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MIXED SUMS OF TRIANGULAR NUMBERS AND CERTAIN BINARY QUADRATIC FORMS

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Abstract
In this paper, we prove that for $d = 3, \ldots, 8$, every natural number can be written as $t_x + t_y + 3t_z + dt_w$, where x, y, z, and w are nonnegative integers and $t_k = k(k+1)/2$ $(k = 0, 1, 2, \ldots)$ is a triangular number. Furthermore, we study mixed sums of triangular numbers and certain binary quadratic forms.

1. Introduction

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$, and denote the set of squares by $\{x^2 : x \in \mathbb{Z}\}$. A *triangular* number is defined as $t_x = x(x+1)/2$, where $x \in \mathbb{N}_0$. Furthermore, for positive integers $j, k, n \in \mathbb{N}$, let $d_{j,k}(n)$ denote the number of positive divisors d of n such that $d \equiv j \pmod{k}$.

A well-known result of Gauss states that every $n \in \mathbb{N}$ can be written as a sum of three triangular numbers; that is, $n = \Delta_1 + \Delta_2 + \Delta_3$, where Δ_j $(1 \leq j \leq 3)$ is a triangular number. In 1862 and 1863, Liouville [7, 8] proved the following result:

Theorem 1.1. (Liouville) *Let a, b, and c be positive integers with $a \leq b \leq c$. Then every $n \in \mathbb{N}_0$ can be written as $at_x + bt_y + ct_z$ for $x, y, z \in \mathbb{N}_0$ if and only if (a, b, c) is among the following vectors:*

$$(1, 1, 1), (1, 1, 2), (1, 1, 4), (1, 1, 5), (1, 2, 2), (1, 2, 3), (1, 2, 4).$$

In this paper, we prove that $n = \Delta_1 + \Delta_2 + 3\Delta_3 + d\Delta_4$, where $3 \leq d \leq 8$ and Δ_j $(1 \leq j \leq 4)$ is a triangular number. To prove this conjecture, we use the results of Barrucand et al. [2] and Adiga et al. [1], which were obtained using Ramanujan's theory of theta functions. For $q \in \mathbb{C}$ such that $|q| < 1$, we introduce

$$\varphi(q) = \sum_{n \in \mathbb{Z}} q^{n^2}, \ \psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}, \ a(q) = \sum_{m,n \in \mathbb{Z}} q^{m^2+mn+n^2}.$$

Note that Williams [10] determined the number of representations of $n \in \mathbb{N}_0$ as $\Delta_1 + \Delta_2 + 2(\Delta_3 + \Delta_4)$ using an entirely arithmetic method.

Our main theorems are as follows:

Theorem 1.2.

(1) *For $d \in \mathbb{N}$ with $3 \le d \le 8$, every $n \in \mathbb{N}_0$ can be represented as $t_x+t_y+3t_z+dt_w$ for $x, y, z, w \in \mathbb{N}_0$.*

(2) *Let a, b, c, and d be positive integers with $a \le b \le c \le d$. Then every $n \in \mathbb{N}_0$ can be written as $at_x+bt_y+ct_z+dt_w$ for $x, y, z, w \in \mathbb{N}_0$ if and only if (a, b, c, d) is among the following vectors:*

$$(1,1,1,d),(1,1,2,d),(1,1,4,d),(1,1,5,d),(1,2,2,d),(1,2,3,d),(1,2,4,d),$$
$$(1,1,3,3),(1,1,3,4),(1,1,3,5),(1,1,3,6),(1,1,3,7),(1,1,3,8).$$

Theorem 1.3. *For fixed positive integers a and c, set*

$$g_c^a(x,y,z) = at_x + c(y^2+yz+z^2), \text{ with } x \in \mathbb{N}_0,\ y,z \in \mathbb{Z}.$$

(1) *The form g_c^a represents all $n \in \mathbb{N}_0$ if and only if $(a, c) = (1, 1)$.*

(2) *If the form g_c^a represents $n = 1, 2, 4, 8$, it represents all $n \in \mathbb{N}_0$.*

Theorem 1.4. *For fixed positive integers, a, b, and c with $a \le b$, set*

$$g_c^{a,b}(x,y,z,w) = at_x + bt_y + c(z^2+zw+w^2), \text{ with } x,y \in \mathbb{N}_0,\ z,w \in \mathbb{Z}.$$

(1) *The form $g_c^{a,b}$ represents all $n \in \mathbb{N}_0$ if and only if*

$$(a,b,c) = \begin{cases} (1,b,1), & (b \in \mathbb{N}), \\ (2,b,1), & (b = 2,3,4,5,6,7,8), \\ (1,b,2), & (b = 1,2,3,4), \\ (1,2,3), & \\ (1,b,4), & (b = 1,2), \\ (1,1,5). & \end{cases}$$

(2) *If the form $g_c^{a,b}$ represents $n = 1, 2, 4, 5, 8$, it represents all $n \in \mathbb{N}_0$.*

The remainder of this paper is organized as follows. In Section 2, we introduce notation that will be used throughout our paper. In Section 3, we prove Theorem 1.2 for $d = 3, 6, 7, 8$, and in Section 4, we prove Theorem 1.3. In Section 5, we apply Theorem 1.2 to obtain the sufficiency of Theorem 1.4(1). In Section 6, we prove Theorem 1.4.

Remark 1

For $d = 4, 5$, Theorem 1.2(1) follows from Theorem 1.1 of Liouville.

Remark 2

For fixed positive integers, a, b, and c, set

$$f^{a,b,c}(x, y, z) = ax^2 + by^2 + cz^2,\ f_c^a(x, y, z) = ax^2 + c(y^2 + yz + z^2),$$

for $x, y, z \in \mathbb{Z}$. In [5, p. 104], Dickson proved that there exist no positive integers a, b, and c such that $f^{a,b,c}$ can represent all nonnegative integers. In [9], we showed that there exist no positive integers a and c such that f_c^a represents all nonnegative integers.

Remark 3

For fixed positive integers a, b, and c, set

$$f_c^{a,b}(x, y, z) = ax^2 + by^2 + c(z^2 + zw + w^2),$$

for $x, y, z, w \in \mathbb{Z}$. In [9], we determined (a, b, c), where $a \le b$, such that $f_c^{a,b}$ represents all nonnegative integers.

2. Notation and Preliminary Results

2.1. Notation

For fixed positive integers a, b, c, and d and each $n \in \mathbb{N}_0$, we define

$$\begin{aligned}
r_{a,b,c}(n) =& \sharp\{(x, y, z) \in \mathbb{Z}^3 \,|\, n = ax^2 + by^2 + cz^2\},\\
r_{a,b,c,d}(n) =& \sharp\{(x, y, z, w) \in \mathbb{Z}^3 \,|\, n = ax^2 + by^2 + cz^2 + dw^2\},\\
t_{a,b,c}(n) =& \sharp\{(x, y, z) \in \mathbb{N}_0^3 \,|\, n = at_x + bt_y + ct_z\},\\
t_{a,b,c,d}(n) =& \sharp\{(x, y, z, w) \in \mathbb{N}_0^4 \,|\, n = at_x + bt_y + ct_z + dt_w\},\\
m_{a\text{-}b,c}(n) =& \sharp\left\{(x, y, z) \in \mathbb{Z} \times \mathbb{N}_0^2 \,|\, n = ax^2 + bt_y + ct_z\right\},\\
m_{a,b\text{-}c}(n) =& \sharp\left\{(x, y, z) \in \mathbb{Z}^2 \times \mathbb{N}_0 \,|\, n = ax^2 + by^2 + ct_z\right\},\\
A_c^a(n) =& \sharp\left\{(x, y, z) \in \mathbb{Z}^3 \,|\, n = ax^2 + c(y^2 + yz + z^2)\right\},\\
A_c^{a,b}(n) =& \sharp\left\{(x, y, z, w) \in \mathbb{Z}^4 \,|\, n = ax^2 + by^2 + c(z^2 + zw + w^2)\right\},\\
B_c^a(n) =& \sharp\left\{(x, y, z) \in \mathbb{N}_0 \times \mathbb{Z}^2 \,|\, n = at_x + c(y^2 + yz + z^2)\right\},\\
B_c^{a,b}(n) =& \sharp\left\{(x, y, z, w) \in \times\mathbb{N}_0^2 \times \mathbb{Z}^2 \,|\, n = at_x + bt_y + c(z^2 + zw + w^2)\right\}.
\end{aligned}$$

2.2. Ramanujan's Theory of Theta Functions

From Baruah, Cooper, and Hirschhorn [3], recall that

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8), \tag{2.1}$$
$$\varphi(q)^2 = \varphi(q^2)^2 + 4q\psi(q^4)^2, \tag{2.2}$$
$$\varphi(q)\psi(q^2) = \psi(q)^2, \tag{2.3}$$
$$\varphi(q)\varphi(q^3) = a(q^4) + 2q\psi(q^2)\psi(q^6), \tag{2.4}$$
$$a(q) = \varphi(q)\varphi(q^3) + 4q\psi(q^2)\psi(q^6), \tag{2.5}$$
$$a(q) = a(q^4) + 6q\psi(q^2)\psi(q^6). \tag{2.6}$$

3. Proof of Theorem 1.2

3.1. Preliminary Results

Using Ramanujan's theory of theta functions, Adiga, Cooper, and Han [1] proved the following theorem:

Theorem 3.1. *Let $n \in \mathbb{N}_0$. Then,*

$$r_{1,1,3}(8n+5) = 16t_{1,1,3}(n).$$

Proof. For a detailed proof, see Barrucand et al. [2] and Adiga et al. [1]. □

Dickson [5, p. 112-113] proved the following result:

Theorem 3.2.

(1) *A nonnegative integer $n \in \mathbb{N}_0$ can be written as $x^2 + y^2 + 3z^2$ for $x, y, z \in \mathbb{Z}$ if and only if $n \neq 9^k(9l+6)$ for $k, l \in \mathbb{N}_0$.*

(2) *Every $n \in \mathbb{N}_0$ can be written as $x^2 + y^2 + 3z^2 + 3w^2$ for $x, y, z, w \in \mathbb{Z}$.*

Using Theorems 3.1 and 3.2, the following theorem is obtained:

Theorem 3.3. *A nonnegative integer $n \in \mathbb{N}_0$ can be written as $t_x + t_y + 3t_z$ for $x, y, z \in \mathbb{N}_0$ if and only if n satisfies one of the following conditions:*

(1) $n \not\equiv 5, 8 \pmod 9$,

(2) $n \equiv 5 \pmod 9$, $8n+5 = 9^k(8N+5)$, *and* $N \not\equiv 5, 8 \pmod 9$ *for* $k \in \mathbb{N}$ *and* $N \in \mathbb{N}_0$.

Proof. From Theorems 3.1 and 3.2, note that $n = t_x + t_y + 3t_z$ for $x, y, z \in \mathbb{N}_0$ if and only if $8n+5 \neq 9^k(9l+6)$ where $k, l \in \mathbb{N}_0$. Moreover, $8n+5 \equiv 0 \pmod 9$ if and only if $n \equiv 5 \pmod 9$.

First, consider the case when $n \not\equiv 5 \pmod 9$. From Theorem 3.2, condition (1) is obtained because $8n+5 \equiv 6 \pmod 9$ if and only if $n \equiv 8 \pmod 9$.

Next, we consider the case when $n \equiv 5 \pmod 9$ and $n = 9N+5$ for $N \in \mathbb{N}_0$, which implies $8n+5 = 9(8N+5)$. Therefore, we set $8n+5 = 9^k(8N'+5)$ for $k \in \mathbb{N}, N' \in \mathbb{N}_0$, where $N' \not\equiv 5 \pmod 9$. From Theorem 3.2 and the discussion in the first paragraph, we obtain condition (2). □

3.2. Proof of Theorem 1.2 (1) for $d = 3$

3.2.1. Preliminary Results

Before proving Theorem 1.2 (1) for $d = 3$, we first obtain a useful number theoretic property for $t_{1,1,3,3}(n)$.

Theorem 3.4. *Let $k \in \mathbb{N}$ and $n, N \in \mathbb{N}_0$. Then,*

$$t_{1,1,3,3}(2^k N + (2^k - 1)) = 2^k t_{1,1,3,3}(N),$$

which implies $t_{1,1,3,3}(n) \equiv 0 (\bmod 2^k)$ if $n \equiv -1 \pmod{2^k}$.

Proof. Multiplying both sides of (2.6) by $\psi(q^2)\psi(q^6)$ yields

$$\psi(q^2)\psi(q^6)a(q) = \psi(q^2)\psi(q^6)a(q^4) + 6q\psi(q^2)^2\psi(q^6)^2,$$

which implies

$$\sum_{n=0}^{\infty} B_1^{2,6}(n)q^n = \sum_{N=0}^{\infty} B_2^{1,3}(N)q^{2N} + 6\sum_{N=0}^{\infty} t_{1,1,3,3}(N)q^{2N+1}. \tag{3.1}$$

Multiplying both sides of (2.5) by $\psi(q^2)\psi(q^6)$ yields

$$\psi(q^2)\psi(q^6)a(q) = \varphi(q)\varphi(q^3)\psi(q^2)\psi(q^6) + 4q\psi(q^2)^2\psi(q^6)^2.$$

From (2.3), we obtain

$$\psi(q^2)\psi(q^6)a(q) = \psi(q)^2\psi(q^3)^2 + 4q\psi(q^2)^2\psi(q^6)^2,$$

which implies

$$\sum_{n=0}^{\infty} B_1^{2,6}(n)q^n = \sum_{n=0}^{\infty} t_{1,1,3,3}(n)q^n + 4\sum_{N=0}^{\infty} t_{1,1,3,3}(N)q^{2N+1}. \tag{3.2}$$

From (3.1) and (3.2), note that

$$B_1^{2,6}(2N+1) = 6t_{1,1,3,3}(N) = t_{1,1,3,3}(2N+1) + 4t_{1,1,3,3}(N),$$

which implies

$$t_{1,1,3,3}(2N+1) = 2t_{1,1,3,3}(N). \tag{3.3}$$

Using induction, we complete the proof of the theorem. Clearly, for $k = 1$, the theorem holds. Now, suppose the theorem holds for k. In this case,

$$\begin{aligned} t_{1,1,3,3}(2^{k+1}N + (2^{k+1}-1)) &= t_{1,1,3,3}\left(2^k \cdot (2N+1) + (2^k-1)\right) \\ &= 2^k t_{1,1,3,3}(2N+1) \\ &= 2^{k+1} t_{1,1,3,3}(N). \end{aligned}$$

□

From Baruah, Cooper, and Hirschhorn [3], recall the following result:

Theorem 3.5. (Baruah, Cooper and Hirschhorn) *For every $n \in \mathbb{N}_0$,*

$$t_{1,1,3,3}(n) = \begin{cases} \frac{1}{4} r_{1,1,3,3}(n+1) & \text{if } n \text{ is even}, \\ \frac{1}{8}\{r_{1,1,3,3}(2n+2) - r_{1,1,3,3}(n+1)\} & \text{if } n \text{ is odd}. \end{cases}$$

3.2.2. Proof of Theorem 1.2 (1) for $d = 3$

Proof. From Theorem 3.2, first note that for every $N \in \mathbb{N}_0$, $r_{1,1,3,3}(N) > 0$. If $n \in \mathbb{N}_0$ is even and $n = 2N$ for $N \in \mathbb{N}_0$, by Theorem 3.5, we obtain

$$t_{1,1,3,3}(n) = t_{1,1,3,3}(2N) = \frac{1}{4} r_{1,1,3,3}(2N+1) > 0.$$

Suppose that $n \in \mathbb{N}_0$ is odd and $n+1 = 2^k(2N+1)$ for $k \in \mathbb{N}$ and $N \in \mathbb{N}_0$. Then,

$$n = 2^k \cdot (2N) + (2^k - 1).$$

From Theorems 3.4 and 3.5, it follows that

$$t_{1,1,3,3}(n) = t_{1,1,3,3}\left(2^k \cdot (2N) + (2^k-1)\right) = 2^k t_{1,1,3,3}(2N) = \frac{2^k}{4} r_{1,1,3,3}(2N+1) > 0.$$

□

From the proof of Theorem 1.2 (1) for $d = 3$, we can improve Theorem 3.5 of Baruah, Cooper, and Hirschhorn [3] when n is an odd number.

Corollary 3.1.

(1) *Suppose that $n \in \mathbb{N}_0$ is even and $n = 2N$ for $N \in \mathbb{N}_0$. Then,*

$$t_{1,1,3,3}(n) = \frac{1}{4} r_{1,1,3,3}(2N+1).$$

(2) *Suppose that $n \in \mathbb{N}_0$ is odd and $n+1 = 2^k(2N+1)$ for $k \in \mathbb{N}$ and $N \in \mathbb{N}_0$. Then,*

$$t_{1,1,3,3}(n) = \frac{2^k}{4} r_{1,1,3,3}(2N+1).$$

Corollary 3.2. *For $k \in \mathbb{N}$ and $N \in \mathbb{N}_0$,*

$$r_{1,1,3,3}(2^k(2N+1)) = r_{1,1,3,3}(2(2N+1)) + 4(2^{k-1}-1) r_{1,1,3,3}(2N+1).$$

Proof. For $k, n \in \mathbb{N}$ and $N \in \mathbb{N}_0$, set

$$n+1 = 2^k(2N+1),$$

which implies $n = 2^k \cdot (2N) + 2^k - 1$.

By Theorem 3.5 and Corollary 3.1 (2), it follows that

$$\begin{aligned} t_{1,1,3,3}(n) =& \frac{1}{8}\{r_{1,1,3,3}(2n+2) - r_{1,1,3,3}(n+1)\} \\ =& \frac{1}{8}\{r_{1,1,3,3}(2^{k+1}(2N+1)) - r_{1,1,3,3}(2^k(2N+1))\} \\ =& \frac{2^k}{4} r_{1,1,3,3}(2N+1), \end{aligned}$$

which implies

$$r_{1,1,3,3}(2^{k+1}(2N+1)) = r_{1,1,3,3}(2^k(2N+1)) + 2^{k+1} r_{1,1,3,3}(2N+1).$$

Solving this recurrence relation with respect to k completes the proof. □

3.3. Proof of Theorem 1.2 (1) for $d = 6$

Proof. From Theorem 3.3, we only need to prove that $n \in \mathbb{N}_0$ can be written as $t_x + t_y + 3t_z + 6t_w$ for $x, y, z, w \in \mathbb{N}_0$ if $n \equiv 5 \,\text{or}\, 8 \pmod 9$.

When $n \equiv 8 \pmod 9$, taking $w = 1$ yields

$$n - 6t_1 \equiv 8 - 6 \cdot 1 \equiv 2 \pmod 9,$$

which implies from Theorem 3.3 that $n - 6t_1$ can be expressed as $t_x + t_y + 3t_z$ for $x, y, z \in \mathbb{N}_0$.

When $n \equiv 5 \pmod 9$, by Theorem 3.3, we assume that

$$8n + 5 = 9^k(8N+5), \ k, N \in \mathbb{N}, \ N \equiv 8 \pmod 9.$$

Taking $w = 2$, note that $n - 6t_2 = n - 6 \cdot 3 \equiv 5 \pmod 9$ and

$$\begin{aligned} 8(n - 6t_2) + 5 =& 8n + 5 - 8 \cdot 2 \cdot 9 \\ =& 9^k(8N+5) - 8 \cdot 2 \cdot 9 \\ =& 9\left\{9^{k-1}(8N+5) - 8 \cdot 2\right\} \\ =& 9\left\{8\left(9^{k-1}N + \frac{9^{k-1} \cdot 5 - 5}{8} - 2\right) + 5\right\}. \end{aligned}$$

When $k = 1$,

$$9^{k-1}N + \frac{9^{k-1} \cdot 5 - 5}{8} - 2 = N - 2 \equiv 8 - 2 \equiv 6 (\operatorname{mod} 9),$$

which implies $n - 6t_2$ can be expressed as $t_x + t_y + 3t_z$ for $x, y, z \in \mathbb{N}_0$.

When $k \geq 2$, we obtain

$$9^{k-1}N + \frac{9^{k-1} \cdot 5 - 5}{8} - 2 \equiv -8 \cdot 5 - 2 \equiv 3 (\operatorname{mod} 9),$$

which implies $n - 6t_2$ can be expressed as $t_x + t_y + 3t_z$ for $x, y, z \in \mathbb{N}_0$. □

3.4. Proof of Theorem 1.2 (1) for $d = 7, 8$

Proof. By Theorem 3.3, we are reduced to proving that $n \in \mathbb{N}_0$ can be written as $t_x + t_y + 3t_z + dt_w$ for $x, y, z, w \in \mathbb{N}_0$ if $n \equiv 5 \operatorname{or} 8 (\operatorname{mod} 9)$.

Taking $w = 1$, we have

$$n - d \cdot t_1 \not\equiv 5, 8 (\operatorname{mod} 9),$$

which implies $n - d \cdot t_1$ can be expressed as $t_x + t_y + 3t_z$ for $x, y, z \in \mathbb{N}_0$. □

3.5. Proof of Theorem 1.2 (2)

Proof. Suppose that every $n \in \mathbb{N}_0$ can be expressed as $at_x + bt_y + ct_z + dt_w$ for $x, y, z, w \in \mathbb{N}_0$. Taking $n = 1, 2$ yields

$$(a, b) = (1, 1), (1, 2).$$

First, we consider the case when $(a, b) = (1, 2)$. Choosing $n = 4$ implies

$$(a, b, c, d) = (1, 2, 2, d), (1, 2, 3, d), (1, 2, 4, d).$$

Next, we consider the case when $(a, b) = (1, 1)$. If $n = 5$, then $c = 1, 2, 3, 4, 5$, which implies

$$(a, b, c, d) = (1, 1, 1, d), (1, 1, 2, d), (1, 1, 3, d), (1, 1, 4, d), (1, 1, 5, d).$$

When $(a, b, c, d) = (1, 1, 3, d)$, taking $n = 8$ yields

$$(a, b, c, d) = (1, 1, 3, 3), (1, 1, 3, 4), (1, 1, 3, 5), (1, 1, 3, 6), (1, 1, 3, 7), (1, 1, 3, 8).$$

Thus, the necessary conditions are obtained; sufficiency follows from Theorems 1.1 and 1.2 (1). □

4. Proof of Theorem 1.3

4.1. Preliminary Results

First, note that for each positive integer $n \in \mathbb{N}$,

$$\sharp\{(x,y) \in \mathbb{Z}^2 | n = x^2 + xy + y^2\} = 6(d_{1,3}(n) - d_{2,3}(n)). \tag{4.1}$$

To prove this formula, we refer to Berndt [4, p. 79]. Formula (4.1) implies $n = 2, 5, 6, 8$ cannot be expressed as $x^2 + xy + y^2$ for $x, y, z \in \mathbb{Z}$.

Next, note that every $n \in \mathbb{N}_0$ can be expressed as $x^2 + 3y^2 + t_z$ for $(x, y, z) \in \mathbb{Z}^2 \times \mathbb{N}_0$, which was proven by Guo, Pan, and Sun [6].

Finally, consider the following formula from Baruah, Cooper, and Hirschhorn [3]:

$$a(q) = \varphi(q)\varphi(q^3) + 4q\psi(q^2)\psi(q^6), \tag{4.2}$$

where

$$\varphi(q) = \sum_{n\in\mathbb{Z}} q^{n^2},\ \psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2},\ a(q) = \sum_{m,n\in\mathbb{Z}} q^{m^2+mn+n^2}.$$

4.2. Proof of Theorem 1.3

Proof. If $n = 1$, then $a = 1$ or $c = 1$. Taking $n = 2$, we obtain

$$(a, c) = (1, 1), (1, 2), (2, 1).$$

Choosing $n = 4$ implies $(a, c) \neq (1, 2)$; choosing $n = 8$ implies $(a, c) \neq (2, 1)$.

From (4.2), we have

$$\psi(q)a(q) = \varphi(q)\varphi(q^3)\psi(q) + 4q\psi(q)\psi(q^2)\psi(q^6),$$

which implies

$$\sum_{n=0}^{\infty} B_1^1(n)q^n = \sum_{n=0}^{\infty} m_{1,3\text{-}1}(n)q^n + 4\sum_{N=0}^{\infty} t_{1,2,6}(N)q^{N+1}.$$

From a result of Guo, Pan, and Sun [6], it follows that $m_{1,3\text{-}1}(n) > 0$. Therefore, $B_1^1(n) > 0$, which means that every $n \in \mathbb{N}_0$ can be expressed as $t_x + (y^2 + yz + z^2)$ for $(x, y, z) \in \mathbb{N}_0 \times \mathbb{Z}^2$; thus, Theorem 1.3 (1) holds.

Theorem 1.3 (2) follows from the above discussion. □

5. Applications of Theorem 1.2

Theorem 1.2 is used to prove the following theorems; in particular, to prove Theorem 5.1, we use the fact that every $n \in \mathbb{N}_0$ can be written as $t_x + t_y + 3(t_z + t_w)$ for $x, y, z, w \in \mathbb{N}_0$.

Theorem 5.1. *Let $b \in \mathbb{N}$ and $2 \leq b \leq 8$. Then every $n \in \mathbb{N}_0$ can be expressed as $2t_x + bt_y + (z^2 + zw + w^2)$ for $(x, y, z, w) \in \mathbb{N}_0^2 \times \mathbb{Z}^2$.*

Theorem 5.2. *Let $b \in \mathbb{N}$ and $1 \leq b \leq 4$. Then every $n \in \mathbb{N}_0$ can be expressed as $t_x + bt_y + 2(z^2 + zw + w^2)$ for $(x, y, z, w) \in \mathbb{N}_0^2 \times \mathbb{Z}^2$.*

Theorem 5.3. *Every $n \in \mathbb{N}_0$ can be expressed as $t_x + 2t_y + 3(z^2 + zw + w^2)$ for $(x, y, z, w) \in \mathbb{N}_0^2 \times \mathbb{Z}^2$.*

Theorem 5.4. *Let $b = 1, 2$. Then every $n \in \mathbb{N}_0$ can be expressed as $t_x + bt_y + 4(z^2 + zw + w^2)$ for $(x, y, z, w) \in \mathbb{N}_0^2 \times \mathbb{Z}^2$.*

Theorem 5.5. *Every $n \in \mathbb{N}_0$ can be expressed as $t_x + t_y + 5(z^2 + zw + w^2)$ for $(x, y, z, w) \in \mathbb{N}_0^2 \times \mathbb{Z}^2$.*

5.1. Proof of Theorem 5.1

5.1.1. Preliminary Results

Consider the following result by Dickson [5, p. 112-113]:

Lemma 5.1. *A nonnegative integer $n \in \mathbb{N}_0$ can be written as $x^2 + 4y^2 + 12z^2$ for $x, y, z \in \mathbb{Z}$ if and only if $n \neq 4l + 2,\ 4l + 3,\ 9^k(9l + 6)$, where $k, l \in \mathbb{N}_0$.*

Lemma 5.1 gives rise to the following proposition:

Proposition 5.1. *A nonnegative integer $n \in \mathbb{N}_0$ can be written as $x^2+4(y^2+yz+z^2)$ for $x, y, z \in \mathbb{Z}$ if and only if $n \neq 4l + 2,\ 4l + 3,\ 9^k(9l + 6)$, where $k, l \in \mathbb{N}_0$.*

Proof. Replacing q by q^4 in (2.6) and (2.5) yields

$$\begin{aligned} a(q^4) =& a(q^{16}) + 6q^4\psi(q^8)\psi(q^{24}), \\ \varphi(q^4)\varphi(q^{12}) =& a(q^{16}) + 2q^4\psi(q^8)\psi(q^{24}). \end{aligned}$$

Multiplying both sides of these equations by $\varphi(q)$ results in

$$\varphi(q)a(q^4) = \varphi(q)a(q^{16}) + 6q^4\varphi(q)\psi(q^8)\psi(q^{24}), \tag{5.1}$$

$$\varphi(q)\varphi(q^4)\varphi(q^{12}) = \varphi(q)a(q^{16}) + 2q^4\varphi(q)\psi(q^8)\psi(q^{24}), \tag{5.2}$$

which implies

$$\begin{aligned} \sum_{n=0}^{\infty} A_4^1(n)q^n =& \sum_{n=0}^{\infty} A_{16}^1(n)q^n + 6q^4 \sum_{N=0}^{\infty} m_{1\text{-}8,24}(N)q^N, \\ \sum_{n=0}^{\infty} r_{1,4,12}(n)q^n =& \sum_{n=0}^{\infty} A_{16}^1(n)q^n + 2q^4 \sum_{N=0}^{\infty} m_{1\text{-}8,24}(N)q^N. \end{aligned}$$

Therefore, it follows that $n \neq 4l+2,\ 4l+3,\ 9^k(9l+6)$, where $k, l \in \mathbb{N}_0$, if and only if $r_{1,4,12}(n) > 0$, if and only if $A_{16}^1(n) > 0$ or $m_{1\text{-}8,24}(n-4) > 0$, and if and only if $A_4^1(n) > 0$, which proves the proposition. $\square$

Using Proposition 5.1, we can prove the following proposition:

Proposition 5.2. *A nonnegative integer $N \in \mathbb{N}_0$ can be written as $2t_x + (y^2 + yz + z^2)$ for $(x, y, z) \in \mathbb{N}_0 \times \mathbb{Z}^2$ if and only if either of the following occurs:*

(1) $N \not\equiv 2, 8 \,(\mathrm{mod}\, 9)$,

(2) $N \equiv 2\,(\mathrm{mod}\ 9)$ *and* $4N+1 = 9^k(4N'+1)$, $N' \not\equiv 2, 8(\mathrm{mod}\ 9)$, $k \in \mathbb{N}, N' \in \mathbb{N}_0$.

Proof. Multiplying both sides of (2.1) by $a(q^4)$ yields

$$\varphi(q)a(q^4) = \varphi(q^4)a(q^4) + 2q\psi(q^8)a(q^4),$$

which implies

$$\sum_{n=0}^{\infty} A_4^1(n)q^n = \sum_{N=0}^{\infty} A_1^1(N)q^{4N} + 2\sum_{N=0}^{\infty} B_1^2(N)q^{4N+1}.$$

We then obtain

$$B_1^2(N) > 0 \text{ if and only if } A_4^1(4N+1) > 0.$$

Therefore, the proposition follows from Proposition 5.1 and the facts that

$$4N + 1 \equiv 0 \,(\mathrm{mod}\, 9) \text{ if and only if } N \equiv 2 \,(\mathrm{mod}\, 9)$$

and

$$4N + 1 \equiv 6 \,(\mathrm{mod}\, 9) \text{ if and only if } N \equiv 8 \,(\mathrm{mod}\, 9).$$

□

5.1.2. Proof of Theorem 5.1 for $b \neq 3, 6$

Proof. Because of Proposition 5.2, we are reduced to proving that $N \in \mathbb{N}_0$ can be written as $2t_x + bt_y + (z^2 + zw + w^2)$ for $(x, y, z, w) \in \mathbb{N}_0^2 \times \mathbb{Z}^2$ if $N \equiv 2$ or $8\,(\mathrm{mod}\ 9)$.

Suppose $b = 2$. When $N \equiv 2(\mathrm{mod}\, 9)$, taking $y = 2$ results in $N - 2 \cdot 3 \equiv 5(\mathrm{mod}\, 9)$, which implies $B_1^2(N - 2 \cdot t_2) > 0$. When $N \equiv 8(\mathrm{mod}\, 9)$, taking $y = 1$ results in $N - 2 \cdot 1 \equiv 6(\mathrm{mod}\, 9)$, which implies $B_1^2(N - 2 \cdot t_1) > 0$.

Next, suppose $b = 4, 5, 7$. Taking $y = 1$ yields $N - b \cdot 1 \not\equiv 2, 8(\mathrm{mod}\, 9)$, which implies $B_1^2(N - b \cdot t_1) > 0$.

Suppose $b = 8$. When $N \equiv 2(\mathrm{mod}\, 9)$, taking $y = 1$ results in $N - 8 \cdot 1 \equiv 3(\mathrm{mod}\, 9)$, which implies $B_1^2(N - 8 \cdot t_1) > 0$. When $N = 8, 17, 26, 35, 44$, we take

$$(x, y, z, w) = (0, 1, 0, 0), (0, 1, 3, 0), (1, 1, 4, 0), (0, 1, 3, 3), (0, 1, 6, 0).$$

When $N \equiv 8(\mathrm{mod}\, 9)$ and $N > 44$, taking $y = 3$ yields $N - 8 \cdot 6 \equiv 5(\mathrm{mod}\, 9)$, which implies $B_1^2(N - 8 \cdot t_3) > 0$. □

5.1.3. Proof of Theorem 5.1 for $b = 3$

Proof. By Proposition 5.2, it suffices to show that $N \in \mathbb{N}_0$ can be written as $2t_x + 3t_y + (z^2 + zw + w^2)$ for $(x, y, z, w) \in \mathbb{N}_0^2 \times \mathbb{Z}^2$ if $N \equiv 2 \,\mathrm{or}\, 8 \pmod 9$.

First, consider the case when $N \equiv 2(\bmod 9)$. By Proposition 5.2, we assume that $4N + 1 = 9^k(4N' + 1)$, where $k \in \mathbb{N}$, $N' \in \mathbb{N}_0$, and $N' \equiv 8(\bmod 9)$. Taking $y = 2$ results in

$$N - 3 \cdot t_2 = N - 3 \cdot 3 \equiv 2(\bmod 9).$$

We then obtain

$$\begin{aligned} 4(N - 3 \cdot t_2) + 1 =& 4(N - 3 \cdot 3) + 1 \\ =& 4N + 1 - 36 \\ =& 9^k(4N' + 1) - 36 \\ =& 9\left\{4\left(9^{k-1}N' + \frac{9^{k-1} - 1}{4} - 1\right) + 1\right\}. \end{aligned}$$

When $k = 1$,

$$9^{k-1}N' + \frac{9^{k-1} - 1}{4} - 1 \equiv 8 - 1 \equiv 7(\bmod 9),$$

which implies $B_1^2(N - 3 \cdot t_2) > 0$.

When $k \geq 2$, we obtain

$$9^{k-1}N' + \frac{9^{k-1} - 1}{4} - 1 \equiv 2 - 1 \equiv 1(\bmod 9),$$

which implies $B_1^2(N - 3 \cdot t_2) > 0$.

Next, we consider the case when $N \equiv 8(\bmod 9)$. Taking $y = 1$ yields

$$N - 3 \cdot t_1 = N - 3 \equiv 5(\bmod 9),$$

which implies $B_1^2(N - 3 \cdot t_1) > 0$. □

5.1.4. Proof of Theorem 5.1 for $b = 6$

Proof. Multiplying both sides of (2.5) by $\psi(q^2)\psi(q^6)$ results in

$$\psi(q^2)\psi(q^6)a(q) = \varphi(q)\varphi(q^3)\psi(q^2)\psi(q^6) + 4q\psi(q^2)^2\psi(q^6)^2.$$

Using (2.3) we obtain

$$\psi(q^2)\psi(q^6)a(q) = \psi(q)^2\psi(q^3)^2 + 4q\psi(q^2)^2\psi(q^6)^2,$$

which implies

$$\sum_{n=0}^{\infty} B_1^{2,6}(n)q^n = \sum_{n=0}^{\infty} t_{1,1,3,3}(n)q^n + 4\sum_{N=0}^{\infty} t_{1,1,3,3}(N)q^{2N+1}. \tag{5.3}$$

Theorem 1.2 implies $t_{1,1,3,3}(n) > 0$, which proves Theorem 5.1 for $b = 6$. □

5.2. Proof of Theorem 5.2

5.2.1. Preliminary Results

Consider the following result by Dickson [5, p. 112]:

Lemma 5.2. *A nonnegative integer* $n \in \mathbb{N}_0$ *can be written as* $x^2 + 16y^2 + 48z^2$ *for* $x, y, z \in \mathbb{Z}$ *if and only if* $n \neq 4l+2, 4l+3, 8l+5, 16l+8, 16l+12, 9^k(9l+6)$*, where* $k, l \in \mathbb{N}_0$.

Proposition 5.3. *A nonnegative integer* $n \in \mathbb{N}_0$ *can be written as* $x^2+16(y^2+yz+z^2)$ *for* $x, y, z \in \mathbb{Z}$ *if and only if* $n \neq 4l+2, 4l+3, 8l+5, 16l+8, 16l+12, 9^k(9l+6)$*, where* $k, l \in \mathbb{N}_0$.

Proof. Replacing q by q^{16} in (2.6) and (2.4) results in

$$\begin{aligned} a(q^{16}) =& a(q^{64}) + 6q^{16}\psi(q^{32})\psi(q^{96}), \\ \varphi(q^{16})\varphi(q^{48}) =& a(q^{64}) + 2q^{16}\psi(q^{32})\psi(q^{96}). \end{aligned}$$

Multiplying both sides of these equations by $\varphi(q)$ yields

$$\varphi(q)a(q^{16}) = \varphi(q)a(q^{64}) + 6q^{16}\varphi(q)\psi(q^{32})\psi(q^{96}), \tag{5.4}$$

$$\varphi(q)\varphi(q^{16})\varphi(q^{48}) = \varphi(q)a(q^{64}) + 2q^{16}\varphi(q)\psi(q^{32})\psi(q^{96}), \tag{5.5}$$

which implies

$$\begin{aligned} \sum_{n=0}^{\infty} A_{16}^1(n)q^n =& \sum_{n=0}^{\infty} A_{64}^1(n)q^n + 6q^{16} \sum_{N=0}^{\infty} m_{1\text{-}32,96}(N)q^N, \\ \sum_{n=0}^{\infty} r_{1,16,48}(n)q^n =& \sum_{n=0}^{\infty} A_{64}^1(n)q^n + 2q^{16} \sum_{N=0}^{\infty} m_{1\text{-}32,96}(N)q^N. \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} r_{1,16,48}(n) > 0 \,\text{if and only if}\, & A_{64}^1(n) > 0 \,\text{or}\, m_{1\text{-}32,96}(n-16) > 0, \\ \text{if and only if}\, & A_{16}^1(n) > 0, \end{aligned}$$

which proves the proposition. ∎

Proposition 5.4. *A nonnegative integer* $N \in \mathbb{N}_0$ *can be written as* $t_x + 2(y^2+yz+z^2)$ *for* $(x, y, z) \in \mathbb{N}_0 \times \mathbb{Z}^2$ *if and only if either of the following occurs:*

(1) $N \not\equiv 1, 4 \pmod 9$,

(2) $N \equiv 1 \pmod 9$ *and* $8N + 1 = 9^k(8N' + 1)$, $k \in \mathbb{N}$, $N' \in \mathbb{N}_0$*, and* $N' \not\equiv 1, 4 \pmod 9$.

Proof. Multiplying both sides of (2.1) by $a(q^{16})$ yields

$$\varphi(q)a(q^{16}) = \varphi(q^4)a(q^{16}) + 2q\psi(q^8)a(q^{16}),$$

which implies

$$\sum_{n=0}^{\infty} A_{16}^1(n)q^n = \sum_{N=0}^{\infty} A_4^1(N)q^{4N} + 2\sum_{N=0}^{\infty} B_2^1(N)q^{8N+1}.$$

We then obtain

$$B_2^1(N) > 0 \text{ if and only if } A_{16}^1(8N+1) > 0.$$

Therefore, the proposition follows from Proposition 5.3 and the facts that

$$8N + 1 \equiv 0 \pmod 9 \text{ if and only if } N \equiv 1 \pmod 9$$

and

$$8N + 1 \equiv 6 \pmod 9 \text{ if and only if } N \equiv 4 \pmod 9.$$

□

5.2.2. Proof of Theorem 5.2 for $b \neq 3$

Proof. From Proposition 5.4, it suffices to show that if $N \equiv 1$ or $4 \pmod 9$, $N \in \mathbb{N}_0$ can be expressed as $t_x + bt_y + 2(z^2 + zw + w^2)$ for $(x, y, z, w) \in \mathbb{N}_0^2 \times \mathbb{Z}^2$. Assume that $N \equiv 1$ or $4 \pmod 9$.

For $b = 1, 2,$ or 4, taking $y = 1$ yields $N - b \cdot t_1 \not\equiv 1, 4 \pmod 9$, which implies $B_2^1(N - b \cdot t_1) > 0$. □

5.2.3. Proof of Theorem 5.2 for $b = 3$

Proof. Multiplying both sides of (2.6) by $\psi(q^2)\psi(q^6)$ results in

$$\psi(q^2)\psi(q^6)a(q) = \psi(q^2)\psi(q^6)a(q^4) + 6q\psi(q^2)^2\psi(q^6)^2,$$

which implies

$$\sum_{n=0}^{\infty} B_1^{2,6}(n)q^n = \sum_{N=0}^{\infty} B_2^{1,3}(N)q^{2N} + 6\sum_{N=0}^{\infty} t_{1,1,3,3}(N)q^{2N+1}. \tag{5.6}$$

Theorem 5.1 implies $B_1^{2,6}(n) > 0$, which allows us to conclude that $B_2^{1,3}(N) > 0$. □

5.3. Proof of Theorem 5.3

5.3.1. Preliminary Results

Consider the following result by Dickson [5, p. 113]:

Lemma 5.3. *A nonnegative integer $n \in \mathbb{N}_0$ can be written as $x^2+24y^2+72z^2$ for $x, y, z \in \mathbb{Z}$ if and only if $n \neq 3l+2, 4l+2, 4l+3, 9l+3, 4^k(8l+5)$, where $k, l \in \mathbb{N}_0$.*

Proposition 5.5. *A nonnegative integer $n \in \mathbb{N}_0$ can be written as $x^2+24(y^2+yz+z^2)$ for $x, y, z \in \mathbb{Z}$ if and only if $n \neq 3l+2, 4l+2, 4l+3, 9l+3, 4^k(8l+5)$, where $k, l \in \mathbb{N}_0$.*

Proof. Replacing q by q^{24} in (2.6) and (2.4) implies

$$\begin{aligned} a(q^{24}) =& a(q^{96}) + 6q^{24}\psi(q^{48})\psi(q^{144}), \\ \varphi(q^{24})\varphi(q^{72}) =& a(q^{96}) + 2q^{24}\psi(q^{48})\psi(q^{144}). \end{aligned}$$

Multiplying both sides of these equations by $\varphi(q)$ results in

$$\varphi(q)a(q^{24}) = \varphi(q)a(q^{96}) + 6q^{24}\varphi(q)\psi(q^{48})\psi(q^{144}), \tag{5.7}$$

$$\varphi(q)\varphi(q^{24})\varphi(q^{72}) = \varphi(q)a(q^{96}) + 2q^{24}\varphi(q)\psi(q^{48})\psi(q^{144}), \tag{5.8}$$

which implies

$$\begin{aligned} \sum_{n=0}^{\infty} A_{24}^{1}(n)q^n =& \sum_{n=0}^{\infty} A_{96}^{1}(n)q^n + 6q^{24}\sum_{N=0}^{\infty} m_{1\text{-}48,144}(N)q^N, \\ \sum_{n=0}^{\infty} r_{1,24,72}(n)q^n =& \sum_{n=0}^{\infty} A_{96}^{1}(n)q^n + 2q^{24}\sum_{N=0}^{\infty} m_{1\text{-}48,144}(N)q^N. \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} r_{1,24,72}(n) > 0 \,\text{if and only if}\, & A_{96}^{1}(n) > 0 \,\text{or}\, m_{1\text{-}48,144}(n-24) > 0, \\ & \text{if and only if}\, A_{24}^{1}(n) > 0, \end{aligned}$$

which proves the proposition. □

Proposition 5.5 gives rise to the following proposition:

Proposition 5.6. *A nonnegative integer $n \in \mathbb{N}_0$ can be written as $t_x+3(y^2+yz+z^2)$ for $(x, y, z) \in \mathbb{N}_0 \times \mathbb{Z}^2$ if and only if $n \not\equiv 2, 5, 7, 8 \pmod 9$.*

Proof. Multiplying both sides of (2.1) by $a(q^{24})$ results in

$$\varphi(q)a(q^{24}) = \varphi(q^4)a(q^{24}) + 2q\psi(q^8)a(q^{24}),$$

which implies

$$\sum_{n=0}^{\infty} A_{24}^{1}(n)q^{n} = \sum_{N=0}^{\infty} A_{6}^{1}(N)q^{4N} + 2\sum_{N=0}^{\infty} B_{3}^{1}(N)q^{8N+1}.$$

We then obtain

$$B_3^1(N) > 0 \text{ if and only if } A_{24}^1(8N+1) > 0.$$

Therefore, the proposition follows from Proposition 5.5 and the facts that

$$8N + 1 \equiv 2 \pmod 3 \text{ if and only if } N \equiv 2 \pmod 3,$$

and

$$8N + 1 \equiv 3 \pmod 9 \text{ if and only if } N \equiv 7 \pmod 9.$$

□

5.3.2. Proof of Theorem 5.3

Proof. From Proposition 5.6, it suffices to show that $N \in \mathbb{N}_0$ can be expressed as $t_x + 2t_y + 3(z^2 + zw + w^2)$ for $(x, y, z, w) \in \mathbb{N}_0^2 \times \mathbb{Z}^2$ if $N \equiv 2, 5, 7 \text{ or } 8 \pmod 9$.

When $N \equiv 2, 5 \text{ or } 8 \pmod 9$, taking $y = 1$ yields

$$N - 2 \cdot t_1 \equiv 0, 3, \text{ or } 6 \pmod 9,$$

which implies $B_3^1(N - 2 \cdot t_1) > 0$.

When $N \equiv 7 \pmod 9$, taking $y = 2$ yields

$$N - 2 \cdot t_2 \equiv 1 \pmod 9,$$

which implies $B_3^1(N - 2 \cdot t_2) > 0$. □

5.4. Proof of Theorem 5.4

5.4.1. Proof of Theorem 5.4 for $b = 1$

Proof. Multiplying both sides of (2.3) by $a(q^4)$ results in

$$\begin{aligned}\psi(q)^2 a(q^4) =&\varphi(q)\psi(q^2)a(q^4)\\ =&(\varphi(q^4) + 2q\psi(q^8))\psi(q^2)a(q^4), \text{ by (2.1)},\\ =&\varphi(q^4)\psi(q^2)a(q^4) + 2q\psi(q^2)\psi(q^8)a(q^4),\end{aligned}$$

which implies

$$\sum_{n=0}^{\infty} B_4^{1,1}(n)q^n = \sum_{N=0}^{\infty} M_2^{2\text{-}1}(N)q^{2N} + 2\sum_{N=0}^{\infty} B_2^{1,4}(N)q^{2N+1}, \tag{5.9}$$

where

$$M_2^{2\text{-}1}(N) = \sharp\left\{(x,y,z,w) \in \mathbb{Z}\times\mathbb{N}_0\times\mathbb{Z}^2 \,|\, N = 2x^2 + t_y + 2(z^2+zw+w^2)\right\}.$$

If n is odd and $n = 2N+1$, by Theorem 5.2,

$$B_4^{1,1}(2N+1) = 2B_2^{1,4}(N) > 0.$$

Consider the case when n is even. By (5.9),

$$B_4^{1,1}(2N) = M_2^{2\text{-}1}(N).$$

If $N \not\equiv 1, 4 (\bmod 9)$, using Proposition 5.4, we obtain $M_2^{2\text{-}1}(N) > 0$. If $N \equiv 1 \,\text{or}\, 4 (\bmod 9)$, taking $x = 1$ yields

$$N - 2x^2 \equiv 8 \,\text{or}\, 2(\bmod 9), \;\text{ giving }\; B_2^1(N - 2\cdot 1^2) > 0, \;\text{ which implies }\; M_2^{2\text{-}1}(N) > 0.$$

□

5.4.2. Proof of Theorem 5.4 for $b = 2$

Proof. Replacing q by q^4 in (2.5) yields

$$a(q^4) = \varphi(q^4)\varphi(q^{12}) + 4q^4\psi(q^8)\psi(q^{24}).$$

Multiplying both sides of this equation by $\psi(q)\psi(q^2)$ results in

$$\psi(q)\psi(q^2)a(q^4) = \varphi(q^4)\varphi(q^{12})\psi(q)\psi(q^2) + 4q^4\psi(q)\psi(q^2)\psi(q^8)\psi(q^{24}),$$

which implies

$$\sum_{n=0}^{\infty} B_4^{1,2}(n)q^n = \sum_{n=0}^{\infty} m_{4,12\text{-}1,2}(n)q^n + 4q^4\sum_{N=0}^{\infty} t_{1,2,8,24}(N)q^N,$$

where

$$m_{4,12\text{-}1,2}(n) = \sharp\left\{(x,y,z,w) \in \mathbb{Z}^2\times\mathbb{N}_0^2 \,|\, n = 4x^2 + 12y^2 + t_z + 2t_w\right\}.$$

From Guo, Pan, and Sun [6], recall that every $n \in \mathbb{N}_0$ can be expressed as $4x^2 + 2t_y + t_z$ for $(x,y,z) \in \mathbb{Z}\times\mathbb{N}_0^2$, which implies that

$$m_{4,12\text{-}1,2}(n) > 0, \;\text{ which gives }\; B_4^{1,2}(n) > 0.$$

□

5.5. Proof of Theorem 5.5

5.5.1. Preliminary Results

Consider the following result by Dickson [5, p. 113]:

Lemma 5.4. *A nonnegative integer $n \in \mathbb{N}_0$ can be written as $x^2+40y^2+120z^2$ for $x, y, z \in \mathbb{Z}$ if and only if $n \neq 4l+2, 4l+3, 9^k(9l+6), 25^k(5l \pm 2), 4^k(8l+5)$, where $k, l \in \mathbb{N}_0$.*

Using Lemma 5.4, we obtain the following proposition:

Proposition 5.7. *A nonnegative integer $n \in \mathbb{N}_0$ can be written as $x^2+40(y^2+yz+z^2)$ for $x, y, z \in \mathbb{Z}$ if and only if $n \neq 4l+2, 4l+3, 9^k(9l+6), 25^k(5l \pm 2), 4^k(8l+5)$, where $k, l \in \mathbb{N}_0$.*

Proof. Replacing q by q^{40} in (2.6) and (2.4) results in

$$\begin{aligned} a(q^{40}) =& a(q^{160}) + 6q^{40}\psi(q^{80})\psi(q^{240}), \\ \varphi(q^{40})\varphi(q^{120}) =& a(q^{160}) + 2q^{40}\psi(q^{80})\psi(q^{240}). \end{aligned}$$

Multiplying both sides of these equations by $\varphi(q)$ yields

$$\varphi(q)a(q^{40}) =\varphi(q)a(q^{160}) + 6q^{40}\varphi(q)\psi(q^{80})\psi(q^{240}), \tag{5.10}$$
$$\varphi(q)\varphi(q^{40})\varphi(q^{120}) =\varphi(q)a(q^{160}) + 2q^{40}\varphi(q)\psi(q^{80})\psi(q^{240}), \tag{5.11}$$

which implies

$$\begin{aligned} \sum_{n=0}^{\infty} A_{40}^{1}(n)q^n =& \sum_{n=0}^{\infty} A_{160}^{1}(n)q^n + 6q^{40}\sum_{N=0}^{\infty} m_{1\text{-}80,240}(N)q^N, \\ \sum_{n=0}^{\infty} r_{1,40,120}(n)q^n =& \sum_{n=0}^{\infty} A_{160}^{1}(n)q^n + 2q^{40}\sum_{N=0}^{\infty} m_{1\text{-}80,240}(N)q^N. \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} r_{1,40,120}(n) > 0 \,\text{if and only if}\, & A_{160}^{1}(n) > 0 \,\text{or}\, m_{1\text{-}80,240}(n-40) > 0, \\ & \text{if and only if}\, A_{40}^{1}(n) > 0, \end{aligned}$$

which proves the proposition. □

Proposition 5.5 gives rise to the following proposition:

Proposition 5.8. *A nonnegative integer $N \in \mathbb{N}_0$ can be written as $t_x + 5(y^2 + yz + z^2)$ for $(x, y, z) \in \mathbb{N}_0 \times \mathbb{Z}^2$ if and only if $8N + 1 \neq 9^k(9l + 6), 25^k(5l \pm 2)$, where $k, l \in \mathbb{N}_0$; in particular, $N \in \mathbb{N}_0$ can be written as $t_x + 5(y^2 + yz + z^2)$ for $(x, y, z) \in \mathbb{N}_0 \times \mathbb{Z}^2$ if $N \not\equiv 1, 4 (\operatorname{mod} 9)$, $N \not\equiv 2, 4 (\operatorname{mod} 5)$, and $N \not\equiv 3 (\operatorname{mod} 25)$.*

Proof. Multiplying both sides of (2.1) by $a(q^{40})$ yields

$$\varphi(q)a(q^{40}) = \varphi(q^4)a(q^{40}) + 2q\psi(q^8)a(q^{40}),$$

which implies

$$\sum_{n=0}^{\infty} A_{40}^1(n)q^n = \sum_{N=0}^{\infty} A_{10}^1(N)q^{4N} + 2\sum_{N=0}^{\infty} B_5^1(N)q^{8N+1}.$$

We then obtain

$$B_5^1(N) > 0 \text{ if and only if } A_{40}^1(8N+1) > 0.$$

The second statement follows from the facts that

$$8N+1 \equiv 0 \text{ or } 6 \pmod 9) \text{ if and only if } N \equiv 1 \text{or } 4 \pmod 9,$$

$$8N+1 \equiv \pm 2 \pmod 5 \text{ if and only if } N \equiv 2 \text{ or } 4 \pmod 5,$$

and

$$8N+1 \equiv 0 \pmod{25} \text{ if and only if } N \equiv 3 \pmod{25}.$$

□

5.5.2. Proof of Theorem 5.5

Proof. By Proposition 5.8, it suffices to show that $n \in \mathbb{N}_0$ can be expressed as $t_x + t_y + 5(z^2 + zw + w^2)$ for $(x, y, z, w) \in \mathbb{N}_0^2 \times \mathbb{Z}^2$ if n satisfies one of the following conditions:

(i) $n \equiv 1 \text{ or } 4 \pmod 9$,

(ii) $n \equiv 2 \text{ or } 4 \pmod 5$,

(iii) $n \equiv 3 \pmod{25}$.

First, suppose that $n \equiv 1 \pmod 9$ and $n = 9N + 1$ for $N \in \mathbb{N}_0$. If $N \not\equiv 1, 3 \pmod 5$ and $N \not\equiv 17 \pmod{25}$, taking $x = 1$ yields

$$n - t_1 = 9N \equiv 0 \pmod 9, \ \not\equiv 2, 4 \pmod 5, \ \not\equiv 3 \pmod{25},$$

which implies $n - t_1$ can be written as $t_y + 5(z^2 + zw + w^2)$ for $(y, z, w) \in \mathbb{N}_0 \times \mathbb{Z}^2$.

If $N \equiv 1 \pmod 5$, taking $x = 4$ yields

$$n - t_4 = 9N - 9 \equiv 0 \pmod 9, \equiv 0 \pmod 5,$$

which implies $n - t_4$ can be written as $t_y + 5(z^2 + zw + w^2)$ for $(y, z, w) \in \mathbb{N}_0 \times \mathbb{Z}^2$.

If $N \equiv 3 \pmod 5$, taking $x = 2$ results in

$$n - t_2 = 9N - 2 \equiv 7 \pmod 9, \ \equiv 0 \pmod 5,$$

which implies $n - t_2$ can be written as $t_y + 5(z^2 + zw + w^2)$ for $(y, z, w) \in \mathbb{N}_0 \times \mathbb{Z}^2$.

If $N \equiv 17 \pmod{25}$, taking $x = 2$ results in

$$n - t_2 = 9N - 2 \equiv 7 \pmod 9,\ \equiv 1 \pmod 5,$$

which implies $n - t_2$ can be written as $t_y + 5(z^2 + zw + w^2)$ for $(y, z, w) \in \mathbb{N}_0 \times \mathbb{Z}^2$.

Now, suppose $n \equiv 4 \pmod 9$ and $n = 9N + 4$ for $N \in \mathbb{N}_0$. In the same way, we prove that n can be written as $t_x + t_y + 5(z^2 + zw + w^2)$ for $(x, y, z, w) \in \mathbb{N}_0^2 \times \mathbb{Z}^2$. If $N \not\equiv 1, 4 \pmod 5$ and $N \not\equiv 0 \pmod{25}$, we take $x = 1$. If $N \equiv 1 \pmod 5$, we take $x = 7$. If $N \equiv 4 \pmod 5$, we take $x = 4$. Finally, if $N \equiv 0 \pmod{25}$, we take $x = 3$.

Suppose $n \equiv 2 \pmod 5$ and $n = 5N + 2$ for $N \in \mathbb{N}_0$. If $N \not\equiv 0, 6 \pmod 9$, we take $x = 1$. If $N \equiv 0 \text{ or } 6 \pmod 9$, we take $x = 3$.

Next, assume $n \equiv 4 \pmod 5$ and $n = 5N + 4$ for $N \in \mathbb{N}_0$. If $N \not\equiv 0, 6 \pmod 9$, we take $x = 3$. If $N \equiv 0 \text{ or } 6 \pmod 9$, we take $x = 7$.

Finally, suppose $n \equiv 3 \pmod{25}$ and $n = 25N + 3$ for $N \in \mathbb{N}_0$. If $N \not\equiv 4, 7 \pmod 9$, we take $x = 2$. If $N \equiv 4 \text{ or } 7 \pmod 9$, we take $x = 4$. $\square$

6. Proof of Theorem 1.4

6.1. Proof of Necessary Conditions

Proof. For fixed positive integers a, b, and c with $a \le b$, suppose every $n \in \mathbb{N}$ can be written as $at_x + bt_y + c(z^2 + zw + w^2)$ for $(x, y, z, w) \in \mathbb{N}_0^2 \times \mathbb{Z}^2$.

First, assume that $c = 1$. Taking $n = 2$ yields $a = 1$ or 2. If $a = 1$, by Theorem 1.3, we see that b is arbitrary. If $a = 2$, taking $n = 8$ yields $2 \le b \le 8$.

Suppose $c = 2$. Taking $n = 1$ results in $a = 1$. The choice of $n = 4$ implies $1 \le b \le 4$.

Next, assume that $c = 3$. Taking $n = 1$ results in $a = 1$. Choosing $n = 2$ implies $b = 1$ or 2. Taking $n = 8$ implies $b = 2$.

Suppose $c = 4$. Taking $n = 1$, we have $a = 1$. Choosing $n = 2$ implies $b = 1$ or 2.

Assume $c = 5$. Taking $n = 1$ yields $a = 1$. Choosing $n = 2$ implies $b = 1$ or 2, and taking $n = 4$ implies $b = 1$.

Finally, suppose $c \ge 6$. Taking $n = 1$ results in $a = 1$. Choosing $n = 2$ implies $b = 1$ or 2; taking $n = 4$ implies $b = 1$. On the other hand, $n = 5$ cannot be expressed as $t_x + t_y + c(z^2 + zw + w^2)$ for $(x, y, z, w) \in \mathbb{N}_0^2 \times \mathbb{Z}^2$, which is a contradiction. $\square$

6.2. Proof of Sufficient Conditions

Proof. Sufficiency follows from Theorems 5.1, 5.2, 5.3, 5.4, and 5.5. Note that Theorem 1.4 (2) follows from the proof of the necessary conditions. $\square$

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