ON ZERO-SUM PARTITIONS OF ABELIAN GROUPS

Xiangneng Zeng[1]
Sino-French Institute of Nuclear Engineering and Technology, Sun Yat-Sen University, Guangzhou, P.R.China
junevab@163.com

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Abstract

In this paper, confirming a conjecture of Kaplan et al., we prove that every abelian group G, which is of odd order or contains exactly three involutions, has the zero-sum-partition property. As a corollary, every tree with $|G|$ vertices and at most one vertex of degree 2 is G-anti-magic.

1. Introduction

In 1990, Hartsfield and Ringel [2] introduced the concept of anti-magic graphs. An *anti-magic labeling* of a graph with m edges and n vertices is a bijection from the set of edges to the set of integers $\{1, 2, \ldots, m\}$ such that all the n vertex-sums are pairwise distinct, where the vertex-sum of a vertex v is the sum of labels of all edges incident with v. A graph is called *anti-magic* if it has an anti-magic labeling. Hartsfield and Ringel showed the anti-magicness of some graphs and conjectured that all connected graphs except K_2 are anti-magic.

The conjecture has received much attention. Many graphs have been proven to be anti-magic, see [1, 3] for example. However, the conjecture is still open.

In [3], Kaplan, Lev and Roditty consider the following generalization of the concept of an anti-magic graph. For an abelian group, let $G^\bullet = G \setminus \{0\}$.

Definition 1.1. ([3]) Let $H = (V, E)$ be a graph, where $|V| = n$, $|E| = m$. Let G be an abelian group and let A be a finite subset of $G^\bullet$ with $|A| = m$. An *A-labeling* of H is a one-to-one mapping $f : E(H) \to A$. Given an A-labeling of H, the *weight* of a vertex $v \in V(G)$ is $w(v) = \sum_{uv\in E(H)} f(uv)$.

(i) We shall say that H is *A-anti-magic* if there is an A-labeling of H such that the weights $\{w(v) : v \in V(H)\}$ are all distinct.

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(ii) In the case that G is finite, we shall say that H is *G-anti-magic* if H is $G^\bullet$-anti-magic.

They conjecture that a tree with $|G|$ vertices is G-anti-magic if and only if G is not a group with a unique involution. To tackle the conjecture, the following definition is introduced.

Definition 1.2. ([3]) Let G be an abelian group and A a finite subset of $G^\bullet$ with $|A| = n$. We say that A has the *zero-sum-partition property (ZSP-property)* if, for every partition $n = r_1 + r_2 + \cdots + r_t$ of n, with $r_i \geq 2$ for $1 \leq i \leq t$, there is a partition of A into pairwise disjoint subsets $A_1, A_2, \ldots, A_t$, such that $|A_i| = r_i$ and $\sum_{a \in A_i} a = 0$ for $1 \leq i \leq t$. In the case that G is finite, we shall say that G has the *ZSP-property* if $A = G^\bullet$ has the ZSP-property.

To verify that A has the ZSP-property, it suffices to show that, for any non-negative integers k and l with $n = 3k + 2l$, there is a partition of A into pairwise disjoint subsets $A_1, \ldots, A_k, B_1, \ldots, B_l$, such that $|A_i| = 3$, $|B_j| = 2$ and $\sum_{a \in A_i} a = \sum_{b \in B_j} b = 0$ for $1 \leq i \leq k$ and $1 \leq j \leq l$. Obviously, $B_j = \{b_j, -b_j\}$ for some $b_j \in A$.

The following proposition shows the relation between the two definitions above. A tree is called a 2-tree if it has at most one vertex of degree 2.

Proposition 1.3. ([3]) *Let G be a finite abelian group with the ZSP-property. Then every 2-tree with $|G|$ vertices is G-anti-magic.*

Therefore, to show that such a 2-tree is G-anti-magic, it suffices to show the ZSP-property of G. G. Kaplan, A. Lev and Y. Roditty suggest the following conjecture.

Conjecture 1.4. ([3]) *Let G be a finite abelian group. Then G has the ZSP-property if and only if either G is of odd order or G contains exactly three involutions.*

The authors of [3] prove the necessity of the conjecture and verify the ZSP-property for cyclic groups of odd order and elementary abelian groups of order $n = p^k$, where p is a prime congruent to 1 modulo 3. As a corollary, every 2-tree with n vertices, where n is odd, is $\mathbb{Z}_n$-anti-magic. In particular, such a tree is anti-magic.

In this paper, we completely prove the sufficiency of the conjecture.

Theorem 1.5. *Let G be a finite abelian group of odd order or with exactly three involutions. Then G has the ZSP-property.*

Together with Proposition 1.3, we have

Corollary 1.6. *Let G be a finite abelian group of odd order or with exactly three involutions. Then every 2-tree with $|G|$ vertices is G-anti-magic.*

For convenience, let $\mathcal{G}$ denote the set consisting of all abelian groups which are of odd order or contain exactly three involutions. Notice that for every $G \in \mathcal{G}$, we have $G \cong \mathbb{Z}_{2^\alpha} \oplus \mathbb{Z}_{2^\beta} \oplus K$, where K is of odd order and $\alpha = \beta = 0$ or $1 \le \alpha \le \beta$. Since the properties and results in this paper are invariant under the isomorphism between groups, we only need to consider a group in an isomorphism class. For an integer a and the residue class group $\mathbb{Z}_n$, we use $\bar{a}$ to denote the corresponding residue class in $\mathbb{Z}_n$.

Throughout the paper, unions will always be disjoint.

2. A Lemma

In this section, we prove a lemma which plays an important role in the proof of the main result.

Lemma 2.1. *Let $G \in \mathcal{G}$ and $\mathrm{Bij}(G)$ denote the set of all bijections from G to itself. Then there exist $\phi, \varphi \in \mathrm{Bij}(G)$ (not necessarily distinct) such that $a+\phi(a)+\varphi(a) = 0$ for every $a \in G$. In particular, we may assume that $\phi(0) = \varphi(0) = 0$.*

Proof. First we prove an assertion: let $G_1, G_2 \in \mathcal{G}$ and suppose we have proven the lemma for G_1, G_2, then the lemma holds for $G = G_1 \oplus G_2$. Indeed, suppose the resulting bijections are ϕ_1 and φ_1 for G_1 and ϕ_2 and φ_2 for G_2. Consider the following maps:

$$\phi = (\phi_1, \phi_2): \ G_1 \oplus G_2 \to G_1 \oplus G_2, \ (a_1, a_2) \mapsto (\phi_1(a_1), \phi_2(a_2))$$

and

$$\varphi = (\varphi_1, \varphi_2): \ G_1 \oplus G_2 \to G_1 \oplus G_2, \ (a_1, a_2) \mapsto (\varphi_1(a_1), \varphi_2(a_2)).$$

It is easy to see that ϕ and φ are the desired bijections and thus the assertion is proven.

By the assertion above and noting that this lemma is invariant under the isomorphism, it suffices to prove the lemma for G of odd order and $G = \mathbb{Z}_{2^\alpha} \oplus \mathbb{Z}_{2^\beta}$ with $1 \le \alpha \le \beta$.

The case when G is of odd order is very simple. Indeed, let $\phi(a) = a$ and $\varphi(a) = -2a$ for every $a \in G$. Then these two maps are the desired bijections and thus we are done.

Now, we tackle the case when $G = \mathbb{Z}_{2^\alpha} \oplus \mathbb{Z}_{2^\beta}$, whose elements are denoted by $(\bar{x}, \bar{y})$ with $x, y \in \mathbb{Z}$. The proof is by induction on $|G|$. We handle three basic cases.

Suppose first that $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then the following table gives the desired bijections.

g	$(\bar{0}, \bar{0})$	$(\bar{1}, \bar{0})$	$(\bar{0}, \bar{1})$	$(\bar{1}, \bar{1})$
$\phi(g)$	$(\bar{0}, \bar{0})$	$(\bar{0}, \bar{1})$	$(\bar{1}, \bar{1})$	$(\bar{1}, \bar{0})$
$\varphi(g)$	$(\bar{0}, \bar{0})$	$(\bar{1}, \bar{1})$	$(\bar{1}, \bar{0})$	$(\bar{0}, \bar{1})$

Next suppose $G = \mathbb{Z}_2 \oplus \mathbb{Z}_4$. Then the following table gives the desired bijections.

g	$(\bar{0},\bar{0})$	$(\bar{0},\bar{1})$	$(\bar{0},\bar{2})$	$(\bar{0},\bar{3})$	$(\bar{1},\bar{0})$	$(\bar{1},\bar{1})$	$(\bar{1},\bar{2})$	$(\bar{1},\bar{3})$
$\phi(g)$	$(\bar{0},\bar{0})$	$(\bar{1},\bar{0})$	$(\bar{0},\bar{1})$	$(\bar{1},\bar{1})$	$(\bar{1},\bar{2})$	$(\bar{0},\bar{2})$	$(\bar{1},\bar{3})$	$(\bar{0},\bar{3})$
$\varphi(g)$	$(\bar{0},\bar{0})$	$(\bar{1},\bar{3})$	$(\bar{0},\bar{1})$	$(\bar{1},\bar{0})$	$(\bar{0},\bar{2})$	$(\bar{1},\bar{1})$	$(\bar{0},\bar{3})$	$(\bar{1},\bar{2})$

Finally, suppose $G = \mathbb{Z}_2 \oplus \mathbb{Z}_8$. We have a disjoint partition

$$G = \{0\} \bigcup \left(\bigcup_{i=1}^{5} \{a_{i,1}, a_{i,2}, a_{i,3}\} \right), \tag{2.1}$$

where $a_{i,j} \in G$ and $a_{i,1} + a_{i,2} + a_{i,3} = 0$. Indeed, the following partition is such a partition.

$$\begin{aligned} G \quad = \quad & \{(\bar{0},\bar{0})\} \bigcup \{(\bar{1},\bar{0}),(\bar{1},\bar{1}),(\bar{0},\bar{7})\} \bigcup \{(\bar{1},\bar{2}),(\bar{1},\bar{4}),(\bar{0},\bar{2})\} \\ & \bigcup \{(\bar{1},\bar{3}),(\bar{1},\bar{7}),(\bar{0},\bar{6})\} \bigcup \{(\bar{1},\bar{5}),(\bar{1},\bar{6}),(\bar{0},\bar{5})\} \bigcup \{(\bar{0},\bar{1}),(\bar{0},\bar{3}),(\bar{0},\bar{4})\}. \end{aligned}$$

Define ϕ and φ as follows.

$$\phi(a) = \begin{cases} 0 & \text{if } a = 0 \\ a_{i,2} & \text{if } a = a_{i,1} \text{ for some } 1 \le i \le 5, \\ a_{i,3} & \text{if } a = a_{i,2} \text{ for some } 1 \le i \le 5, \\ a_{i,1} & \text{if } a = a_{i,3} \text{ for some } 1 \le i \le 5, \end{cases}$$

and

$$\varphi(a) = \begin{cases} 0 & \text{if } a = 0 \\ a_{i,3} & \text{if } a = a_{i,1} \text{ for some } 1 \le i \le 5, \\ a_{i,1} & \text{if } a = a_{i,2} \text{ for some } 1 \le i \le 5, \\ a_{i,2} & \text{if } a = a_{i,3} \text{ for some } 1 \le i \le 5. \end{cases}$$

It is easy to see that such ϕ and φ are as desired.

Now we proceed to the induction part.

Suppose first that $G = \mathbb{Z}_{2^\alpha} \oplus \mathbb{Z}_{2^\beta}$ with $\alpha \ge 2$. Then there exists a subgroup $G_0 \in \mathcal{G}$ such that $G/G_0 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. By the induction hypothesis, there are $\phi_0, \varphi_0 \in \mathrm{Bij}(G_0)$ such that $a + \phi_0(a) + \varphi_0(a) = 0$ for every $a \in G_0$. Choose a set of coset representatives for G/G_0 to be $\{0, c, d, -c-d\}$. Note that

$$(c + G_0) \bigcup (d + G_0) \bigcup (-c - d + G_0) = \bigcup_{b \in G_0} \{c + b, d + \phi_0(b), -c - d + \varphi_0(b)\},$$

and every subset $\{c + a, d + \phi_0(a), -c - d + \varphi_0(a)\}$ is zero-sum. Thus define ϕ and φ as follows.

$$\phi(a) = \begin{cases} \phi_0(a) & \text{if } a \in G_0 \\ d + \phi_0(b) & \text{if } a = c + b \text{ for some } b \in G_0, \\ -c - d + \varphi_0(b) & \text{if } a = d + \phi_0(b) \text{ for some } b \in G_0, \\ c + b & \text{if } a = -c - d + \varphi_0(b) \text{ for some } b \in G_0, \end{cases}$$

and

$$\varphi(a)=\begin{cases}\varphi_0(a) & \text{if } a\in G_0\\ -c-d+\varphi_0(b) & \text{if } a=c+b \text{ for some } b\in G_0,\\ c+b & \text{if } a=d+\phi_0(b) \text{ for some } b\in G_0,\\ d+\phi_0(b) & \text{if } a=-c-d+\varphi_0(b) \text{ for some } b\in G_0.\end{cases}$$

Such ϕ and φ are as desired. Thus it remains to handle the case when $\alpha=1$.

Suppose now that $G=\mathbb{Z}_2\oplus\mathbb{Z}_{2^\beta}$. The cases when $\beta\le 3$ have been shown in the base of the induction. Thus we may assume that $\beta\ge 4$ and there is a subgroup $G_0\in\mathcal{G}$ such that $G/G_0\cong\mathbb{Z}_8$. By the induction hypothesis, there are $\phi_0,\varphi_0\in\mathrm{Bij}(G_0)$ such that $a+\phi_0(a)+\varphi_0(a)=0$ for every $a\in G_0$. Since

$$\mathbb{Z}_8=\{\bar{0},\bar{4}\}\bigcup\{\bar{1},\bar{2},\bar{5}\}\bigcup\{-\bar{1}=\bar{7},-\bar{2}=\bar{6},-\bar{5}=\bar{3}\},$$

we can choose a set of coset representatives for G/G_0 to be $\{0,e,c,d,-c-d,-c,-d,c+d\}$ with $2e\in G_0$. Since $G_1=G_0\cup(e+G_0)$ is a group and $G_0<G_1<G$, we have $G_1\in\mathcal{G}$. Thus there are $\phi_1,\varphi_1\in\mathrm{Bij}(G_1)$ such that $a+\phi_0(a)+\varphi_0(a)=0$ for every $a\in G_1$. Similarly as above, we have

$$(c+G_0)\bigcup(d+G_0)\bigcup(-c-d+G_0)=\bigcup_{b\in G_0}\{c+b,d+\phi_0(b),-c-d+\varphi_0(b)\}$$

and

$$(-c+G_0)\bigcup(-d+G_0)\bigcup(c+d+G_0)=\bigcup_{b\in G_0}\{-c+b,-d+\phi_0(b),c+d+\varphi_0(b)\}.$$

Thus $G=G_1\cup(\cup_{i=1}^{2|G_0|}\{a_{i,1},a_{i,2},a_{i,3}\})$ where $a_{i,1}+a_{i,2}+a_{i,3}=0$ for $1\le i\le 2|G_0|$. Define ϕ and φ as follows.

$$\phi(a)=\begin{cases}\phi_1(a) & \text{if } a\in G_1\\ a_{i,2} & \text{if } a=a_{i,1} \text{ for some } 1\le i\le 2|G_0|,\\ a_{i,3} & \text{if } a=a_{i,2} \text{ for some } 1\le i\le 2|G_0|,\\ a_{i,1} & \text{if } a=a_{i,3} \text{ for some } 1\le i\le 2|G_0|,\end{cases}$$

and

$$\varphi(a)=\begin{cases}\varphi_1(a) & \text{if } a\in G_1\\ a_{i,3} & \text{if } a=a_{i,1} \text{ for some } 1\le i\le 2|G_0|,\\ a_{i,1} & \text{if } a=a_{i,2} \text{ for some } 1\le i\le 2|G_0|,\\ a_{i,2} & \text{if } a=a_{i,3} \text{ for some } 1\le i\le 2|G_0|.\end{cases}$$

Such ϕ and φ are as desired. This completes the proof of the main part of the lemma.

Finally, suppose we have found $\phi,\varphi\in\mathrm{Bij}(G)$ such that $a+\phi(a)+\varphi(a)=0$ for every $a\in G$. Consider new bijections ϕ',φ' defined by $\phi'(a)=\phi(a)-\phi(0)$ and $\varphi'(a)=\varphi(a)-\varphi(0)$ for every $a\in G$. The final statement of the lemma follows. $\square$

3. Some cases

When G is a cyclic group of odd order, it is a result of Kaplan et al.

Theorem 3.1. ([3]) *Let $G = \mathbb{Z}_n$, where n is odd. Then G has the ZSP-property.*

Using the same method, we can obtain the following lemma.

Lemma 3.2. *Let t be a positive integer and*

$$T = \{-t, -(t-1), \ldots, -1, 1, 2, \ldots, t\} \subset \mathbb{Z}.$$

Then T has the ZSP-property.

Proof. Let k and l be nonnegative integers with $2t = 3k + 2l$. We shall show that there are pairwise disjoint subsets $A_1, A_2, \ldots, A_k, B_1, \ldots, B_l$ which form a partition of T, such that $|A_i| = 3$, $|B_j| = 2$ and $\sum_{a \in A_i} a = \sum_{b \in B_j} b = 0$ for every $1 \le i \le k$ and $1 \le j \le l$.

If $k = 0$, that is, $t = l$, set $B_i = \{i, -i\}$ for $1 \le i \le t$, and the lemma easily follows. Thus we may assume from now on that $k \ge 1$. Note that $3|(t-l)$. We define the following 3-subsets of T:

$$\begin{aligned}
A_1 &= \{1, \tfrac{t-l}{3} + l + 1, -\tfrac{t-l}{3} - l - 2\},\\
A_2 &= \{2, \tfrac{t-l}{3} + l + 2, -\tfrac{t-l}{3} - l - 4\},\\
&\ \vdots\\
A_{\frac{t-l}{3}} &= \{\tfrac{t-l}{3}, \tfrac{2(t-l)}{3} + l, -t\},\\
A_{\frac{t-l}{3}+1} &= \{\tfrac{2(t-l)}{3} + l + 1, -\tfrac{t-l}{3} - l - 1, -\tfrac{t-l}{3}\},\\
A_{\frac{t-l}{3}+2} &= \{\tfrac{2(t-l)}{3} + l + 2, -\tfrac{t-l}{3} - l - 3, -\tfrac{t-l}{3} + 1\},\\
&\ \vdots\\
A_{\frac{2(t-l)}{3}-1} &= \{\tfrac{2(t-l)}{3} + l + \tfrac{t-l}{3} - 1, -\tfrac{t-l}{3} - l - (\tfrac{2(t-l)}{3} - 3), -\tfrac{t-l}{3} + \tfrac{t-l}{3} - 2\}\\
&= \{t-1, -t+3, -2\},\\
A_k = A_{\frac{2(t-l)}{3}} &= \{t, -t+1, -1\},
\end{aligned}$$

and the following 2-subsets of T:

$$\begin{aligned}
B_1 &= \{\tfrac{t-l}{3} + 1, -\tfrac{t-l}{3} - 1\},\\
B_2 &= \{\tfrac{t-l}{3} + 2, -\tfrac{t-l}{3} - 2\},\\
&\ \vdots\\
B_l &= \{\tfrac{t-l}{3} + l, -\tfrac{t-l}{3} - l\}.
\end{aligned}$$

One can observe that T is the disjoint union of $A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_l$. Moreover, all these subsets are zero-sum. Thus the lemma follows. $\square$

Corollary 3.3. *Let G be an abelian group and $g \in G$ of order n. Then the set*

$$A = \{tg, 2tg, \ldots, itg, (n-it)g, (n-(i-1)t)g, \ldots, (n-t)g\}$$

has the ZSP-property, where t and i are positive integers with $it < (n-it)$.

Proof. Note that

$$\begin{aligned} A &= \{tg, 2tg, \ldots, itg, (n-it)g, (n-(i-1)t)g, \ldots, (n-t)g\} \\ &= \{-i, -(i-1), \ldots, -1, 1, 2, \ldots, i\}tg. \end{aligned}$$

The corollary follows directly from Lemma 3.2. □

To prove the ZSP-property for general abelian groups, it is useful to introduce a definition. We call a 6-subset C of an abelian group G *good* if $C = \{c, d, -c-d, -c, -d, c+d\}$ for some distinct $c, d \in G$. Since

$$C = \{c, d, -c-d\} \cup \{-c, -d, c+d\} = \{c, -c\} \cup \{d, -d\} \cup \{c+d, -c-d\},$$

our main idea is to obtain a partition of $G^\bullet$ with as many good 6-subsets as possible. Now we give some results in this direction.

Lemma 3.4. *Let $G = \mathbb{Z}_p \oplus \mathbb{Z}_p$ where p is an odd prime.*

(i) If $p \equiv 2 \bmod 3$, then

$$G^\bullet = \bigcup_{i=1}^{(p^2-1)/6} C_i,$$

where $C_1, \ldots, C_{(p^2-1)/6}$ are pairwise disjoint good 6-subsets.

(ii) If $p \equiv 1 \bmod 3$, then

$$G^\bullet = G_{-1}^\bullet \bigcup G_0^\bullet \bigcup \left(\bigcup_{i=1}^{(p-1)^2/6} C_i \right),$$

where G_{-1} and G_0 are distinct proper subgroups of order p and $C_1, \ldots, C_{(p-1)^2/6}$ are pairwise disjoint good 6-subsets.

Moreover, in both cases, G has the ZSP-property.

Proof. Let $G_{-1} = \langle(\bar{0}, \bar{1})\rangle$ and $G_i = \langle(\bar{1}, \bar{i})\rangle$ for $0 \le i \le p-1$. Obviously, $G^\bullet$ is the disjoint union of $G_i^\bullet$, $-1 \le i \le p-1$.

We assert that, for all $0 \le j \le p-2$, $G_{j-1}^\bullet \cup G_j^\bullet \cup G_{j+1}^\bullet$ can be partitioned into the disjoint union of good 6-subsets. Indeed, for $j = 0$,

$$\begin{aligned} G_{-1}^\bullet \bigcup G_0^\bullet \bigcup G_1^\bullet &= \langle(\bar{0}, \bar{1})\rangle^\bullet \bigcup \langle(\bar{1}, \bar{0})\rangle^\bullet \bigcup \langle(\bar{1}, \bar{1})\rangle^\bullet \\ &= \langle(\bar{0}, \bar{1})\rangle^\bullet \bigcup \langle(\bar{1}, \bar{0})\rangle^\bullet \bigcup \langle(-\bar{1}, -\bar{1})\rangle^\bullet \\ &= \bigcup_{i=1}^{(p-1)/2} \{(\bar{0}, \bar{i}), (\bar{i}, \bar{0}), (-\bar{i}, -\bar{i}), (\bar{0}, -\bar{i}), (-\bar{i}, \bar{0}), (\bar{i}, \bar{i})\}, \end{aligned}$$

and for $j \neq 0$,

$$
\begin{aligned}
& G^\bullet_{j-1} \bigcup G^\bullet_j \bigcup G^\bullet_{j+1} \\
= \ & \langle(\bar{1}, \overline{j-1})\rangle^\bullet \bigcup \langle(\bar{1}, \bar{j})\rangle^\bullet \bigcup \langle(\bar{1}, \overline{j+1})\rangle^\bullet \\
= \ & \langle(\bar{1}, \overline{j-1})\rangle^\bullet \bigcup \langle(-\bar{2}, -\overline{2j})\rangle^\bullet \bigcup \langle(\bar{1}, \overline{j+1})\rangle^\bullet \\
= \ & \bigcup_{i=1}^{(p-1)/2} \{(\bar{i}, \overline{i(j-1)}), (-\overline{2i}, -\overline{2ij}), (\bar{i}, \overline{i(j+1)}), \\
& \qquad (-\bar{i}, -\overline{i(j-1)}), (\overline{2i}, \overline{2ij}), (-\bar{i}, -\overline{i(j+1)})\}.
\end{aligned}
$$

The assertion follows.

Note that there are $p+1$ G_i's. Also note that $3|(p+1)$ if $p \equiv 2 \bmod 3$ and $3|(p-1)$ if $p \equiv 1 \bmod 3$. Thus Statements (i) and (ii) follow.

Finally, we show that G has the ZSP-property. Let k and l be nonnegative integers with $|G^\bullet| = 3k + 2l$. Obviously, k is even.

Suppose $p \equiv 2 \bmod 3$. We only need to choose $k/2$ good 6-subsets, each of which is partitioned into the union of two zero-sum 3-subsets, and then partition each of other 6-subsets into the union of three zero-sum 2-subsets.

Suppose then $p \equiv 1 \bmod 3$. If $k \leq (p-1)^2/3$, we first choose $k/2$ good 6-subsets, each of which is partitioned into the union of two zero-sum 3-subsets; then partition each of the others into the union of three zero-sum 2-subsets; finally partition $G^\bullet_{-1}$ and $G^\bullet_0$ into the union of $(p-1)/2$ zero-sum 2-subsets respectively. If $k > (p-1)^2/3$, then by Theorem 3.1, G_{-1} and G_0 have the ZSP-property. Thus $G^\bullet_{-1}$ and $G^\bullet_0$ can be partitioned into the union of $(p-1)/3$ zero-sum 3-subsets respectively. Then we choose $k/2 - (p-1)/3$ good 6-subsets, each of which is partitioned into the union of two zero-sum 3-subsets, and then partition each of other 6-subsets into the union of three zero-sum 2-subsets.

This completes the proof of the lemma. □

Lemma 3.5. *Let $G = \mathbb{Z}_8$. Then*

$$G = \{\bar{0}, \bar{4}\} \bigcup \{\bar{1}, \bar{2}, \bar{5}, -\bar{1}, -\bar{2}, -\bar{5}\},$$

where the latter subset is a good 6-subset.

Proof. Obvious. □

Lemma 3.6. *Let $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_n$ where $n \geq 1$ is odd. Then*

$$G^\bullet = \{e_1, e_2, e_1 + e_2\} \bigcup K^\bullet \bigcup \left(\bigcup_{i=1}^{(n-1)/2} C_i \right),$$

where e_1, e_2, *and* e_1+e_2 *are the three involutions,* $K = \langle(\bar{0},\bar{0},\bar{1})\rangle$ *is the cyclic subgroup of order* n *and* $C_1,\ldots,C_{(n-1)/2}$ *are pairwise disjoint good 6-subsets. Moreover,* G *has the ZSP-property.*

Proof. If $n=1$, then the lemma follows trivially. Now assume that $n>1$. Let $d=(\bar{0},\bar{0},\bar{1})$. Note that

$$\begin{aligned} G\setminus K &= (e_1+K)\bigcup(e_2+K)\bigcup(e_1+e_2+K) \\ &= \{e_1,e_2,e_1+e_2\}\bigcup \\ &\quad \bigcup_{i=1}^{(n-1)/2}\{e_1+id,e_2+id,e_1+e_2-2id,e_1-id,e_2-id,e_1+e_2+2id\}. \end{aligned}$$

The partition is as desired.

Now we prove that G has the ZSP-property. Let k and l be nonnegative integers with $|G^{\bullet}|=3k+2l$. Obviously, k is odd. If $k\le n$, then choose $(k-1)/2$ good 6-subsets, each of which is partitioned into the union of two zero-sum 3-subsets. Together with $\{e_1,e_2,e_1+e_2\}$, we have k zero-sum 3-subsets. Note that the set of remainder elements is the union of zero-sum 2-subsets. The partition is as desired. If $k>n$, all the good 6-subsets are partitioned into the union of zero-sum 3-subsets, and then $K^{\bullet}$ can be partitioned into the union of $k-n$ 3-subsets and l 2-subsets by Theorem 3.1.

This completes the proof of the lemma. □

Lemma 3.7. *Let* $G=\mathbb{Z}_2\oplus\mathbb{Z}_4\oplus\mathbb{Z}_n$ *where* $n\ge 1$ *is odd. Then* G *has the ZSP-property.*

Proof. We consider the following partition of $G_0=\mathbb{Z}_2\oplus\mathbb{Z}_{4n}$, which is isomorphic to G:

$$\begin{aligned} G_0^{\bullet} &= \{(\bar{1},\bar{1}),(\bar{1},\overline{4n-1})\}\bigcup\{(\bar{0},\overline{2n-1}),(\bar{0},\overline{2n+1})\} \\ &\quad \bigcup\{(\bar{1},\overline{2n}),(\bar{1},\bar{0}),(\bar{0},\overline{2n})\}\bigcup A\bigcup\left(\bigcup_{i=1}^{n-1}C_i\right), \end{aligned}$$

where

$$\begin{aligned} C_i &= \{(\bar{1},\overline{1+i}),(\bar{1},\overline{2n+i}),(\bar{0},\overline{2n-2i-1}), \\ &\quad (\bar{1},\overline{4n-(1+i)}),(\bar{1},\overline{4n-(2n+i)}),(\bar{0},\overline{4n-(2n-2i-1)})\} \end{aligned}$$

is a good 6-subset for every $1\le i\le n-1$ and

$$A=\{(\bar{0},\bar{2}),(\bar{0},\bar{4}),\ldots,(\bar{0},\overline{2n-2}),(\bar{0},\overline{2n+2})),(\bar{0},\overline{2n+4})),\ldots,(\bar{0},\overline{4n-2})\},$$

which has the ZSP-property by Corollary 3.3. Thus, for any nonnegative integers k and l which satisfy $|G_0^\bullet| = 3k + 2l$ and $l \geq 2$, there are pairwise disjoint zero-sum subsets $A_1, \ldots, A_k, B_1, \ldots, B_l$, which form a partition of $G_0^\bullet$, such that $|A_i| = 3$ for $1 \leq i \leq k$ and $|B_j| = 2$ for $1 \leq j \leq l$. Since G is isomorphic to G_0, the conclusion above also holds for G. To prove the lemma, it remains to deal with the cases $l \leq 1$.

Suppose first that $l = 0$. Then $|G| \equiv 1 \bmod 3$ and $n \equiv 2 \bmod 3$. We have the following partition of $G = \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_n$:

$$\begin{aligned} G^\bullet \quad = \quad & \bigcup_{i=0}^{(n-3)/2} \{(\bar{0}, \bar{1}, \bar{i}), (\bar{0}, \bar{1}, \overline{i + (n+1)/2}), (\bar{0}, \bar{2}, \overline{(n-1)/2 - 2i})\} \\ & \bigcup \left(\bigcup_{i=1}^{(n-1)/2} \{(\bar{1}, \bar{3}, \bar{i}), (\bar{1}, \bar{3}, \overline{i + (n-1)/2}), (\bar{0}, \bar{2}, \overline{(n+1)/2 - 2i})\} \right) \\ & \bigcup \left(\bigcup_{i=1}^{n-1} \{(\bar{0}, \bar{3}, \bar{i}), (\bar{1}, \bar{1}, \overline{i + (n+1)/2}), (\bar{1}, \bar{0}, \overline{(n-1)/2 - 2i})\} \right) \\ & \bigcup \{(\bar{0}, \bar{1}, \overline{(n-1)/2}), (\bar{0}, \bar{3}, \bar{0}), (\bar{0}, \bar{0}, \overline{(n+1)/2})\} \\ & \bigcup \{(\bar{1}, \bar{1}, \overline{(n+1)/2}), (\bar{1}, \bar{3}, \bar{0}), (\bar{0}, \bar{0}, \overline{(n-1)/2})\} \\ & \bigcup \{(\bar{0}, \bar{2}, \overline{(n+1)/2}), (\bar{1}, \bar{0}, \overline{(n-1)/2}), (\bar{1}, \bar{2}, \bar{0})\} \bigcup B, \end{aligned}$$

where

$$B = \{(\bar{0}, \bar{0}, \bar{i}) : 1 \leq i \leq n-1, i \neq (n-1)/2, (n+1)/2\} \bigcup \{(\bar{1}, \bar{2}, \bar{j}) : 1 \leq j \leq n-1\}.$$

Note that the subset $\{(\bar{0}, \bar{0}, \bar{i}) : 0 \leq i \leq n-1\} \bigcup \{(\bar{1}, \bar{2}, \bar{j}) : 0 \leq j \leq n-1\}$ is a subgroup isomorphic to $\mathbb{Z}_{2n}$. Let χ be an isomorphism between these two groups. Moreover we may assume that

$$\chi(B) = \{\bar{i} : 1 \leq i \leq 2n-1, i \neq n-1, n, n+1\}.$$

Since $\chi(B)$ has the ZSP-property by Corollary 3.3, so does B. Note that $|B| = 2n-4$ is a multiple of 3. Thus B can be partitioned into the union of zero-sum 3-subsets. Therefore $G^\bullet$ is the union of zero-sum 3-subsets, as desired.

Now suppose $l = 1$. Then $3 \mid |G|$ and there is a subgroup $L \leq G$ with $G/L \cong \mathbb{Z}_3$. Let $\{c, 0, -c\}$ be a set of coset representatives for G/L. Note that $L \in \mathcal{G}$. By Lemma 2.1, there are $\phi, \varphi \in \mathrm{Bij}(L)$ such that $a + \phi(a) + \varphi(a) = 0$ for every $a \in L$. Thus we have the following partition:

$$G^\bullet = \{c, -c\} \bigcup \left(\bigcup_{a \in L^\bullet} \{a, c + \phi(a), -c + \varphi(a)\} \right),$$

as desired.

This completes the proof of the lemma. $\square$

Lemma 3.8. *Let $G = \mathbb{Z}_2 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_n$ where $n \geq 1$ is odd. Then G has the ZSP-property.*

Proof. Similarly as in the proof of Lemma 3.7, we have the following partition of $G_0 = \mathbb{Z}_2 \oplus \mathbb{Z}_{8n}$, which is isomorphic to G:

$$\begin{aligned} G_0^\bullet &= \{(\bar{1},\bar{1}),(\bar{1},\overline{8n-1})\}\bigcup\{(\bar{0},\overline{4n-1}),(\bar{0},\overline{4n+1})\} \\ &\quad \bigcup\{(\bar{1},\overline{4n}),(\bar{1},\bar{0}),(\bar{0},\overline{4n})\}\bigcup A\bigcup\left(\bigcup_{i=1}^{2n-1} C_i\right), \end{aligned}$$

where

$$\begin{aligned} C_i &= \{(\bar{1},\overline{1+i}),(\bar{1},\overline{4n+i}),(\bar{0},\overline{4n-2i-1}), \\ &\quad (\bar{1},\overline{8n-(1+i)}),(\bar{1},\overline{8n-(4n+i)}),(\bar{0},\overline{8n-(4n-2i-1)})\} \end{aligned}$$

is a good 6-subset for every $1 \leq i \leq 2n-1$ and

$$A = \{(\bar{0},\bar{2}),(\bar{0},\bar{4}),\ldots,(\bar{0},\overline{4n-2}),(\bar{0},\overline{4n+2})),(\bar{0},\overline{4n+4})),\ldots,(\bar{0},\overline{8n-2})\}$$

which has the ZSP-property by Corollary 3.3. Let k and l be nonnegative integers with $|G_0^\bullet| = 3k+2l$. The cases $l \geq 2$ have been done by the partition above. To prove the lemma, it remains to deal with the cases $l \leq 1$.

Suppose first that $l = 0$. Then $|G| \equiv 1 \bmod 3$ and $n \equiv 1 \bmod 3$. If $n = 1$, such a partition is given in (2.1). If $n > 1$, let L be the subgroup satisfying $G/L \cong \mathbb{Z}_2 \oplus \mathbb{Z}_8$. Thus we can choose a set of coset representatives, say A, such that

$$A = \{0\}\bigcup\left(\bigcup_{i=1}^{5}\{b_i, c_i, -b_i - c_i\}\right).$$

Since $L \in \mathcal{G}$, there are $\phi, \varphi \in \mathrm{Bij}(L)$ such that $a + \phi(a) + \varphi(a) = 0$ for every $a \in L$ by Lemma 2.1. We have the following partition:

$$G^\bullet = L^\bullet\bigcup\left(\bigcup_{i=1}^{5}\bigcup_{a\in L}\{b_i + a, c_i + \phi(a), -b_i - c_i + \varphi(a)\}\right).$$

Note that L is isomorphic to $\mathbb{Z}_n$, which has the ZSP-property by Theorem 3.1. Thus L also has the ZSP-property. Since $n \equiv 1 \bmod 3$, $L^\bullet$ can be partitioned into the union of zero-sum 3-subsets. Therefore $G^\bullet$ is the union of zero-sum 3-subsets, as desired.

Now suppose $l = 1$. The case is the same as the corresponding case in the proof of Lemma 3.7.

This completes the proof of the lemma. □

Lemma 3.9. *Let $G = \mathbb{Z}_4 \oplus \mathbb{Z}_4$, $G = \mathbb{Z}_4 \oplus \mathbb{Z}_8$ or $G = \mathbb{Z}_8 \oplus \mathbb{Z}_8$. Then G has the ZSP-property.*

Proof. First we consider the case $G = \mathbb{Z}_4 \oplus \mathbb{Z}_4$. It easily follows from the following partition:

$$\begin{aligned} G^\bullet &= \{(\bar{0},\bar{2}),(\bar{2},\bar{0}),(\bar{2},\bar{2})\} \bigcup \{(\bar{0},\bar{1}),(\bar{1},\bar{2}),(\bar{3},\bar{1}),(\bar{0},\bar{3}),(\bar{3},\bar{2}),(\bar{1},\bar{3})\} \\ & \bigcup \{(\bar{1},\bar{0}),(\bar{1},\bar{1}),(\bar{2},\bar{3}),(\bar{3},\bar{0}),(\bar{3},\bar{3}),(\bar{2},\bar{1})\}, \end{aligned}$$

where the latter two 6-subsets are both good.

Next we consider the case $G = \mathbb{Z}_8 \oplus \mathbb{Z}_8$. Let L be the subgroup of G satisfying $L \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $G/L \cong \mathbb{Z}_4 \oplus \mathbb{Z}_4$. As seen above, we can choose a set of coset representatives, say A, such that

$$A = \{0, e_1, e_2, -e_1 - e_2\} \bigcup \left(\bigcup_{i=1}^{2} \{b_i, c_i, -b_i - c_i, -b_i, -c_i, b_i + c_i\} \right),$$

where $2e_1, 2e_2 \in L$. By Lemma 2.1, there are $\phi, \varphi \in \mathrm{Bij}(L)$ such that $a + \phi(a) + \varphi(a) = 0$ for every $a \in L$. Thus

$$\begin{aligned} G^\bullet &= L^\bullet \bigcup (e_1 + L) \cup (e_2 + L) \cup (-e_1 - e_2 + L) \bigcup \\ & \bigcup_{\substack{1 \le i \le 2 \\ a \in L}} \{b_i + a, c_i + \phi(a), -b_i - c_i + \varphi(a), -b_i - a, -c_i - \phi(a), b_i + c_i - \varphi(a)\}, \end{aligned}$$

where the latter 6-subsets are good. Since $L \cup (e_1 + L) \cup (e_2 + L) \cup (-e_1 - e_2 + L)$ is a subgroup isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_4$, it has the ZSP-property by the first paragraph, and so the ZSP-property of G follows.

Finally, we consider the case $G = \mathbb{Z}_4 \oplus \mathbb{Z}_8$. Let k and l be nonnegative integers such that $|G^\bullet| = 3k + 2l$. Consider the following partition:

$$\begin{aligned} G^\bullet &= \{(\bar{0},\bar{1}),(\bar{1},\bar{2}),(\bar{3},\bar{5}),(\bar{0},\bar{7}),(\bar{3},\bar{6}),(\bar{1},\bar{3})\} \\ & \bigcup \{(\bar{1},\bar{1}),(\bar{3},\bar{2}),(\bar{0},\bar{5}),(\bar{3},\bar{7}),(\bar{1},\bar{6}),(\bar{0},\bar{3})\} \\ & \bigcup \{(\bar{3},\bar{1}),(\bar{0},\bar{2}),(\bar{1},\bar{5}),(\bar{1},\bar{7}),(\bar{0},\bar{6}),(\bar{3},\bar{3})\} \\ & \bigcup \{(\bar{0},\bar{4}),(\bar{2},\bar{0}),(\bar{2},\bar{4})\} \bigcup \{(\bar{1},\bar{0}),(\bar{3},\bar{0})\} \bigcup \{(\bar{1},\bar{4}),(\bar{3},\bar{4})\} \\ & \bigcup \{(\bar{2},\bar{1}),(\bar{2},\bar{7})\} \bigcup \{(\bar{2},\bar{2}),(\bar{2},\bar{6})\} \bigcup \{(\bar{2},\bar{3}),(\bar{2},\bar{5})\}, \end{aligned}$$

where the first three 6-subsets are good. When $k = 1, 3, 5, 7$, the desired partitions follow from the partition above by partitioning some of the good 6-subsets into zero-sum 3-subsets and others 2-subsets. It remains to show the case when $k = 9$. Let L be the subgroup of G satisfying that $L \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4$ and $G/L \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Let

$A = \{0, e_1, e_2, -e_1 - e_2\}$ be a set of coset representatives. Since $L \in \mathcal{G}$, there are $\phi, \varphi \in \mathrm{Bij}(L)$ such that $a + \phi(a) + \varphi(a) = 0$ for every $a \in L$ by Lemma 2.1. Thus

$$G^\bullet = L^\bullet \bigcup \left(\bigcup_{a \in L} \{e_1 + a, e_2 + \phi(a), -e_1 - e_2 + \varphi(a)\} \right).$$

Since L has the ZSP-property by Lemma 3.7, $L^\bullet$ can be partitioned into the union of a zero-sum 3-subset and two zero-sum 2-subsets. Therefore, $G^\bullet$ is the union of nine zero-sum 3-subsets and two zero-sum 2-subsets, as desired. □

4. Proof of the Main Result

Now we are in a position to give the proof of the main result.

Proof. Suppose the theorem is false and let G be the smallest group in $\mathcal{G}$ without the ZSP-property.

`Case 1.` Suppose there is a subgroup $L < G$ such that $G/L \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$, where p is an odd prime congruent to 2 modulo 3. By Lemma 3.4, we can choose a set of coset representatives, say A, such that

$$A = \{0\} \bigcup \left(\bigcup_{i=1}^{(p^2-1)/6} C_i \right),$$

where $C_i = \{b_i, c_i, -b_i - c_i, -b_i, -c_i, b_i + c_i\}, i = 1, \ldots, (p^2-1)/6$ are pairwise disjoint good 6-subsets. Since $L \in \mathcal{G}$, there are $\phi, \varphi \in \mathrm{Bij}(L)$ such that $a + \phi(a) + \varphi(a) = 0$ for every $a \in L$ by Lemma 2.1. Thus we have the following partition of $G^\bullet$:

$$G^\bullet = L^\bullet \bigcup \bigcup_{\substack{1 \le i \le (p^2-1)/6 \\ a \in L}} \{b_i + a, c_i + \phi(a), -b_i - c_i + \varphi(a), -b_i - a, c_i - \phi(a), b_i + c_i - \varphi(a)\},$$

where the latter 6-subsets are good. Since $|L| < |G|$, L has the ZSP-property, which implies that G also has the ZSP-property, a contradiction.

`Case 2.` Suppose there is a subgroup $L < G$ such that $G/L \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$, where p is an odd prime congruent to 1 modulo 3. By Lemma 3.4, we can choose a set of coset representatives, say A, such that

$$A = \{0\} \bigcup G_{-1}^\bullet \bigcup G_0^\bullet \bigcup \left(\bigcup_{i=1}^{(p-1)^2/6} C_i \right),$$

where the images of G_{-1} and G_0 in G/L are distinct subgroups of order p, $G_{-1}^\bullet = \{a \in G_{-1} : a \notin L\}$, $G_0^\bullet = \{a \in G_0 : a \notin L\}$ and $C_i = \{b_i, c_i, -b_i - c_i, -b_i, -c_i, b_i +$

$c_i\}, i = 1, \ldots, (p-1)^2/6$ are pairwise disjoint good 6-subsets. The same as in the Case 1, we have the following partition of $G^\bullet$:

$$\begin{aligned} G^\bullet &= L^\bullet \bigcup \left(\bigcup_{a \in G_{-1}^\bullet} (a+L) \right) \bigcup \left(\bigcup_{a \in G_0^\bullet} (a+L) \right) \\ & \bigcup \bigcup_{\substack{1 \le i \le (p-1)^2/6 \\ a \in L}} \{b_i + a, c_i + \phi(a), -b_i - c_i + \varphi(a), \\ & \qquad\qquad -b_i - a, c_i - \phi(a), b_i + c_i - \varphi(a)\}. \end{aligned} \tag{4.1}$$

We remark that $\cup_{a \in G_{-1}}(a+L) = L \cup (\cup_{a \in G_{-1}^\bullet}(a+L))$ and $\cup_{a \in G_0}(a+L) = L \cup (\cup_{a \in G_0^\bullet}(a+L))$ are both subgroups of G. Now we want to show that G has the ZSP-property, which will lead to a contradiction. Let k and l be nonnegative integers with $|G^\bullet| = 3k + 2l$.

Suppose $k \le (p-1)|L|/3$. Since $\cup_{a \in G_{-1}}(a+L)$ is a smaller group in $\mathcal{G}$, it has the ZSP-property. Thus $\left(\cup_{a \in G_{-1}}(a+L)\right)^\bullet$ is the union of k zero-sum 3-subsets and $(p|L| - 1 - 3k)/2$ zero-sum 2-subsets. Note that the remaining elements of $G^\bullet$ are the union of zero-sum 2-subsets. Thus $G^\bullet$ is the union of k zero-sum 3-subsets and l zero-sum 2-subsets. Now suppose $k > (p-1)|L|/3$ instead. As a set of all non-zero elements of a cyclic group of order p, the image of $G_0^\bullet$ in G/L is the union of $(p-1)/3$ zero-sum 3-subsets in G/L by Theorem 3.1. Hence we may assume that $G_0^\bullet = \cup_{i=1}^{(p-1)/3}\{g_i, h_i, -g_i - h_i\}$. Thus

$$\bigcup_{a \in G_0^\bullet} (a+L) = \bigcup_{i=1}^{(p-1)/3} \bigcup_{a \in L} \{g_i + a, h_i + \phi(a), -g_i - h_i + \varphi(a)\}.$$

Note that the rest elements of G is the union of a subgroup with the ZSP-property and several good 6-subsets, thus they can be partitioned into the union of $k - (p-1)|L|/3$ zero-sum 3-subsets and l zero-sum 2-subsets. Thus, G has the ZSP-property, a contradiction to the hypothesis.

`Case 3.` Suppose $G = \mathbb{Z}_{2^\alpha} \oplus \mathbb{Z}_{2^\beta} \oplus H$, where $|H|$ is odd, $1 \le \alpha \le \beta$ and $\beta \ge 4$. Then there is a subgroup $L \in \mathcal{G}$ such that $L \cong \mathbb{Z}_{2^\alpha} \oplus \mathbb{Z}_{2^{\beta-3}} \oplus H$ and $G/L \cong \mathbb{Z}_8$. By Lemma 3.5, we can choose a set of coset representatives, say A, such that $A = \{0, e\} \cup \{b, c, -b-c, -b, -c, b+c\}$, where $2e \in L$. Since $L \in \mathcal{G}$, there are $\phi, \varphi \in \mathrm{Bij}(L)$ such that $a + \phi(a) + \varphi(a) = 0$ for every $a \in L$ by Lemma 2.1. Thus

$$G^\bullet = L^\bullet \bigcup (e+L) \bigcup \bigcup_{a \in L} \{b+a, c+\phi(a), -b-c+\varphi(a), -b-a, -c-\phi(a), b+c-\varphi(a)\}.$$

Note that $L \cup (e+L)$ is a subgroup of G. So, $L \cup (e+L)$ has the ZSP-property and thus G has the ZSP-property, a contradiction.

`Case 4.` Suppose $G = \mathbb{Z}_{2^\alpha} \oplus \mathbb{Z}_{2^\beta} \oplus H$, where $|H| > 1$ is odd and $2 \le \alpha \le \beta$. Then there is a subgroup $L \in \mathcal{G}$ such that $G/L \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_n$ where $n > 1$ is odd. By

Lemma 3.6, we can choose a set of coset representatives, say A, such that

$$A = \{0, e_1, e_2, -e_1 - e_2\} \bigcup K^\bullet \bigcup \left(\bigcup_{i=1}^{(n-1)/2} C_i \right),$$

where $2e_1, 2e_2 \in L$, the image of K in G/L is a cyclic group of order n and $C_i = \{b_i, c_i, -b_i - c_i, -b_i, -c_i, b_i + c_i\}$ is a good 6-subset for every $1 \leq i \leq (n-1)/2$. Since $L \in \mathcal{G}$, there are $\phi, \varphi \in \mathrm{Bij}(L)$ such that $a + \phi(a) + \varphi(a) = 0$ for every $a \in L$ by Lemma 2.1. Thus,

$$\begin{aligned} G^\bullet \;=\; & L^\bullet \bigcup (e_1 + L) \bigcup (e_2 + L) \bigcup (-e_1 - e_2 + L) \bigcup \bigcup_{a \in K^\bullet} (a + L) \\ & \bigcup \bigcup_{\substack{1 \leq i \leq (n-1)/2 \\ a \in L}} \{b_i + a, c_i + \phi(a), -b_i - c_i + \varphi(a), \\ & \qquad -b_i - a, c_i - \phi(a), b_i + c_i - \varphi(a)\}. \end{aligned}$$

We now prove that G has the ZSP-property. Let k and l be nonnegative integers with $|G^\bullet| = 3k + 2l$.

Suppose $k \leq |L|$. Since $L \cup (e_1 + L) \cup (e_2 + L) \cup (-e_1 - e_2 + L)$ is a smaller group in $\mathcal{G}$, it has the ZSP-property. Thus $L^\bullet \cup (e_1 + L) \cup (e_2 + L) \cup (-e_1 - e_2 + L)$ is the union of k zero-sum 3-subsets and $(4|L| - 1 - 3k)/2$ zero-sum 2-subsets. Note that the remaining elements of $G^\bullet$ are the union of zero-sum 2-subsets. Thus, $G^\bullet$ is the union of k zero-sum 3-subsets and l zero-sum 2-subsets. Now we instead suppose $k > |L|$. We consider the following partition:

$$(e_1 + L) \bigcup (e_2 + L) \bigcup (-e_1 - e_2 + L) = \bigcup_{a \in L} \{e_1 + a, e_2 + \phi(a), -e_1 - e_2 + \varphi(a)\}.$$

Note that $\cup_{a \in K} (a + L)$ is a smaller group in $\mathcal{G}$. Hence it has the ZSP-property. Thus, the remaining elements of G are the union of a subgroup with the ZSP-property and several good 6-subsets, and can be partitioned into the union of $k - |L|$ zero-sum 3-subsets and l zero-sum 2-subsets. Therefore, G has the ZSP-property, a contradiction to the hypothesis.

Case 5. Suppose $G = \mathbb{Z}_{2^\alpha} \oplus \mathbb{Z}_{2^\beta} \oplus H$, where $|H| > 3$ is odd, $3 \mid |H|$ and $\alpha = \beta = 0$ or $1 \leq \alpha \leq \beta$.

First we consider the case $\alpha = \beta = 0$. Now G contains a subgroup L such that $G/L \cong \mathbb{Z}_3$. Let $\{b, 0, -b\}$ be a set of coset representatives. Let $I \subset L$ be such that $L^\bullet$ is the disjoint union of I and $-I$. Then we have the following partition:

$$G^\bullet = \{b, -b\} \bigcup \left(\bigcup_{c \in I} \{b + c, -b + c, -2c, b - c, -b - c, 2c\} \right),$$

where the latter 6-subsets are good. Thus G has the ZSP-property, a contradiction.

Then we consider the case $1 \le \alpha \le \beta$. Let T be the subgroup isomorphic to $\mathbb{Z}_{2^\alpha} \oplus \mathbb{Z}_{2^\beta}$. Since $T \in \mathcal{G}$, there are $\phi, \varphi \in \mathrm{Bij}(T)$ such that $\phi(0) = \varphi(0) = 0$ and $a + \phi(a) + \varphi(a) = 0$ for every $a \in T$ by Lemma 2.1. Since $T < G$, T has the ZSP-property. From the above paragraph, we have

$$H^\bullet = \{b, -b\} \bigcup \left(\bigcup_{c \in I} \{b + c, -b + c, -2c, b - c, -b - c, 2c\} \right)$$

for some $b \in H$, and thus

$$G^\bullet = T^\bullet \bigcup (b + T) \bigcup (-b + T) \bigcup \left(\bigcup_{c \in I} \bigcup_{a \in T} C_{a,c} \right).$$

where the $C_{a,c}$ are good 6-subsets for all $c \in I$ and $a \in T$. Let k and l be nonnegative integers with $|G^\bullet| = 3k + 2l$.

If $k < |T|$, then $T^\bullet$ provides a zero-sum 3-subset and $(|T| - 4)/2$ zero-sum 2-subsets; $(b+T) \cup (-b+T)$ only provides zero-sum 2-subsets; $(k-1)/2$ good 6-subsets provide $k - 1$ zero-sum 3-subsets and the other good 6-subsets provide zero-sum 2-subsets. This shows that $G^\bullet$ is the union of k zero-sum 3-subsets and l zero-sum 2-subsets. Now suppose $k \ge |T|$. Note that

$$T^\bullet \bigcup (b + T) \bigcup (-b + T) = \{b, -b\} \bigcup \left(\bigcup_{a \in T^\bullet} \{b + a, -b + \phi(a), \varphi(a)\} \right).$$

Thus, good 6-subsets provide $k + 1 - |T|$ zero-sum 3-subsets and $l - 1$ zero-sum 2-subsets, and now $G^\bullet$ is the union of k zero-sum 3-subsets and l zero-sum 2-subsets. Therefore, G has the ZSP-property, a contradiction.

Now we summarize what we have done and what is left. Let $G = \mathbb{Z}_{2^\alpha} \oplus \mathbb{Z}_{2^\beta} \oplus H$, where $|H|$ is odd and $0 = \alpha = \beta$ or $1 \le \alpha \le \beta$. Cases 1,2 and 5 show that H must be cyclic or trivial. In particular, if $0 = \alpha = \beta$, then G has the ZSP-property by Theorem 3.1. Case 3 shows that $\alpha \le \beta \le 3$, and Case 4 that if H is not trivial, then $\alpha = 1$. Therefore, it remains to consider the cases: $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_n$, $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_n$, $\mathbb{Z}_2 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_n$, $\mathbb{Z}_4 \oplus \mathbb{Z}_4$, $\mathbb{Z}_4 \oplus \mathbb{Z}_8$ and $\mathbb{Z}_8 \oplus \mathbb{Z}_8$, where $n \ge 1$ is odd. These cases have been tackled in Lemmas 3.6, 3.7, 3.8 and 3.9 of Section 3. This completes the proof of the theorem. $\square$

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