MONOCHROMATIC SOLUTIONS OF EXPONENTIAL EQUATIONS

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Abstract

We show that for every 2-coloring of $\mathbb{N}$ and every $k \in \mathbb{N}$, there are infinitely many monochromatic solutions of the system of k^2 equations $z_{ij} = x_i^{y_j}, 1 \le i, j \le k$, where $x_1, \ldots, x_k, y_1, \ldots, y_k$ are distinct positive integers greater than 1. We give similar, but somewhat weaker, results for more than two colors.

– Dedicated to the memory of Paul Erdős.

1. Introduction

Using ultrafilters and results from [3, 4], Alessandro Sisto [5] showed that every 2-coloring of $\mathbb{N}$ gives infinitely many monochromatic sets of the form $\{a, b, a^b\}$, where $a, b > 1, a \neq b$, and he raised the question of whether there is an elementary proof of this fact.

We use van der Waerden's Theorem on arithmetic progressions to give an elementary proof of a generalization of Sisto's result. We show that for any 2-coloring of $\mathbb{N}$ and any $k \in \mathbb{N}$, there are infinitely many monochromatic sets of the form

$$\{a_1, a_2, \cdots, a_k, e_1, e_2 \cdots, e_k\} \cup \{a_i^{e_j} : 1 \le i, j \le k\},$$

where $a_1, \ldots, a_k, e_1, \ldots, e_k$ are distinct positive integers greater than 1.

We also show that for any 3-coloring of $\mathbb{N}$ and any $k \in \mathbb{N}$, either there are monochromatic sets as just mentioned, or there are monochromatic sets of the form

$$\{c^{a_1}, c^{a_2}, \cdots, c^{a_k}, c^{e_1}, c^{e_2}, \cdots, c^{e_k}\} \cup \{c^{a_i^{e_j}} : 1 \le i, j \le k\},$$

where c is a power of 3.

Analogous results hold for more than 3 colors. For example, for any 4-coloring of $\mathbb{N}$ and any $k \in \mathbb{N}$, either there are monochromatic sets of one of the previous two types, or there are monochromatic sets of the form

$$\{b^{c^{a_1}}, b^{c^{a_2}}, \cdots, b^{c^{a_k}}, b^{c^{e_1}}, b^{c^{e_2}}, \cdots, b^{c^{e_k}}\} \cup \{(b^{c^{a_i^{e_i}}} : 1 \le i, j \le k\},$$

where b, c are powers of 3.

In each case, $a_1, \ldots, a_k, e_1, \ldots, e_k$ are distinct positive integers greater than 1.

A somewhat different result was proved (using non-elementary methods) by Beiglböck et al [1, 2]: For every finite coloring of $\mathbb{N}$ and $k \in \mathbb{N}$, there are $a, b, d \in \mathbb{N}$ such that $\{b(a+id)^j : 0 \leq i, j \leq k\} \cup \{bd^j : 0 \leq j \leq k\} \cup \{a + id : 0 \leq i \leq k\}$ is monochromatic.

2. Two Colors

Definition 1. For $k \in \mathbb{N}$, an *exponential k-set* is a set of the form

$$\{a_1, a_2, \cdots, a_k, e_1, e_2 \cdots, e_k\} \cup \{a_i^{e_j} : 1 \leq i, j \leq k\},$$

where $a_1, \ldots, a_k, e_1, \ldots, e_k$ are distinct positive integers greater than 1.

Thus, an exponential k-set can be viewed as a non-trivial solution in $\mathbb{N}$, with distinct $x_1, \ldots, x_k, y_1, \ldots, y_k$, of the system of equations

$$z_{ij} = x_i^{y_j},\ 1 \leq i, j \leq k.$$

Theorem 1. *For every 2-coloring of* $\mathbb{N}$ *and* $k \in \mathbb{N}$ *there exists a monochromatic exponential* k*-set, that is, there exist distinct positive integers* $a_1, a_2, \cdots, a_k, e_1, e_2, \cdots, e_k$*, all greater than 1, such that*

$$\{a_1, a_2, \cdots, a_k, e_1, e_2, \cdots, e_k\} \cup \{a_i^{e_j} : 1 \leq i, j \leq k\ \}$$

is monochromatic.

Proof. Let us first carry out the proof for $k = 1$, which illustrates, without the complications which will come later, the basic scheme of the proof.

Let f be a 2-coloring of $\mathbb{N}$, using the colors 0 and 1. We seek a monochromatic set $\{a_1, e_1, a_1^{e_1}\}$ where $a_1, e_1 > 1,\ a_1 \neq e_1$. We define

$$g(x) = f(2^{3^x}), x \geq 1.$$

By van der Waerden's Theorem on arithmetic progressions, there are $p, d' \in \mathbb{N}$ with g constant on $\{p, p + d', p + 2d', \ldots, p + 16d'\}$.

In particular, with $d = 2d', g$ is constant on $\{p, p + d, p + 2d, \ldots, p + 8d\}$, and $d \geq 2$ Thus,

$$\{2^{3^{p+jd}} : 0 \leq j \leq 8\}$$

is monochromatic with respect to f, say with colour 0, and $d \geq 2$. There are now two cases to consider.

`Case 1.` There exists $x, 1 \le x \le 8$, with $f(3^{xd}) = 0$. Set $a_1 = 2^{3^p}, e_1 = 3^{xd}$. Then $\{a_1, e_1, a_1^{e_1}\}$ is monochromatic, and $a_1 \ne e_1$.

`Case 2.` $f(3^{xd}) = 1, 1 \le x \le 8$. If there exists $x, 2 \le x \le 8$, with $f(x) = 1$, then $\{3^d, x, 3^{xd}\}$ is monochromatic and $3^d \ne x$, since $x \le 8 < 3^d$. If no such x exists, then $f(x) = 0, 2 \le x \le 8$, and $\{2, 3, 8\}$ is monochromatic.

Now we turn to the general proof for $k > 1$. Let k be fixed, with $k > 1$.

Let f be a 2-coloring of $\mathbb{N}$, using the colors 0 and 1, and define

$$g(x) = f(2^{3^x}),\ x \ge 1.$$

We require g to be constant on an arithmetic progression with $w+1$ terms, where w is defined as follows.

Definition 2. The numbers $t_0, t_1, \ldots, t_{2k-1}$ are defined inductively by setting

$$t_0 = 1,\ t_{q+1} = (t_q + k)^{2(t_q+2k)},\ 0 \le q \le 2k-2.$$

Then we set

$$w = 2t_{2k-1}.$$

By van der Waerden's Theorem there are $p, d' \in \mathbb{N}$ so that g is constant on $\{p, p+d', p+2d', \ldots, p+ewd'\}$, where e is large enough that $3^e \ge w$. Then in particular, with $d = ed'$, g is constant on $\{p, p+d, p+2d, \ldots, p+wd\}$, where $3^d \ge 3^e \ge w$. (The inequality $3^d \ge w$ will be used below only in "Subcase 2a.")

Hence,

$$\{2^{3^p}, 2^{3^{p+d}}, 2^{3^{p+2d}}, \ldots, 2^{3^{p+wd}}\}$$

is monochromatic with respect to f, say with color 0, and $3^d \ge w$.

Let

$$T = \{j \in [1, w/2] : f(3^{jd}) = 0\}.$$

There are now two cases to consider.

`Case 1.` $|T| \ge k$. Let $x_1, \ldots, x_k \in T$. Then $x_j \le w/2, 1 \le j \le k$, and

$$f(3^{x_j d}) = 0,\ 1 \le j \le k.$$

In this case, we take

$$a_i = 2^{3^{p+x_i d}}, 1 \le i \le k,$$

$$e_j = 3^{x_j d}, 1 \le j \le k.$$

Then

$$a_i^{e_j} = 2^{3^{p+(x_i+x_j)d}},\ \text{and } x_i + x_j \le 2(w/2) = w,$$

hence

$$\{a_1, a_2, \cdots, a_k, e_1, e_2, \cdots, e_k\} \cup \{a_i^{e_j} : 1 \le i, j \le k\}$$

is monochromatic, with color 0, and clearly $a_1, a_2, \cdots, a_k, e_1, e_2, \cdots, e_k$ are distinct and greater than 1.

`Case 2.` $|T| < k$. Thus,

$$f(3^{xd}) = 1,\ x \in [1, w/2] - T,\ \text{and}\ \ |T| \le k - 1.$$

`Subcase 2a.` There exist $y_1, y_2, \ldots, y_k$, with the following two properties:

$$\{y_1, y_2, \ldots, y_k\} \cup \{y_i y_j : 1 \le i, j \le k\} \subset [1, w/2] - T$$

and

$$f(y_i) = 1,\ 1 \le i \le k.$$

Then we have $f(3^{xd}) = 1$ whenever $x = y_i$ or $x = y_i y_j$, $1 \le i, j \le k$, and $f(y_i) = 1, 1 \le i \le k$. Hence f is constant, with color 1, on the set

$$\{3^{y_1 d}, 3^{y_2 d}, \ldots, 3^{y_k d}, y_1, y_2, \ldots, y_k\} \cup \{(3^{y_i d})^{y_j} : 1 \le i, j \le k\}.$$

We may assume $y_1 < y_2 < \cdots < y_k$, so that $3^{y_1 d} < 3^{y_2 d} < \cdots < 3^{y_k d}$. To show that $3^{y_1 d}, 3^{y_2 d}, \ldots, 3^{y_k d}, y_1, y_2, \ldots, y_k$ are distinct, we simply note that $y_k < w \le 3^d \le 3^{y_1 d}$, and hence

$$y_1 < y_2 < \cdots < y_k < 3^{y_1 d} < 3^{y_2 d} < \cdots < 3^{y_k d}.$$

`Subcase 2b.` There do not exist numbers $y_1, y_2, \ldots, y_k$ as in Subcase 2a. This means that for any

$$\{y_1, y_2, \ldots, y_k\} \cup \{y_i y_j : 1 \le i, j \le k\} \subset [1, w/2] - T$$

there is at least one $i, 1 \le i \le k$, such that $f(y_i) = 0$.

Now we make explicit use of the numbers $t_0, \ldots, t_{2k-1}$ defined above.

Definition 3. The sets $A_q \subset B_q, 1 \le q \le 2k-1$, are defined inductively by setting

$$A_1 = [t_0 + 1, (t_0 + k)^{t_0 + 2k}],\ B_1 = [t_0 + 1, (t_0 + k)^{2(t_0 + 2k)}] = [t_0 + 1, t_1],$$

$$A_{q+1} = [t_q + 1, (t_q + k)^{t_q + 2k}],\ B_{q+1} = [t_q + 1, (t_q + k)^{2(t_q + 2k)}] = [t_q + 1, t_{q+1}].$$

Note that for each q, $0 \leq q \leq 2k-2$, we can write

$$A_{q+1} = [t_q+1, t_q+2k] \cup [t_q+2k+1, (t_q+k)^{t_q+2k}],$$

so that if we take

$$[t_q+1, t_q+2k] = [a_1, a_2, \ldots, a_k, e_1, e_2, \ldots, e_k],$$

then A_{q+1} contains the exponential k-set

$$\{a_1, \ldots, a_k, e_1, \ldots, e_k\} \cup \{a_i^{e_j} : 1 \leq i, j \leq k\}.$$

Also, recall that $\max B_{2k-1} = t_{2k-1} = w/2$, hence

$$B_1 \cup \cdots \cup B_{2k-1} \subset [1, w/2].$$

We shall show that one of the sets A_i is monochromatic under f, with color 0.

We have $2k-1$ pairwise disjoint subsets $B_1, \ldots, B_{2k-1}$ of $[1, w/2]$, and, since $|T| < k$, at most $k-1$ of them can meet T. Hence, there are k sets $B_{i_1}, \ldots, B_{i_k}$ (we assume that $i_1 < \cdots < i_k$) such that

$$B_{i_1} \cup \cdots \cup B_{i_k} \subset [1, w/2] - T.$$

(In Lemma 2 below, we consider the corresponding union $A_{i_1} \cup \cdots \cup A_{i_k}$.)

Lemma 1. *If $i \leq j$ and $y \in A_i, z \in A_j$, then $yz \in B_j$.*

Proof. From the definitions of A_j, B_j, we can simplify the notation to write $A_j = [a, b], B_j = [a, b^2]$. Then $y \in A_i, z \in A_j$ implies $2 \leq y \leq b$ and $a \leq z \leq b$, therefore $a \leq z < yz \leq b^2$, hence $yz \in B_j$. □

Lemma 2. *Let $S = \{y \in A_{i_1} \cup \cdots \cup A_{i_k} : f(y) = 1\}$, and assume that $B_{i_1} \cup \cdots \cup B_{i_k} \subset [1, w/2] - T$. Then $|S| < k$.*

Proof. Suppose $|S| \geq k$. Let $y_1, \ldots, y_k \in S \subset A_{i_1} \cup \cdots \cup A_{i_k}$. Then by Lemma 2.1,

$$\{y_1, y_2, \ldots, y_k\} \cup \{y_i y_j : 1 \leq i, j \leq k\} \subset B_{i_1} \cup \cdots \cup B_{i_k} \subset [1, w/2] - T.$$

But since we are in Subcase 2b, this immediately implies that $f(y_i) = 0$ for some i, a contradiction. □

Thus, we now have $|S| < k$ and

$$\text{if } y \in A_{i_1} \cup \cdots \cup A_{i_k} - S \text{ then } f(y) = 0.$$

Since S can meet at most $k-1$ of the intervals $A_{i_1}, \cdots, A_{i_k}$, there is some $q, 1 \leq q \leq k$, such that

$$f(y) = 0, \ y \in A_{i_q}.$$

Since A_{i_q} contains an exponential k-set, this finishes the proof of Theorem 1. □

Corollary 1. *Given $A, k \in \mathbb{N}$, there exists $M(A,k) \in \mathbb{N}, M(A,k) > A$, such that for every 2-coloring of $[A, M(A,k))$ there exists a monochromatic exponential k-set.*

Proof. The exponential $(A+k)$-set

$$\{a_1, \ldots, a_{A+k}, e_1, \ldots, e_{A+k}\} \cup \{a_i^{e_j} : 1 \leq i, j \leq A+k\},$$

where we can assume that $a_1 < \cdots < a_{A+k}$ and $e_1 < \cdots < e_{A+k}$, contains the exponential k-set

$$\{a_{A+1}, \ldots, a_{A+k}, e_{A+1}, \ldots, e_{A+k}\} \cup \{a_i^{e_j} : A+1 \leq i, j \leq A+k\},$$

which is contained in $[A, \infty)$.

Thus, given any 2-coloring f of $[A, \infty)$, extend f arbitrarily to a 2-coloring of $\mathbb{N}$. By Theorem 1, there exists a monochromatic exponential $(A+k)$-set, which contains an exponential k-set in $[A, \infty)$. By compactness, the result follows. □

Corollary 1 is the basis of the proofs for the results involving more than 2 colors.

Corollary 2. *For every 2-coloring of $\mathbb{N}$ and $k \in \mathbb{N}$ there exist infinitely many monochromatic exponential k-sets.*

Proof. This follows immediately from Corollary 1. □

3. Three and Four Colors

Theorem 2. *For every 3-coloring of $\mathbb{N}$ and $k \in \mathbb{N}$ there exist distinct $a_1, a_2, \cdots, a_k, e_1, e_2, \cdots, e_k$, all greater than 1, and $c = 3^d > 1$ such that*

$$\{a_1, a_2, \cdots, a_k, e_1, e_2, \cdots, e_k\} \cup \{a_i^{e_j} : 1 \leq i, j \leq k\ \}$$

or

$$\{c^{a_1}, c^{a_2}, \cdots, c^{a_k}, c^{e_1}, c^{e_2}, \cdots, c^{e_k}\} \cup \{c^{a_i^{e_j}} : 1 \leq i, j \leq k\ \}$$

is monochromatic.

Proof. Let $k \in \mathbb{N}$ and let f be a 3-coloring of $\mathbb{N}$, using the colors $0, 1, 2$. Using the notation of Corollary 1, define $W_q, 1 \leq q \leq k$ by setting

$$W_1 = M(1,k), W_{q+1} = M(W_q, k), 1 \leq q \leq k-1.$$

We follow closely the first part of the proof of Theorem 1.

By van der Waerden's Theorem, there are $p, d \in \mathbb{N}$ so that

$$\{2^{3^p}, 2^{3^{p+d}}, 2^{3^{p+2d}}, \ldots, 2^{3^{p+2W_k d}}\}$$

is monochromatic with respect to f, say with color 0.

Let

$$T = \{j \in [1, W_k) : f(3^{jd}) = 0\},$$

so that

$$f(3^{jd}) \in \{1, 2\}, \ \forall j \in [1, W_k) - T.$$

There are now two cases to consider.

Case 1. $|T| \geq k$. We proceed exactly as in Case 1 of the proof of Theorem 1, to obtain a monochromatic set of colour 0

$$\{a_1, a_2, \cdots, a_k, e_1, e_2, \cdots, e_k\} \cup \{a_i^{e_j} : 1 \leq i, j \leq k \ \}.$$

Case 2. $|T| < k$. We make use of Corollary 1. Consider the intervals

$$[1, W_1), [W_1, W_2), \ldots, [W_{k-1}, W_k).$$

The set T can meet at most $k-1$ of these intervals, so for some q we have

$$[W_q, W_{q+1}) \subset [1, W_k] - T.$$

Thus $g(j) = f(3^{jd}), j \in [W_q, W_{q+1})$ is a 2-coloring of $[W_q, M(W_q, k))$ and by the definition of $M(W_q, k)$ there is an exponential k-set

$$\{a_1, a_2, \cdots, a_k, e_1, e_2, \cdots, e_k\} \cup \{a_i^{e_j} : 1 \leq i, j \leq k \ \}$$

which is monochromatic with respect to g.

Hence, with $c = 3^d$, we have $g(a_1) = f(c^{a_1}), g(a_2) = f(c^{a_2}), \ldots$, so that

$$\{c^{a_1}, c^{a_2}, \cdots, c^{a_k}, c^{e_1}, c^{e_2}, \cdots, c^{e_k}\} \cup \{c^{a_i^{e_j}} : 1 \leq i, j \leq k \ \}$$

is monochromatic with respect to f. □

Corollary 3. *Given $A, k \in \mathbb{N}$, there exists $M(A, k) \in \mathbb{N}$ such that for every 3-coloring of $[A, M(A, k))$ there are distinct $a_1, a_2, \cdots, a_k, e_1, e_2, \cdots, e_k$, all greater than 1, and $c = 3^d, d \in \mathbb{N}$, such that*

$$\{a_1, a_2, \cdots, a_k, e_1, e_2, \cdots, e_k\} \cup \{a_i^{e_j} : 1 \leq i, j \leq k \ \}$$

or

$$\{c^{a_1}, c^{a_2}, \cdots, c^{a_k}, c^{e_1}, c^{e_2}, \cdots, c^{e_k}\} \cup \{c^{a_i^{e_j}} : 1 \leq i, j \leq k \ \}$$

is monochromatic.

Proof. The proof is exactly the same as the proof of Corollary 1. □

Theorem 3. *For every 4-coloring of* $\mathbb{N}$ *and* $k \in \mathbb{N}$ *there exist distinct* $a_1, a_2, \cdots, a_k, e_1, e_2, \cdots, e_k$*, all greater than 1,* $c = 3^d, b = 3^{d'}, d, d' \in \mathbb{N}$ *such that*

$$\{a_1, a_2, \cdots, a_k, e_1, e_2, \cdots, e_k\} \cup \{a_i^{e_j} : 1 \leq i, j \leq k\ \}$$

or

$$\{c^{a_1}, c^{a_2}, \cdots, c^{a_k}, c^{e_1}, c^{e_2}, \cdots, c^{e_k}\} \cup \{c^{a_i^{e_j}} : 1 \leq i, j \leq k\ \}$$

or

$$\{b^{c^{a_1}}, b^{c^{a_2}}, \cdots, b^{c^{a_k}}, b^{c^{e_1}}, b^{c^{e_2}}, \cdots, b^{c^{e_k}}\} \cup \{(b^{c^{a_i^{e_i}}} : 1 \leq i, j \leq k\ \}$$

is monochromatic.

Proof. The proof is virtually the same as the proof of Theorem 2, here using Corollary 3 instead of Corollary 1.

□

4. The General Case

Theorem 4. *Let* $r, k \in \mathbb{N}$ *and let an* r*-coloring of* $\mathbb{N}$ *be given. If* $r = 2$ *there exist infinitely many monochromatic exponential* k*-sets. If* $r > 2$*, either there are infinitely many monochromatic exponential* k*-sets, or there exist* $1 \leq s \leq r-2$*, and infinitely many monochromatic sets of the form*

$$\{c_1^{c_2^{\cdot^{\cdot^{\cdot^{c_s^{a_1}}}}}}, c_1^{c_2^{\cdot^{\cdot^{\cdot^{c_s^{a_2}}}}}}, \ldots, c_1^{c_2^{\cdot^{\cdot^{\cdot^{c_s^{a_k}}}}}}, c_1^{c_2^{\cdot^{\cdot^{\cdot^{c_s^{e_1}}}}}}, c_1^{c_2^{\cdot^{\cdot^{\cdot^{c_s^{e_2}}}}}}, \ldots, c_1^{c_2^{\cdot^{\cdot^{\cdot^{c_s^{e_k}}}}}}\} \cup \{c_1^{c_2^{\cdot^{\cdot^{\cdot^{c_s^{a_i^{e_j}}}}}}} : 1 \leq i, j \leq k\},$$

where $a_1, a_2, \cdots, a_k, e_1, e_2, \cdots, e_k$ *are distinct positive integers greater than 1, and the* $c_i, 1 \leq i \leq s$*, are (not necessarily distinct) powers of* 3.

In fact, given $A, r \in \mathbb{N}$*, there is* $M(A, k, r)$ *such that, for every* r*-coloring of the interval* $[A, M(A, k, r))$*, there exists a monochromatic set of one of these types.*

Proof. The proof is by induction on r, following the methods of the proofs of Theorem 2 and Corollary 1. □

We conclude this paper by proposing the following questions.

Questions. Does every r-coloring of $\mathbb{N}$ give a monochromatic set $\{a, b, a^b\}$, with $a, b > 1, a \neq b$? Does every 2-coloring of $\mathbb{N}$ give a monochromatic solution of

$$w = x^{y^z}?$$

Let $h(k)$ denote the smallest n such that for any 2-coloring of $[1, n]$ there exists a monochromatic exponential k-set. Let $W(k)$ denote the smallest n such that for any 2-coloring of $[1, n]$ there exists a monochromatic k-term arithmetic progression. The proof of Theorem 2.1 shows that $h(1) \leq 2^{3^{W(16)}}$ and that $h(2)$ is bounded above, roughly speaking, by $2^{3^{W(s\cdot 3^s)}}$, where $s = 40 \cdot 3^{20\cdot 3^{10}}$. Perhaps these bounds can be decreased a bit.

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