QUADRATIC DIOPHANTINE EQUATIONS WITH INFINITELY MANY SOLUTIONS IN POSITIVE INTEGERS

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Abstract
For an arbitrary positive integers k, any integer a for which the equation $x^2 - axy + y^2 + kx = 0$ has infinitely many solutions in positive integers satisfies either $a = k+2$ or $3 \leq a \leq \lfloor \frac{k+5}{2} \rfloor$. Moreover, each of 3, $\lfloor \frac{k+5}{2} \rfloor$, and $k + 2$ has this property.

1. Introduction

For fixed integers a and k, we shall call the equation

$$x^2 - axy + y^2 + kx = 0 \tag{1}$$

admissible if it has infinitely many solutions in positive integers. For a positive integer k, let $S(k)$ denote the set of integers a for which the above equation is admissible.

To the best of our knowledge, the earliest study of such a set is due to Marlewski and Zarzycki [2]. In the present terminology, their main result can be phrased as follows.

Theorem 1. ([2]) $S(1) = \{3\}$.

In an equivalent form, this has already been found by Owings [5] in his solution to a completely different problem. More sets of the kind are determined by Yuan and Hu [7].

Theorem 2. ([7]) $S(2) = \{3, 4\}$ *and* $S(4) = \{3, 4, 6\}$.

Important qualitative information has been provided by Feng, Yuan, and Hu [1], who have shown that $S(k)$ is a finite set for any positive integer k. Moreover, the arguments employed to justify this statement provide the inequality $a \leq 4k + 2$ for any $a \in S(k)$. The same bound is explicitly stated in Corollary 1.1 from [3].

The aim in this paper is to give the best possible result in this vein. Additionally, we show that the extreme values always belong to the set in question.

Theorem 3. *Let k be a positive integer. Then, for any $a \in S(k)$ one has either $a = k+2$ or $3 \le a \le \lfloor \frac{k+5}{2} \rfloor$. Moreover, 3, $\lfloor \frac{k+5}{2} \rfloor$, and $k+2$ belong to $S(k)$.*

Note that Theorems 1 and 2 are obtained by specialization from our result, which also gives $\{3,4,5\} \subseteq S(3) \subseteq \{3,4,5\}$, $\{3,5,7\} \subseteq S(5) \subseteq \{3,4,5,7\}$, and $\{3,6,9\} \subseteq S(7) \subseteq \{3,4,5,6,9\}$. From what we shall prove in Section 4 it will result $S(5) = \{3,5,7\}$, $S(7) = \{3,6,9\}$, in accordance with [1].

2. Preparations for the Proof of the Main Result

Throughout the paper k will denote a positive integer.

A key remark of Marlewski and Zarzycki (see [2, Theorem 1]) in their study of the equation $x^2 - kxy + y^2 + x = 0$ has been extended by Feng *et al.* so as to apply to the more general equation (1).

Lemma 1. ([1, Lemma 3]) *Suppose positive integers a, k x, y satisfy* (1). *If* $\gcd(x, y, k) = 1$ *then there exist positive integers c, e such that $x = c^2$, $y = ce$, and* $\gcd(c, e) = 1$.

Let $a \in S(k)$. Since k has finitely many divisors, there exists a divisor d of k such that $\gcd(x, y, k) = k/d$ for infinitely many solutions (x, y) to the equation (1). Hence, both relations $u^2 - auv + v^2 + du = 0$ and $\gcd(u, v, d) = 1$ are satisfied by infinitely many pairs (u, v) of positive integers. Having in view Lemma 1, we obtain the following result.

Lemma 2. *For any $a \in S(k)$ there exists a positive divisor d of k such that the equation*

$$c^2 - ace + e^2 + d = 0, \quad \gcd(c, e) = 1,$$

has infinitely many solutions in positive integers.

Another remark valid for all sets of interests is the following.

Lemma 3. *If k, K are positive integers with K multiple of k, then $S(k) \subset S(K)$.*

Proof. If $K = uk$ and (ξ, ρ) is a solution in positive integers to $x^2 - axy + y^2 + kx = 0$, then $(u\xi, u\rho)$ is a solution in positive integers to $x^2 - axy + y^2 + Kx = 0$. □

We record a direct consequence of this result for future reference.

Corollary 1. *For any positive integers k one has*

$$\bigcup_{\substack{d|k \\ d<k}} S(d) \subseteq S(k).$$

A key ingredient in our proofs is a well-known bound on the fundamental solutions to generalized Pell equations, due to Nagell, and its analogue proved by Stolt.

Lemma 4. ([4, Theorem 108a]) *Let N, D be positive integers with D not a square. Suppose that $x_0+y_0\sqrt{D}$ is the fundamental solution of the Pell equation $X^2-DY^2 = 1$ and the equation*

$$U^2 - DV^2 = -N \tag{2}$$

is solvable in coprime integers. Then (2) *has a solution $u_0+v_0\sqrt{D}$ with the following property:*

$$0 < v_0 \leq \frac{y_0\sqrt{N}}{\sqrt{2(x_0-1)}}, \quad 0 \leq u_0 \leq \sqrt{\frac{1}{2}(x_0-1)N}.$$

Lemma 5. ([6, Theorem 2]) *Let N, D be odd positive integers with D not a square. Suppose that the equation $X^2 - DY^2 = 4$ has solutions in coprime integers and let $x_0 + y_0\sqrt{D}$ be the least solution in positive integers. If the Diophantine equation*

$$U^2 - DV^2 = -4N, \quad \gcd(U, V)|2, \tag{3}$$

is solvable, then (3) *has a solution $u_0 + v_0\sqrt{D}$ with the following property:*

$$0 < v_0 \leq \frac{y_0\sqrt{N}}{\sqrt{x_0-2}}, \quad 0 \leq u_0 \leq \sqrt{(x_0-2)N}.$$

3. Proof of the Main Result

First of all, note that if $a \leq 2$ then the left side of (1) is a sum of non-negative integers for positive x, y, so that $a \notin S(k)$. Now we proceed to establish a tight upper bound on elements from $S(k)$.

Since the conclusion of Theorem 3 is true for $k = 1$ by Theorem 1, we suppose below $k \geq 2$. According to Lemma 2, there exists a positive divisor d of k with the property that the equation

$$c^2 - ace + e^2 + d = 0 \tag{4}$$

has infinitely many solutions in coprime positive integers.

Take an arbitrary $a \in S(k)$.

If a is even, say, $a = 2b$, then Equation (4) is equivalent to $(c-be)^2-(b^2-1)e^2 = -d$, whose associated Pell equation $X^2-(b^2-1)Y^2 = 1$ has the fundamental solution $(x_0, y_0) = (b, 1)$. Lemma 4 ensures the existence of a solution (c_0, e_0) to (4) with

$$1 \le e_0 \le \frac{\sqrt{d}}{\sqrt{2(b-1)}} = \frac{\sqrt{d}}{\sqrt{a-2}}.$$

This inequality implies $a \le d + 2 \le k + 2$. If $d < k$ then $a \le \frac{k}{2} + 2 \le \lfloor \frac{k+5}{2} \rfloor$, while if $e_0 \ge 2$ then $a \le \frac{k}{4} + 2 < \lfloor \frac{k+5}{2} \rfloor$. In the remaining case $e_0 = 1$ and $d = k$, we have to show that for $a < k + 2$ one actually has $a \le \lfloor \frac{k+5}{2} \rfloor$. From Equation (4), which becomes $c_0^2 - ac_0 + 1 + k = 0$ in this case, we obtain

$$a = c_0 + \frac{k+1}{c_0},$$

so that c_0 is a (positive) divisor of $k + 1$. On noting that $a = k + 2$ is tantamount to $c_0 \in \{1, k + 1\}$, it results that

$$a = c_0 + \frac{k+1}{c_0} \le 2 + \frac{k+1}{2},$$

whence $a \le \lfloor \frac{k+5}{2} \rfloor$.

If a is odd then the equation (4) is rewritten as

$$(2c - ae)^2 - (a^2 - 4)e^2 = -4d. \tag{5}$$

Since either none or two of the first three terms in (4) can be even, d must be odd. The least positive solution to the equation

$$X^2 - (a^2 - 4)Y^2 = 4, \quad \gcd(X, Y) = 1,$$

is $(x_0, y_0) = (a, 1)$. According to Lemma 5, the equation (5) has a solution (c_0, e_0) with $1 \le e_0 \le \frac{\sqrt{d}}{\sqrt{a-2}}$. As above, we conclude that in this case one also has either $a = k + 2$ or $a \le \lfloor \frac{k+5}{2} \rfloor$.

It remains to show that the three extremal elements indicated in the statement of Theorem 3 always belong to $S(k)$.

By Corollary 1 and Theorem 1, one has $3 \in S(1) \subseteq S(k)$.

To prove $k+2 \in S(k)$, note that from the seed solution $(x_0, y_0) = \big((k+1)^2, k+1\big)$ one obtains infinitely many solutions in positive integers to the equation $x^2 - (k + 2)xy + y^2 + kx = 0$ by applying the transformations $(x, y) \mapsto \big(x, (k + 2)x - y\big)$ and $(x, y) \mapsto \big((k+2)y-x-k, y\big)$. Explicitly, they are given by the recursion $x_{2n+1} = x_{2n}$, $y_{2n+1} = (k + 2)x_{2n} - y_{2n}$, $x_{2n+2} = (k + 2)y_{2n+1} - x_{2n+1} - k$, $y_{2n+2} = y_{2n+1}$.

From Corollary 1 and what we have just proved, it results that

$$\left\lfloor \frac{2k+5}{2} \right\rfloor = k + 2 \in S(k) \subseteq S(2k),$$

so we are left to show that the equation $x^2 - (t+2)xy + y^2 + (2t-1)x = 0$ is admissible. Indeed, it has a positive solution $(x_0, y_0) = (t^2, t)$ to which one can apply the transformations $(x, y) \mapsto \big(x, (t+2)x - y\big)$ and $(x, y) \mapsto \big((t+2)y - x - 2t + 1, y\big)$.

4. Other Results

Hu and Le characterize in [3] the elements from $S(k)$ in terms of solvability of certain generalized Pell equations. Since deciding this solvability can occasionally be very costly, we give several criteria for establishing whether a given integer can or can not appear in a specific set.

As a direct application of our main theorem we have the next result.

Proposition 1. *For each positive integer k one has*

$$\bigcup_{d|k} \left\{ \left\lfloor \frac{d+5}{2} \right\rfloor, d+2 \right\} \subseteq S(k).$$

In [1] one can find a list with $S(k)$ for $k \leq 33$. Comparison with the sets produced with the help of Proposition 1 reveals that sometimes all elements of $S(k)$ are thus produced. For instance, one has

$$\begin{aligned}
S(24) &= \{3,4,5,6,8,10,14,26\} = \bigcup_{d|24} \{d+2\}, \\
S(25) &= \{3,5,7,15,27\} = \bigcup_{d|25} \left\{ \left\lfloor \frac{d+5}{2} \right\rfloor, d+2 \right\}, \\
S(30) &= \{3,4,5,7,8,10,12,17,32\} = \bigcup_{d|30} \left\{ \left\lfloor \frac{d+5}{2} \right\rfloor, d+2 \right\}.
\end{aligned}$$

However, there are examples where additional elements pop up:

$$\begin{aligned}
S(20) &= \{3,4,5,6,7,10,12,22\} = \bigcup_{d|20} \left\{ \left\lfloor \frac{d+5}{2} \right\rfloor, d+2 \right\} \cup \{10\}, \\
S(32) &= \{3,4,6,10,14,18,34\} = S(16) \cup \{14,34\} \\
&= \bigcup_{d|32} \left\{ \left\lfloor \frac{d+5}{2} \right\rfloor, d+2 \right\} \cup \{14\}, \\
S(33) &= \{3,4,5,7,8,13,19,35\} = S(3) \cup S(11) \cup \{19,35\} \\
&= \bigcup_{d|33} \left\{ \left\lfloor \frac{d+5}{2} \right\rfloor, d+2 \right\} \cup \{7\}.
\end{aligned}$$

Our next results help to decide in certain situations whether an integer is or is not in a set of interest.

Proposition 2. *If all prime divisors of k are congruent to* 1 *modulo* 3*, then each element of $S(k)$ is divisible by* 3*.*

Proof. Consider $a \in S(k)$. From Lemma 2 we deduce that there exists a divisor d of k such that the equation $c^2 - ace + e^2 + d = 0$ is solvable in positive integers. Since d inherits the properties of k, any solution (c, e) consists of integers coprime with 3. Reduction modulo 3 yields $1 \pm a + 1 + 1 \equiv 0 \pmod 3$, hence the desired conclusion. □

Perusal of the list given in [1] reveals that the hypothesis in the previous proposition cannot be weakened (see, for instance, $k = 5$, 6, 10, 15, 25).

Proposition 3. *If* $\gcd(k, 6) > 1$ *then* $4 \in S(k)$*.*

Proof. If $\gcd(k, 6)$ is even, then from Theorem 2 and Corollary 1 we get that $\{3, 4\} = S(2) \subseteq S(\gcd(k, 6)) \subseteq S(k)$. If $\gcd(k, 6)$ is multiple of 3, then $\{3, 4, 5\} = S(3) \subseteq S(k)$. □

Proposition 4. *If all prime divisors of k are congruent to* 1 *modulo* 4*, then no element of $S(k)$ is even.*

Proof. We argue by contradiction. Suppose that one can find a positive integer k satisfying the hypothesis of our proposition and for which at least one element a of $S(k)$ is even. As argued before, there exists a divisor d of k such that the equation $c^2 - ace + e^2 + d = 0$ is solvable in positive integers. Since $d \equiv 1 \pmod 4$, by reducing this relation modulo 2 one sees that precisely one component of each solution (c, e) is even. Reducing modulo 4, one obtains the contradiction $1 + 1 \equiv 0 \pmod 4$. □

Propositions 2 and 4 have the next consequence, which allows one to determine $S(5)$ and $S(7)$ without further explicit computations.

Corollary 2. *If all prime divisors of k are congruent to either* 1 *modulo* 6 *or to* 5 *modulo* 12 *then* $4 \notin S(k)$*.*

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