SOME APPLICATIONS OF SYMMETRIC FUNCTIONS

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Abstract

In this paper, we propose an alternative approach for the determination of the Fibonacci numbers and some results of Foata, Ramanujan and other results on Tchebychev polynomials of the first and second kinds. This approach is based on the action of the symmetrizing operator $L^k_{e_1e_2}$ on the series $\sum_{j=0}^{\infty} a_j z^j$. Obtained results confirm the effectiveness of the proposed approach.

1. Preliminaries and Notation

Here, we recall some basic definitions and theorems that are needed in the sequel. Given two sets of indeterminates A and B (called alphabets), we define a symmetric function $S_j(A-B)$ as in [1]:

$$\frac{\Pi_{b\in B}(1-zb)}{\Pi_{a\in A}(1-za)} = \sum_{j=0}^{\infty} S_j(A-B)z^j = \sum_{j=0}^{\infty} S_j(A)z^j \sum_{j=0}^{\infty} S_j(-B)z^j, \tag{1}$$

where $S_j(A-B)=0$ for $j<0$.

If $A=B$, then formula (1) can be written as

$$1 = \sum_{j=0}^{\infty} S_j(A)z^j \sum_{j=0}^{\infty} S_j(-A)z^j;$$

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therefore,

$$\sum_{j=0}^{\infty} S_j(A)z^j = \frac{1}{\sum\limits_{j=0}^{\infty} S_j(-A)z^j}.$$

On the other hand, if $A = \Phi$ and $F = B + D$, then formula (1) becomes

$$\sum_{j=0}^{\infty} S_j(-F)z^j = \sum_{j=0}^{\infty} S_j(-B)z^j \sum_{j=0}^{\infty} S_j(-D)z^j,$$

which can be rewritten as

$$S_j(-F) = \sum_{k=0}^{j} S_{j-k}(-B)S_k(-D), \text{ for all } j \in \mathbb{N}.$$

Definition 1. *[7] Given a function g on $\mathbb{R}^n$, the divided difference operator is defined as follows:*

$$\partial_{x_i x_{i+1}}(g) = \frac{g(x_1, \cdots, x_i, x_{i+1}, \cdots x_n) - g^{\sigma}(x_1, \cdots, x_i, x_{i+1}, \cdots x_n)}{x_i - x_{i+1}},$$

where g^{σ} is given by

$$g^{\sigma}(x_1, \cdots, x_i, x_{i+1}, \cdots x_n) = g(x_1, \cdots x_{i-1}, x_{i+1}, x_i, x_{i+2} \cdots x_n).$$

Definition 2. *[2] The symmetrizing operator L^k_{xy} is defined by*

$$L^k_{xy} f = \frac{x^{k+1} f - y^{k+1} f^{\sigma}}{x - y}, \text{ for all } k \in \mathbb{N}. \tag{2}$$

2. The Main Results

Proposition 1. *[2] Let E be an alphabet such that $E = \{e_1, e_2\}$. The operator $L^k_{e_1 e_2}$ is defined as follows:*

$$L^k_{e_1 e_2} f(e_1) = S_{k-1}(e_1 + e_2) f(e_1) + e_2^k \partial_{e_1 e_2} f(e_1), \quad \text{for all } k \in \mathbb{N}.$$

Theorem 1. *Given an alphabet $E = \{e_1, e_2\}$, and two series $\sum_{j=0}^{\infty} a_j z^j$ and $\sum_{j=0}^{\infty} b_j z^j$ such that $(\sum_{j=0}^{\infty} a_j z^j)(\sum_{j=0}^{\infty} b_j z^j) = 1$, we have*

$$\sum_{j=0}^{\infty} a_j S_{k+j-1}(e_1 + e_2) z^j$$

$$= \frac{\sum\limits_{j=0}^{k-1} b_j e_1^j e_2^j S_{k-j-1}(e_1 + e_2) z^j - e_1^k e_2^k z^{k+1} \sum\limits_{j=0}^{\infty} b_{j+k+1} S_j(e_1 + e_2) z^j}{\sum\limits_{j=0}^{\infty} b_j e_1^j z^j \sum\limits_{j=0}^{\infty} b_j e_2^j z^j}. \tag{3}$$

Proof. Let $\sum\limits_{j=0}^{\infty} a_j z^j$ and $\sum\limits_{j=0}^{\infty} b_j z^j$ be two sequences such that $\sum\limits_{j=0}^{\infty} a_j z^j = \frac{1}{\sum\limits_{j=0}^{\infty} b_j z^j}$. The left-hand side of the formula (3) can be written as:

$$L_{e_1 e_2}^{k} f\ (e_1) = L_{e_1 e_2}^{k} \left(\sum_{j=0}^{\infty} a_j e_1^j z^j \right) = \sum_{j=0}^{\infty} a_j S_{k+j-1}(e_1+e_2) z^j,$$

while the right-hand side can be expressed as:

$$\begin{aligned}
& S_{k-1}(e_1+e_2) f\ (e_1) + e_2^k \partial_{e_1 e_2} f(e_1) \\
= \; & \frac{S_{k-1}(e_1+e_2)}{\sum\limits_{j=0}^{\infty} b_j e_1^j z^j} + e_2^k \partial_{e_1 e_2} \frac{1}{\sum\limits_{j=0}^{\infty} b_j e_1^j z^j} \\
= \; & \frac{S_{k-1}(e_1+e_2)}{\sum\limits_{j=0}^{\infty} b_j e_1^j z^j} - \frac{\sum\limits_{j=0}^{\infty} b_j S_{j-1}(e_1+e_2) z^j}{\sum\limits_{j=0}^{\infty} b_j e_1^j z^j \sum\limits_{j=0}^{\infty} b_j e_2^j z^j} \\
= \; & \frac{\sum\limits_{j=0}^{\infty} b_j \left[e_2^j S_{k-1}(e_1+e_2) - e_2^k S_{j-1}(e_1+e_2) \right] z^j}{\sum\limits_{j=0}^{\infty} b_j e_1^j z^j \sum\limits_{j=0}^{\infty} b_j e_2^j z^j} \\
= \; & \frac{\sum\limits_{j=0}^{k-1} b_j \left[e_2^j S_{k-1}(e_1+e_2) - e_2^k S_{j-1}(e_1+e_2) \right] z^j}{\sum\limits_{j=0}^{\infty} b_j e_1^j z^j \sum\limits_{j=0}^{\infty} b_j e_2^j z^j} \\
& + \frac{\sum\limits_{j=k+1}^{\infty} b_j \left[e_2^j S_{k-1}(e_1+e_2) - e_2^k S_{j-1}(e_1+e_2) \right] z^j}{\sum\limits_{j=0}^{\infty} b_j e_1^j z^j \sum\limits_{j=0}^{\infty} b_j e_2^j z^j} \\
= \; & \frac{\sum\limits_{j=0}^{k-1} b_j e_1^j e_2^j S_{k-j-1}(e_1+e_2) z^j - e_1^k e_2^k z^{k+1} \sum\limits_{j=0}^{\infty} b_{j+k+1} S_j(e_1+e_2) z^j}{\left(\sum\limits_{j=0}^{\infty} b_j e_1^j z^j \right) \left(\sum\limits_{j=0}^{\infty} b_j e_2^j z^j \right)},
\end{aligned}$$

and the proof is complete. □

3. Applications

3.1. The Case $\sum_{j=0}^{\infty} z^j = \frac{1}{1-z}$

Corollary 1. *Given an alphabet $E = \{e_1, e_2\}$, we have*

$$\sum_{j=0}^{\infty} S_{k+j-1}(e_1+e_2)z^j = \frac{S_{k-1}(e_1+e_2) - e_1e_2S_{k-2}(e_1+e_2)z}{(1-ze_1)(1-ze_2)}, \text{ for all } k \in \mathbb{N}.$$

If $k = 1$, then

$$\sum_{j=0}^{\infty} S_j(e_1+e_2)z^j = \frac{1}{(1-ze_1)(1-ze_2)}. \tag{4}$$

Replacing e_2 by $(-e_2)$ in (4), we obtain

$$\sum_{j=0}^{\infty} S_j(e_1+[-e_2])z^j = \frac{1}{(1-ze_1)(1+ze_2)}. \tag{5}$$

Choosing e_1 and e_2 such that

$$\begin{cases} e_1e_2 = 1, \\ e_1 - e_2 = 1, \end{cases}$$

and substituting in (5) we end up with

$$\sum_{j=0}^{\infty} S_j(e_1+[-e_2])z^j = \frac{1}{1-z-z^2},$$

which represents a generating function for Fibonacci numbers, such that $F_j = S_j(e_1+[-e_2])$.

On the other hand, when replacing e_1 and e_2 by $2e_1$ and $(-2e_2)$, respectively, in (5), and under the condition $4e_1e_2 = -1$, we obtain, for $y = e_1 - e_2$,

$$\sum_{j=0}^{\infty} S_j(2e_1+[-2e_2])z^j = \frac{1}{1-2yz+z^2}, \tag{6}$$

which represents a generating function for Tchebychev polynomials of the second kind, such that $U_j(y) = S_j(2e_1+[-2e_2])$. Moreover, we deduce from (6) that

$$\sum_{j=0}^{\infty} \left[S_j(2e_1+[-2e_2]) - yS_{j-1}(2e_1+[-2e_2])\right] z^j = \frac{1-yz}{1-2yz+z^2},$$

which represents a generating function for Tchebychev polynomials of the first kind, such that

$$T_j(y) = \left[S_j(2e_1+[-2e_2]) - yS_{j-1}(2e_1+[-2e_2])\right]. \tag{7}$$

3.2. The Case $\frac{1}{\sum_{j=0}^{\infty}\frac{z^j}{j!}} = e^{-z}$

Corollary 2. *Given an alphabet $E = \{e_1, e_2\}$, we have*

$$\sum_{j=0}^{\infty} S_{k+j-1}(e_1+e_2)\frac{z^j}{j!} = \frac{\sum\limits_{j=0}^{k-1}(-1)^j e_1^j e_2^j S_{k-j-1}(e_1+e_2)\frac{z^j}{j!}}{\prod\limits_{a\in A}(1-ae_1z)\prod\limits_{a\in A}(1-ae_2z)} - \frac{e_1^k e_2^k z^{k+1}\sum\limits_{j=0}^{\infty}(-1)^{j+k+1}S_j(e_1+e_2)\frac{z^j}{j!}}{\prod\limits_{a\in A}(1-ae_1z)\prod\limits_{a\in A}(1-ae_2z)}, \text{ for all } k\in\mathbb{N}.$$

3.3. The Case $\sum_{j=0}^{\infty} S_j(A)z^j = \frac{1}{\sum_{j=0}^{\infty} S_j(-A)z^j}$

Corollary 3. *Given two alphabets $E = \{e_1, e_2\}$ and $A = \{a_1, a_2, \ldots\}$, we have*

$$\sum_{j=0}^{\infty} S_j(A)S_{k+j-1}(e_1+e_2)z^j = \frac{\sum\limits_{j=0}^{k-1}S_j(-A)e_1^j e_2^j S_{k-j-1}(e_1+e_2)z^j}{\prod\limits_{a\in A}(1-ae_1z)\prod\limits_{a\in A}(1-ae_2z)} - \frac{e_1^k e_2^k z^{k+1}\sum\limits_{j=0}^{\infty}S_{j+k+1}(-A)S_j(e_1+e_2)z^j}{\prod\limits_{a\in A}(1-ae_1z)\prod\limits_{a\in A}(1-ae_2z)}. \quad (8)$$

If $k = 1$ and $A = \{a_1, a_2\}$, then

$$\sum_{j=0}^{\infty} S_j(a_1+a_2)S_j(e_1+e_2)z^j = \frac{1-a_1a_2e_1e_2z^2}{\Pi_{a\in A}(1-ae_1z)\Pi_{a\in A}(1-ae_2z)}. \quad (9)$$

`Case 1`: Substituting $e_1 = a_1 = 1$, $e_2 = x$ and $a_2 = y$ in (9), we obtain the following identity of Ramanunja [4]:

$$\sum_{j=0}^{\infty} S_j(1+x)S_j(1+y)z^j = \frac{1-xyz^2}{(1-z)(1-zx)(1-zy)(1-zxy)}.$$

`Case 2`: Replacing e_2 by $(-e_2)$ and a_2 by $(-a_2)$ in (9) yields

$$\sum_{j=0}^{\infty} S_j(a_1+[-a_2])S_j(e_1+[-e_2])z^j = \frac{1-e_1e_2a_1a_2z^2}{(1-a_1e_1z)(1+a_2e_1z)(1+a_1e_2z)(1-a_2e_2z)}. \quad (10)$$

This case consists of three related parts.

Firstly, the substitutions of

$$\begin{cases} a_1 + a_2 = 0, \\ a_1 a_2 = 1, \end{cases} \quad \text{and} \quad \begin{cases} e_1 - e_2 = 1, \\ e_1 e_2 = 1, \end{cases}$$

in (10) give

$$\begin{aligned} \sum_{j=0}^{\infty} S_j(a_1 + [-a_2]) S_j(e_1 + [-e_2]) z^j &= \frac{1 - z^2}{1 - z - 4z^2 - z^3 + z^4} \\ &= \sum_{j=0}^{\infty} F_j^2 z^j, \end{aligned}$$

which represents a generating function for Fibonacci numbers of second order (see [6]), such that $F_j^2 = S_j(a_1 + [-a_2]) S_j(e_1 + [-e_2])$.

Secondly, the substitution of

$$\begin{cases} e_1 - e_2 = 1, \\ e_1 e_2 = 1, \\ 4a_1 a_2 = -1, \end{cases}$$

in (10) and set for ease on notations $x = a_1 - a_2$, we reach

$$\frac{1 + z^2}{1 - 2xz + (3 - 4x^2)z^2 + 2xz^3 + z^4} = \sum_{j=0}^{\infty} F_j U_j(x) z^j,$$

which corresponds to a generating function for the combined Fibonacci numbers and Tchebychev polynomials of the second kind.

In the last case, recall that for $y = e_1 - e_2$, the substitution of

$$\begin{cases} 4e_1 e_2 = -1, \\ 4a_1 a_2 = -1, \end{cases}$$

in (10) results in

$$\sum_{j=0}^{\infty} U_j(y) U_j(x) z^j = \frac{1 - z^2}{1 - 4yxz + (4x^2 + 4y^2 - 2)z^2 - 4yxz^3 + z^4},$$

which represents a generating function for Tchebychev polynomials of the second kind.

According to formulas (7), (8), and to the fact that

$$S_{j-1}(2a_1 + [-2a_2]) = \frac{(2a_1)^j - (-2a_2)^j}{2a_1 + 2a_2},$$

we have

$$\sum_{j=0}^{\infty} U_j(y)T_j(x)z^j = \frac{1 - 2yxz + (2x^2 - 1)z^2}{1 - 4yxz + (4x^2 + 4y^2 - 2)z^2 - 4yxz^3 + z^4},$$

which represents a generating function for the combined Tchebychev polynomials of the second and first kinds.

Finally, we have

$$\sum_{j=0}^{\infty} T_j\,(y)\,T_j(x)z^j = \frac{1 - 3yxz + (2x^2 + 2y^2 - 1)z^2 - yxz^3}{1 - 4yxz + (4x^2 + 4y^2 - 2)z^2 - 4yxz^3 + z^4},$$

which corresponds to a generating function for Tchebychev polynomials of the first kind.

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References

[1] A. Abderrezzak, Généralisation de la transformation d'Euler d'une série formelle, *Adv. Math.* **103** (1994), 180–195.

[2] A. Boussayoud, A. Abderrezzak and M. Kerada, A Generalization of some orthogonal polynomials, *Springer Proc. Math. Stat.* **41** (2013), 229–235.

[3] A. Lascoux, Inversion des matrices de Hankel, *Linear Algebra Appl.* **129** (1990), 77–102.

[4] A. Lascoux, Addition of ±1 : application to arithmetic, *Sémin. Lothar. Comb.* **52** (2004), 1–9.

[5] A. Lascoux and M.P. Schützenberger, Symmetry and flag manifolds, *Lect. Notes Math.* **996** (1983), 118–144.

[6] D. Foata, G-N. Han, Nombres de Fibonacci et polynômes orthogonaux, *Leonardo Fibonacci: il tempo, le opere, l'eredit scientifica* [*Pisa. 23–25 Marzo 1994, Marcello Morelli e Marco Tangheroni, ed.*], (1990), 179–200.

[7] I.G. Macdonald, Symmetric functions and Hall polynomias, *Oxford University Press*, (1979).