} W(2n - 1)q^n = \frac{q}{1-q} H_1(q^2).$$

Summing the generating functions for $W(2n + 1)$ and $W(2n - 1)$ yields

$$\frac{1}{1-q} H_1(q^2) + \frac{q}{1-q} H_1(q^2) = \frac{1+q}{1-q} H_1(q^2)$$

and this is the generating function for $W(2n)$ as noted above. Therefore, for all $n \geq 1$, we have $W(2n + 1) + W(2n - 1) = W(2n)$ and this is equivalent to the desired result. □

3. Divisibility Properties

Numerous authors have considered divisibility properties satisfied by various binary partition functions. See, for example, [2], [3], [4], [5], [7], [8], and [9]. In this section, we wish to do the same for $W(n)$.

The first goal is to prove the following theorem that provides a characterization of $W(n)$ modulo 2. In order to state the theorem, we define M_j to be the j^{th} value of the Moser-de Bruijn sequence. This is the sequence of integers that can be written as a sum of distinct powers of 4 [6, Sequence A000695]. The sequence includes the values 0, 1, 4, 5, 16, 17, 20, 21, 64, 65, 68, 69, 80, . . .

Theorem 4. *For all $n \geq 0$,*

$$W(n) \equiv \begin{cases} 1 \pmod{2} & \text{if } n = 3M_j \text{ or } n = 3M_j + 1 \\ 0 \pmod{2} & \text{otherwise} \end{cases}$$

where M_j is an element of the Moser-de Bruijn sequence defined above.

Proof. Thanks to Theorem 1 and elementary generating function manipulations, we know

$$\begin{aligned} \sum_{n \geq 0} W(n)q^n &= \prod_{j \geq 1} \frac{(1 - q^{3 \times 2^{2j-2}})}{(1 - q^{2^{j-1}})} \\ &\equiv \prod_{j \geq 1} \frac{(1 - q^{3 \times 2^{2j-2}} + 2q^{3 \times 2^{2j-2}})}{(1 + q^{2^{j-1}})} \pmod{2} \\ &= \frac{\prod_{j \geq 1} (1 + q^{3 \times 2^{2j-2}})}{\frac{1}{1-q}} \\ &\quad \text{since every integer has a unique representation in base 2} \\ &= (1 - q) \prod_{j \geq 1} (1 + q^{3 \times 2^{2j-2}}) \\ &\equiv (1 + q) \prod_{j \geq 1} (1 + q^{3 \times 2^{2j-2}}) \pmod{2}. \end{aligned}$$

The result follows. □

From Theorem 4, we can easily write down specific Ramanujan-like congruences modulo 2.

Corollary 1. *Let $R = \{2, 5, 6, 7, 8, 9, 10, 11\}$. For all $n \geq 0$ and any $r \in R$,*

$$W(12n + r) \equiv 0 \pmod{2}.$$

Proof. Since $M_j \equiv 0, 1 \pmod{4}$ for any $j \geq 1$, we know that $3M_j \equiv 0, 3 \pmod{12}$ and $3M_j + 1 \equiv 1, 4 \pmod{12}$ for any $j \geq 1$. In particular, none of the values of the form $12n + r$, $r \in R$, are congruent to $0, 1, 3, 4 \pmod{12}$. The result follows from Theorem 4. \square

Using similar tools and techniques, we now transition to a characterization result modulo 4 for a subset of the values of $W(n)$.

Theorem 5. *For all $n \geq 0$,*

$$W(4n + 2) \equiv \begin{cases} 2 \pmod{4} & \text{if } n = 3M_j \\ 0 \pmod{4} & \text{otherwise} \end{cases}$$

where M_j is an element of the Moser-de Bruijn sequence defined above.

Proof. Recall that

$$\sum_{n \geq 0} W(4n + 2)q^n = \frac{2}{1 - q} H_1(q) = \frac{2(1 - q^3)(1 - q^{12})(1 - q^{48}) \dots}{(1 - q)^2(1 - q^2)(1 - q^4) \dots}.$$

Thus,

$$\begin{aligned} \sum_{n \geq 0} \frac{1}{2} W(4n + 2)q^n &= \frac{(1 - q^3)(1 - q^{12})(1 - q^{48}) \dots}{(1 - q)^2(1 - q^2)(1 - q^4) \dots} \\ &\equiv \frac{(1 + q^3)(1 + q^{12})(1 + q^{48}) \dots}{(1 - q)(1 + q)(1 + q^2)(1 + q^4) \dots} \pmod{2} \\ &= \frac{(1 + q^3)(1 + q^{12})(1 + q^{48}) \dots}{(1 - q) \frac{1}{1 - q}} \\ &= (1 + q^3)(1 + q^{12})(1 + q^{48}) \dots \end{aligned}$$

The result follows. \square

With Theorem 5 in hand, it is a straightforward matter to write down specific Ramanujan-like congruences modulo 4.

Corollary 2. *For all $n \geq 0$,*

$$\begin{aligned} W(12n + 6) &\equiv 0 \pmod{4} \text{ and} \\ W(12n + 10) &\equiv 0 \pmod{4}. \end{aligned}$$

Proof. From Theorem 5, it is clear that, for n not divisible by 3,

$$W(4n + 2) \equiv 0 \pmod{4}.$$

Thus,

$$W(4(3n + 1) + 2) \equiv 0 \pmod{4} \text{ and } W(4(3n + 2) + 2) \equiv 0 \pmod{4}.$$

\square

Corollary 3. For all $n \geq 0$,

$$\begin{aligned} W(16n + 6) &\equiv 0 \pmod{4} \text{ and} \\ W(16n + 10) &\equiv 0 \pmod{4}. \end{aligned}$$

Proof. Given that the values of the form $3M_j$ are congruent to either 0 or 3 modulo 4, we see from Theorem 5 that congruences modulo 4 must occur when $n \equiv 1 \pmod{4}$ and $n \equiv 2 \pmod{4}$. Thus,

$$\begin{aligned} W(4(4n + 1) + 2) &\equiv 0 \pmod{4} \text{ and} \\ W(4(4n + 2) + 2) &\equiv 0 \pmod{4}. \end{aligned}$$

□

Next, we look at $W(n)$ modulo 3. It is clear from Theorem 1 that

$$\begin{aligned} \sum_{n \geq 0} W(n)q^n &\equiv \prod_{j \geq 1} \frac{(1 - q^{2^{2j-2}})^3}{(1 - q^{2^{2j-1}})} \pmod{3} \\ &= \prod_{j \geq 1} \frac{(1 - q^{2^{2j-2}})^2}{(1 - q^{2^{2j-1}})} \\ &= \frac{(1 - q)^2(1 - q^2)^2(1 - q^4)^2 \dots}{(1 - q^2)(1 - q^3)(1 - q^5) \dots} \\ &= \frac{(1 - q)^2(1 - q^4)^2(1 - q^{16})^2 \dots}{(1 - q^2)(1 - q^8)(1 - q^{32}) \dots} \\ &= \frac{(1 - q)(1 - q^4)(1 - q^{16}) \dots}{(1 + q)(1 + q^4)(1 + q^{16}) \dots} \\ &:= F(q). \end{aligned}$$

We now perform classical generating function dissections (with the goal of eventually considering $W(8n + 4)$ modulo 3). We have

$$\begin{aligned} \sum_{n \geq 0} W(2n)q^{2n} &\equiv \frac{1}{2} (F(q) + F(-q)) \pmod{3} \\ &= \frac{1}{2} \left(\frac{(1 - q^4)(1 - q^{16}) \dots}{(1 + q^4)(1 + q^{16}) \dots} \right) \left(\frac{1 - q}{1 + q} + \frac{1 + q}{1 - q} \right) \\ &= \frac{1}{2} \left(\frac{(1 - q^4)(1 - q^{16}) \dots}{(1 + q^4)(1 + q^{16}) \dots} \right) \left(\frac{(1 - q)^2 + (1 + q)^2}{1 - q^2} \right) \\ &= \frac{(1 + q^2)(1 - q^4)(1 - q^{16}) \dots}{(1 - q^2)(1 + q^4)(1 + q^{16}) \dots}. \end{aligned}$$

So

$$\sum_{n \geq 0} W(2n)q^n \equiv \frac{(1+q)(1-q^2)(1-q^8)(1-q^{32}) \dots}{(1-q)(1+q^2)(1+q^8)(1+q^{32}) \dots} \pmod{3}.$$

We call the right-hand side of the above $G(q)$. Then we know

$$\begin{aligned} \sum_{n \geq 0} W(4n)q^{2n} &\equiv \frac{1}{2} (G(q) + G(-q)) \pmod{3} \\ &= \frac{1}{2} \left(\frac{(1-q^2)(1-q^8)(1-q^{32}) \dots}{(1+q^2)(1+q^8)(1+q^{32}) \dots} \right) \left(\frac{1+q}{1-q} + \frac{1-q}{1+q} \right) \\ &= \frac{1}{2} \left(\frac{(1-q^2)(1-q^8)(1-q^{32}) \dots}{(1+q^2)(1+q^8)(1+q^{32}) \dots} \right) \left(\frac{(1+q)^2 + (1-q)^2}{1-q^2} \right) \\ &= \frac{(1+q^2)(1-q^2)(1-q^8)(1-q^{32}) \dots}{(1-q^2)(1+q^2)(1+q^8)(1+q^{32}) \dots} \\ &= \frac{(1-q^8)(1-q^{32})(1-q^{128}) \dots}{(1+q^8)(1+q^{32})(1+q^{128}) \dots} \end{aligned}$$

after some fortuitous cancellations! Therefore, we have

$$\sum_{n \geq 0} W(4n)q^n \equiv \frac{(1-q^4)(1-q^{16})(1-q^{64}) \dots}{(1+q^4)(1+q^{16})(1+q^{64}) \dots} \pmod{3}. \tag{3}$$

Two very important observations can be made. First, note that the right-hand side of (3) is a function of q^4 . This is extremely significant as it means that, after the right-hand side has been written as a power series, many powers of q must have coefficients that are congruent to 0 modulo 3. This leads to our first Ramanujan-like congruences modulo 3:

Theorem 6. For all $n \geq 0$,

$$\begin{aligned} W(16n+4) &\equiv 0 \pmod{3}, \\ W(16n+8) &\equiv 0 \pmod{3}, \text{ and} \\ W(16n+12) &\equiv 0 \pmod{3}. \end{aligned}$$

Proof. In (3), replace n by $4n+1$, $4n+2$, and $4n+3$ to obtain the results in this theorem. These congruences must follow because the right-hand side of (3) is a function of q^4 . □

The second observation provides us with an important “internal congruence” modulo 3 that is satisfied by W . Namely, the right-hand side of (3) is actually $F(q^4)$! This then yields the following theorem:

Theorem 7. For all $n \geq 0$, $W(16n) \equiv W(n) \pmod{3}$.

Proof. Thanks to (3), we know that

$$\sum_{n \geq 0} W(4n)q^n \equiv F(q^4) \pmod{3}. \tag{4}$$

Since

$$F(q^4) \equiv \sum_{n \geq 0} W(n)q^{4n} \pmod{3},$$

(4) implies that

$$\sum_{n \geq 0} W(4n)q^n \equiv \sum_{n \geq 0} W(n)q^{4n} \pmod{3}.$$

Since the right-hand side of this congruence only contains powers of q of the form q^{4n} , we know that

$$\sum_{n \geq 0} W(4(4n))q^{4n} \equiv \sum_{n \geq 0} W(n)q^{4n} \pmod{3}.$$

Equating the corresponding coefficients on both sides of this congruence yields the result. \square

With Theorems 6 and 7 in hand, we can now prove the following infinite family of congruences modulo 3.

Theorem 8. *For all $j \geq 0$ and $n \geq 0$,*

$$\begin{aligned} W(2^{4j+3}n + 2^{4j+2}) &\equiv 0 \pmod{3} \text{ and} \\ W(2^{4j+4}n + 2^{4j+3}) &\equiv 0 \pmod{3}. \end{aligned}$$

Proof. The proof follows easily via induction on j . The basis step follows from Theorem 6 and the induction step follows from Theorem 7. \square

4. Closing Thoughts

We close with a number of thoughts, mostly related to potential future work. First, computational evidence indicates that additional Ramanujan-like congruences are satisfied by $W(n)$ modulo slightly larger moduli (including 8 and 9). This may prove to be the subject of future work. We also note that several other natural choices for the sets S_1 and S_2 exist. In particular, we could choose $S_1 = \{m^{2n-2}\}_{n \geq 1}$ and $S_2 = \{m^{2n-1}\}_{n \geq 1}$ where m is a fixed integer greater than 1. This would then lead to a family of restricted m -ary partition functions. One could then attempt to pursue a study similar to the one above.

Acknowledgements The authors gratefully acknowledge George Andrews for extremely valuable input shared during the development of this work.

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