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BEATTY SEQUENCES AND TRIGONOMETRIC FUNCTIONS

Clark Kimberling
Department of Mathematics, University of Evansville, Evansville, Indiana
ck6@evansville.edu

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Abstract

Beatty sequences are used to solve certain inequalities involving the tangent and sine functions. Fraenkel's theorem on nonhomogeneous Beatty sets is then used to determine integers for which certain products of sines are positive (or negative). Pairs of complementary nonhomogeous Beatty sets are recast as two pairs of complementary nonhomogeneous Beatty sequences. The final section poses a general question regarding the underlying connections between a broad class of functions and Beatty sequences.

1. Introduction

Beatty sequences occur in pairs in accord with Beatty's theorem that if $r > 1$ is an irrational number, then the pair $(\lfloor nr \rfloor)_{n\in\mathbb{N}}$ and $(\lfloor nr/(r-1) \rfloor)_{n\in\mathbb{N}}$ partition the set $\mathbb{N}$ of positive integers. As a first glimpse of a relationship between such a pair and a trigonometric function, let $t(k) = k\ \tan(1/k)$. Since $t(k) > 1$ on $[1, \infty)$ and $t(k) \to 1$ as $k \to \infty$, it is natural to consider the function

$$k(n) = \text{ least } k \in \mathbb{N} \text{ such that } t(k) < 1 + 1/n^2.$$

Let a be the sequence of numbers n for which $k(n+1) = k(n)+1$, and let b be the sequence for which $k(n+1) = k(n)$. The first few terms of a and b are shown here:

$$a = (1, 3, 5, 6, 8, 10, 12, 13, 15, 17, 19, 20, 22, 24, 25, 27, 29, 31, \ldots),$$
$$b = (2, 4, 7, 9, 11, 14, 16, 18, 21, 23, 26, 28, 30, 33, 35, 37, 40, 42, \ldots),$$

We shall show in Section 2 that a and b are the Beatty sequences for $r = \sqrt{3}$ and $r/(r-1) = (3+\sqrt{3})/2$, indexed in the Online Encyclopedia of Integer Sequences [2] as A022838 and A054406, respectively. In Section 2 a similar inequality involving the sine function is solved using Beatty sequences for $\sqrt{6}$ and $(6 + \sqrt{6})/5$.

In Section 3, nonhomogeneous Beatty sequences ($\lfloor nr + h \rfloor$), in which the domain is the set $\mathbb{Z}$ of all integers rather than $\mathbb{N}$, are proved to have connections with certain products involving the sine function. For example, if

$$f(x) = \sin(\frac{x\pi}{\sqrt{2}} + \frac{\pi}{4})\sin(\frac{(x+1)\pi}{\sqrt{2}} + \frac{\pi}{4}),$$

then the integers n for which $f(n) < 0$, and those for which $f(n) > 0$, are, respectively, these sets:

$$\{\left\lfloor (n - \frac{1}{4})\sqrt{2}) \right\rfloor\}_{n\in\mathbb{Z}} = \{\ldots, -8, -7, -5, -4, -2, -1, 1, 2, 3, 5, 6, \ldots\}$$
$$\{\left\lfloor (n + \frac{1}{4})(2 + \sqrt{2}) \right\rfloor\}_{n\in\mathbb{Z}} = \{\ldots, -13, -10, -6, -3, 0, 4, 7, 11, 14, \ldots\}.$$

2. The Tangent Inequality and $(\left\lfloor \sqrt{3}n \right\rfloor)$

We begin with a lemma, an easy proof of which is omitted. Thoughout Sections 3 and 4, the letters n, k, and h represent numbers in $\mathbb{N}$.

Lemma 1. *Suppose that t in $(0, 1)$ is irrational, and let $s(n) = \lceil nt \rceil$ or $s(n) = \lfloor nt \rfloor$. Let a be the sequence of numbers n such that $s(n+1) = s(n)$, and b the sequence of those n such that $s(n+1) = s(n) + 1$. Then a is the Beatty sequence of $1/(1-t)$, and b is the Beatty sequence of $1/t$).*

Theorem 1. *Let $k(n)$ be the least k such that $k\tan(1/k) < 1 + 1/n^2$. Then $k(n) = \lceil n/\sqrt{3} \rceil = \lfloor n/\sqrt{3} \rfloor + 1$.*

Proof. Throughout, let $k = \lceil n/\sqrt{3} \rceil$ and $f(x) = \tan x$. First, if $n = 1$, then $k = 1$, as asserted. For $n \geq 2$, the proof is in two parts: (i) $k\tan(1/k) - 1 < 1/n^2$, and (ii) $(k-1)\tan(1/(k-1)) - 1 > 1/n^2$. Part (i) depends on five lemmas, stated here and proved later: □

Lemma 2. *If $n \geq 2$, then*

$$\frac{1}{k^2} < \frac{3}{n^2} - \frac{5}{n^4}.$$

Lemma 3. *If $n \geq 1$, then*

$$\frac{1}{5k^4} < \frac{2}{n^4}.$$

Lemma 4. *If* $0 < u \le 1$, *then*

$$u > \tan \frac{u}{1+u^2}.$$

Lemma 5. *If* $x \ge 8$, *then*

$$xf^{(6)}(1/x) < 300.$$

Lemma 6. *If* $n \ge 6$, *then*

$$\frac{5}{4k^6} < \frac{1}{n^4}.$$

Assuming the lemmas, for $h \ge 1$ and x in $(0, 1/h]$, we have

$$\tan x = x + x^3/3 + 2x^5/15 + R_5(x), \tag{1}$$

where

$$R_5(x) \le \frac{M}{6!}x^6,$$

$M = \sup\{f^{(6)}(x),\ 0 < x \le 1/h\}$. Let $u = \sec x$ and $v = \tan x$. Then

$$f^{(6)}(x) = 272u^6v + 416u^4v^3 + 32u^2v^5$$

is strictly increasing on $(0, 1/h]$, so that $M = f^{(6)}(1/h)$. In (1), put $x = 1/h$ and multiply by h to get

$$h \tan \frac{1}{h} = 1 + \frac{1}{3h^2} + \frac{2}{15h^4} + \frac{f^{(6)}(1/h)}{6!h^5}. \tag{2}$$

Now putting $h = k = \lceil n/\sqrt{3} \rceil$, by (2) it suffices to prove that

$$\frac{1}{3k^2} + \frac{2}{15k^4} + \frac{f^{(6)}(1/k)}{6!k^5} < \frac{1}{n^2}, \tag{3}$$

or equivalently, that

$$\frac{1}{k^2} + \frac{2}{5k^4} + \frac{f^{(6)}(1/k)}{240k^5} < \frac{3}{n^2}. \tag{4}$$

By Lemmas 2-6, for $k \ge 8$ (and equivalently, for $n \ge 13$), the left side of (4) is bounded above by

$$\frac{3}{n^2} - \frac{5}{n^4} + \frac{4}{n^4} + \frac{1}{n^4} = \frac{3}{n^2},$$

so that by (3), $k \tan(1/k) - 1 < 1/n^2$ for $n \ge 13$, and it is easy to check that this inequality also holds for $n < 13$.

We turn next to the proof of part (ii) of the theorem. Let $\{\}$ denote fractional part. Trivially,

$$\{\frac{n}{\sqrt{3}}\}^2 \le \frac{n}{\sqrt{3}}\{\frac{n}{\sqrt{3}}\}$$

for $n \geq 1$. Consequently,

$$3(\frac{n}{\sqrt{3}} - \{\frac{n}{\sqrt{3}}\})^2 < n^2,$$

so that

$$3(k-1)^2 = 3\left\lfloor n/\sqrt{3}\right\rfloor^2 < n^2,$$

and

$$1 + \frac{1}{n^2} < 1 + \frac{1}{3(k-1)^2}.$$

Since

$$(k-1)\tan\frac{1}{k-1} = 1 + \frac{1}{3(k-1)^2} + \frac{2}{15(k-1)^4} + \cdots,$$

we have $(k-1)\tan(1/(k-1)) - 1 > 1/n^2$, so that all that remains is to prove the lemmas.

To prove Lemma 2, we start with two easily verified facts: $3k^2 - n^2 > 0$, and $3k^2 - n^2$ is not congruent to 1 mod 3. Thus, $3k^2 - n^2 \geq 2$, so that

$$k^2 - \frac{n^2}{3} \geq \frac{2}{3}. \tag{5}$$

Now suppose that $n \geq 4$. Then $n^2 > 10$, whence $6(3n^2 - 5) > 15n^2$, so that by (5),

$$k^2 - \frac{n^2}{3} > \frac{5n^2}{3(3n^2-5)}.$$

This implies $(3n^2 - 5)k^2 > n^4 + 5n^2 - 5n^2/3 > n^4$, so that $1/k^2 < 3/n^2 - 5/n^4$. It is easy to check that this also holds for $n = 2$ and $n = 3$.

Lemma 3 follows from

$$n^4 < 10n^4/9 = 10(n/\sqrt{3})^4 < 10k^4.$$

To prove Lemma 4, for $0 \leq u \leq 1$, let

$$g(u) = \arctan u - \frac{u}{1+u^2}.$$

Then $g(0) = 0$, and

$$g'(u) = \frac{2u^2}{(1+u^2)^2} > 0$$

on $(0, 1)$. Thus, $g(u) > 0$ on $(0, 1)$, so that

$$u > \tan\frac{u}{1+u^2}$$

on $(0, 1)$, and this inequality holds also for $u = 1$.

For the proof of Lemma 5, let $g(x) = xf^{(6)}(1/x)$. Let $s = \sec(1/x)$ and $t = \tan(1/x)$. Then

$$g'(x) = -\frac{16s^2}{x}(17s^6 - 17xs^4t + 180s^4t^2 - 26xs^2t^3 + 114s^2t^4 - 2xt^5 + 4t^6) \quad (6)$$

Putting $u = \tan(1/x)$ in Lemma 4 gives

$$\tan\frac{1}{x} > \tan\frac{u}{1+u^2},$$

so that $s^2 - xt > 0$. Consequently,

$$\begin{aligned} 17s^6 - 17xs^4t &> 0, \\ 180s^4t^2 - 26xs^2t^3 &> 0, \\ 114s^2t^4 - 2xt^5 + 4t^6 &> 0, \end{aligned}$$

so that by (6), $g'(x) < 0$ for all x satisfying $0 < \tan(1/x) < 1$, hence for $x \geq 8$. Therefore, g is strictly decreasing on $[8, \infty)$, so that $g(x) < 8f^{(6)}(1/8) < 300$ for $x \geq 8$.

A proof of Lemma 6 follows:

$$\frac{5}{4k^6} < \frac{5}{4(n/\sqrt{3})^6} < \frac{1}{n^4}$$

for $n \geq 6$.

Corollary 1. *Let $t(k) = k\tan(1/k)$, and let $k(n)$ be the least k for which $t(k) < 1 + 1/n^2$. Let a be the sequence of numbers n such that $k(n+1) = k(n)$, and let b be the sequence such that $k(n+1) = k(n) + 1$. Then a and b are the Beatty sequences given by $a(n) = \lfloor n(3+\sqrt{3})/2 \rfloor$ and $b(n) = \lfloor n\sqrt{3} \rfloor$.*

Proof. Apply Lemma 1 to the result in Theorem 1. □

3. A Sine Inequality and $(\lfloor \sqrt{6}n \rfloor)$

Theorem 2. *Let $k(n)$ be the least k such that $1 - k\sin(1/k) < 1/n^2$. Then $k(n) = \lceil n/\sqrt{6} \rceil$.*

Proof. Throughout, let $k = k(n) = \lceil n/\sqrt{6} \rceil$. Let $h \geq 1$, and put $x = 1/h$ in the Maclaurin series for $\sin x$ to find that

$$\begin{aligned} 1 - h\sin\frac{1}{h} &= \frac{1}{6h^2} - \frac{1}{h^4}\left(\frac{1}{5!} - \frac{1}{7!h^2}\right) - \frac{1}{h^8}\left(\frac{1}{9!} - \frac{1}{11!h^2}\right) - \cdots \\ &< \frac{1}{6h^2}. \end{aligned}$$

The least h such that $1/(6h^2) < 1/n^2$ is clearly k, so that $1 - k\sin(1/k) < 1/n^2$. Now, to show that k is the *least* number k' such that $1 - k'\sin(1/k') < 1/n^2$, we shall show that

$$\frac{1}{n^2} < 1 - (k-1)\sin\frac{1}{k-1}$$

for $k \geq 2$. (For $k = 1$, note that $n = 1$ or $n = 2$, and in both cases, $k = 1$ is the least number satisfying $1 - k\sin(1/k) < 1/n^2$.) Suppose that $k \geq 2$, and let $k_1 = \lfloor n/\sqrt{6} \rfloor = k - 1$. The Maclaurin series for sine gives

$$1 - k_1 \sin\frac{1}{k_1} > \frac{1}{6k_1^2} - \frac{1}{120k_1^4}.$$

Also, $n^2 \geq 6k_1^2 + 1$, so that

$$n^2\left(1 - k_1\sin\frac{1}{k_1}\right) > (6k_1^2 + 1)\left(\frac{1}{6k_1^2} - \frac{1}{120k_1^4}\right) > 1,$$

as desired. $\square$

Corollary 2. *Let $t(k) = k\sin(1/k)$, and let $k(n)$ be the least k for which $1 - t(k) < 1/n^2$. Let a be the sequence of numbers n such that $k(n+1) = k(n)$, and let b be the sequence such that $k(n+1) = k(n) + 1$. Then a and b are the Beatty sequences A022840 and A138235 given by $a(n) = \lfloor n(6+\sqrt{6})/5 \rfloor$ and $b(n) = \lfloor n\sqrt{6} \rfloor$.*

Proof. Apply Lemma 1 to the result in Theorem 2.

$\square$

4. Fraenkel's Theorem and Sine Products

In the preceding sections, the notation $(\lfloor n\alpha \rfloor)$ is used for the Beatty sequence of a number α, where $n \in \mathbb{N}$. A set of the form $\{\lfloor n\alpha + \gamma \rfloor\}$, where $\gamma \neq 0$ and n ranges through $\mathbb{Z}$, is called a nonhomogenous Beatty set. The definitive version of Beatty's theorem for complementary pairs of such sets (Fraenkel [1], Theorem XI, p. 10) can be stated as follows:

Theorem 3 (Fraenkel's theorem.). *Let α and β be positive irrational numbers. The Beatty sets $\{\lfloor n\alpha + \gamma \rfloor\}_{n\in\mathbb{Z}}$ and $\{\lfloor n\beta + \delta \rfloor\}_{n\in\mathbb{Z}}$ are complementary in $\mathbb{Z}$ if and only if the following three conditions hold:*

(i) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = 1$;

(ii) $\dfrac{\gamma}{\alpha} + \dfrac{\delta}{\beta} \in \mathbb{Z}$;

(iii) *if* $n \in \mathbb{Z}$, *then* $n\beta + \delta \notin \mathbb{Z}$.

Theorem 4. *Suppose that $r > 1$ is an irrational number and that t is a real number not in $\mathbb{Z}$. Let*

$$f(x) = \sin(\frac{x\pi}{r} - t\pi)\sin(\frac{(x+1)\pi}{r} - t\pi).$$

Then for every $m \in \mathbb{Z}$,

$$f(m) < 0 \text{ if and only if } m \in \{\lfloor (n+t)r \rfloor\}_{n\in\mathbb{Z}};$$
$$f(m) > 0 \text{ if and only if } m \in \{\left\lfloor (n-t)\frac{r}{r-1} \right\rfloor\}_{n\in\mathbb{Z}}$$

Moreover, either $1 = \lfloor (n+t)r \rfloor$ where $n = \lfloor 2/r - t \rfloor$ or else $1 = \lfloor (n-t)r/(r-2) \rfloor$ where $n = \lfloor 2 + t - 2/r \rfloor$.

Proof. Let $\alpha = r$, $\beta = r/(r-1)$, and $\gamma = tr$. With $\delta = -tr/(r-1)$, Fraenkel's theorem implies that the resulting nonhomogeneous Beatty sets are complementary. The zeros of f form the chain

$$\cdots < (-2+t)r < (-1+t)r - 1 < (-1+t)r < tr - 1 < tr < 2tr - 1 < 2tr < \cdots$$

Since each interval

$$((n+t)r - 1, (n+t)r)$$

has length 1, it contains exactly one integer, specifically $\lfloor (n+t)r \rfloor$. By Fraenkel's theorem, the set of integers complementary to $\{\lfloor (n+t)r \rfloor\}$ is $\{\lfloor (n-t)r/(r-1) \rfloor\}$, so that the latter integers are in the intervals $((n+t)r, (n+1+t)r)$.

The integer 1 must be in one of the two sets. Consider first the possilbility that, for some n,

$$(n+t)r - 1 < 1 < (n+t)r.$$

Then

$$nr < 2 - tr < nr + 1,$$

so that $n = \lfloor 2/r - t \rfloor$. On the other hand, if $1 = \lfloor (n-t)r/(r-1) \rfloor$, we find $n = \lfloor 2 + t - 2/r \rfloor$. □

Example 5. Taking $r = \pi$ and $t = 1/2$ gives

$$\begin{aligned} f(x) &= \sin(x - \pi/2)\sin((x+1) - \pi/2) \\ &= \cos(x)\cos(x+1). \end{aligned}$$

By Theorem 11, the integers k satisfying $f(k) < 0$ are given by the nonhomogeneous Beatty set

$$\{\lfloor (n+1/2)\pi \rfloor\}_{n\in\mathbb{Z}} = \{\ldots, -11, -8, -5, -2, 1, 4, 7, 10, 14, \ldots\}$$

and those satisfying $f(k) > 0$, by the complementary nonhomogeneous Beatty set

$$\{\lfloor (n-1/2)\pi/(\pi-1) \rfloor\}_{n\in\mathbb{Z}} = \{\ldots, -10, -9, -7, -6, -4, -3, -1, 0, 2, 3, 5, 6, 8, 9 \ldots\}.$$

If $t = 0$ in Theorem 4, then $\delta = 0$, so that (iii) in the hypothesis of Fraenkel's theorem does not hold. Nevertheless, a separate proof very similar to that of Theorem 4, using Beatty's theorem instead of Fraenkel's, gives the following theorem.

Theorem 6. *Suppose that $r > 1$ is an irrational number. Let*

$$f(x) = \sin\frac{x\pi}{r}\sin\frac{(x+1)\pi}{r}.$$

Then $f(m) < 0$ or $f(m) > 0$ according as $m \in \{\lfloor nr \rfloor\}$ or $m \in \{\lfloor nr/(r-1) \rfloor\}$, respectively. Moreover, either $1 = \lfloor nr \rfloor$ where $n = \lfloor 2/r \rfloor$ or else $1 = \lfloor nr/(r-1) \rfloor$, where $n = \lfloor 2 - 2/r \rfloor$.

5. Nonhomogeneous Beatty Sequences

A distinction between (homogeneous) Beatty *sequences* and (nonhomogeneous) Beatty *sets* has already been made. We turn now to *nonhomogeneous Beatty sequences*, $(nr + h)$, where $n \in \mathbb{N}$. The basic idea is that a pair of sets that partition all the integers contain two subsets that partition the positive integers, and also two other subsets that partition the negative integers – and that the latter can be modified to yield a partition of the positive integers. These four subsets can be expressed as sequences indexed by $\mathbb{N}$.

Lemma 7. *Suppose that $B_1 = \{\lfloor n\alpha + \gamma \rfloor\}_{n\in\mathbb{Z}}$ and $B_2 = \{\lfloor n\beta + \delta \rfloor\}_{n\in\mathbb{Z}}$ are a complementary pair of nonhomogeneous Beatty sets (as in Theorem 4, so that $\delta = -\gamma/(\alpha-1)$). Then $1 \in B_1$ if and only if*

$$\lfloor (2-\gamma)/\alpha \rfloor < 1 - \gamma/\alpha, \tag{7}$$

in which case $1 = \lfloor n_1\alpha + \gamma \rfloor$, where $n_1 = \lfloor (2-\gamma)/\alpha \rfloor$.

Proof. First, assume (7). Then the integer $n_1 = \lfloor (2-\gamma)/\alpha \rfloor$ satisfies

$$\frac{1-\gamma}{\alpha} < n_1 < \frac{2-\gamma}{\alpha},$$

so that $1 = \lfloor n_1\alpha + \gamma \rfloor \in B_1$. The converse clearly holds. □

In the next theorem, we assume without loss of generality that $1 \in B_1$, because otherwise $1 \in B_2$, by Fraenkel's theorem, so that the roles of B_1 and B_2 could be reversed in the statement and proof of the theorem. □

Theorem 7. *Suppose that $B_1 = \{\lfloor n\alpha + \gamma \rfloor\}_{n\in\mathbb{Z}}$ and $B_2 = \{\lfloor n\beta + \delta \rfloor\}_{n\in\mathbb{Z}}$ are a complementary pair of nonhomogeneous Beatty sets and that $1 \in B_1$. If $0 \in B_2$, let $1 = \lfloor n_1\alpha + \gamma \rfloor$ and $0 = \lfloor n_0\beta + \delta \rfloor$; then*

$$(\lceil n\alpha - n_1\alpha - \gamma \rceil)_{n\in\mathbb{N}} \quad \textit{and} \quad (\lceil n\beta - n_0\beta - \delta \rceil)_{n\in\mathbb{N}} \tag{8}$$

are a pair of complementary nonhomogeneous Beatty sequences. On the other hand, if $0 \in B_1$, let n_2 be the least integer satisfying $\lfloor n_2\beta + \delta \rfloor > 0$; then

$$(\lceil n\alpha - n_1\alpha + \alpha - \gamma \rceil)_{n\in\mathbb{N}} \quad \textit{and} \quad (\lceil n\beta - n_2\beta - \delta \rceil)_{n\in\mathbb{N}} \tag{9}$$

are a pair of complementary nonhomogeneous Beatty sequences.

Proof. In this first case, the sequences $(\lfloor n\alpha + \gamma \rfloor)_{n\geqslant n_1}$ and $(\lfloor n\beta + \delta \rfloor)_{n\geqslant n_0+1}$ partition $\mathbb{N}$. Equivalently, $(\lfloor n\alpha + (n_1-1)\alpha + \gamma \rfloor)_{n\geqslant 1}$ and $(\lfloor n\beta + n_0 + \delta \rfloor)_{n\geqslant 1}$ partition $\mathbb{N}$. Consequently, the sets

$$(\lfloor n\alpha + n_1\alpha - \alpha + \gamma \rfloor)_{n\leq 0} \quad \text{and} \quad (\lfloor n\beta + n_0 + \delta \rfloor)_{n\leq -1}$$

partition $-\mathbb{N}$. Multiplying all terms by -1, using the identity $-\lfloor x \rfloor = \lceil x \rceil$ for irrational x, and using $\lceil \;\; \rceil$ instead of $\lfloor \;\; \rfloor$, we conclude that the sequences in (8) are as claimed.

A proof for the second case is slightly different. The sequences $(\lfloor n\alpha + \gamma \rfloor)_{n\geqslant n_1}$ and $(\lfloor n\beta + \delta \rfloor)_{n\geqslant n_2}$ partition $\mathbb{N}$. Consequently,

$$(\lfloor (n + n_1 - 1)\alpha + \gamma \rfloor)_{n\leq -1} \quad \text{and} \quad (\lfloor (n + n_2 - 1)\beta + \delta \rfloor)_{n\leq 0}$$

partition $-\mathbb{N}$. Equivalently,

$$(\lceil -n\alpha - n_1\alpha + \alpha - \gamma \rceil)_{n\leq -1} \quad \text{and} \quad (\lceil -n\beta - n_2\beta + \beta - \delta \rceil)_{n\leq 0}$$

partition $\mathbb{N}$. Consequently,

$$(\lceil n\alpha - n_1\alpha + \alpha - \gamma \rceil)_{n\geqslant 1} \quad \text{and} \quad (\lceil n\beta - n_2\beta + \beta - \delta \rceil)_{n\geqslant 0}$$

partition $\mathbb{N}$, so that, after adjusting the index in the second sequence, the sequences in (9) are as claimed. ■

The condition given in Lemma 7 for $1 \in B_1$ can be supplemented by a condition for $0 \in B_2$, as in the first case in Theorem 7. We have $0 = \lfloor n\beta + \delta \rfloor$ for some integer n, and necessarily, $-\delta/\beta < n < (1-\delta)/\beta$, which is equivalent to $-\delta/\beta < \lfloor (1-\delta)/\beta \rfloor$. Replacing β by $\alpha/(\alpha-1)$ and δ by $-\gamma/(\alpha-1)$ yields

$$\frac{\gamma}{\alpha} < \left\lceil \frac{\gamma - 1}{\alpha} \right\rceil . \tag{10}$$

The steps are reversible, so that if (10) fails, then $0 \in B_1$, as in the second case in Theorem 7. □

Example 8. Taking $\alpha = \sqrt{2}$ and $\gamma = \sqrt{1/2}$ gives $\beta = 2+\sqrt{2}$ and $\delta = -1-\sqrt{1/2}$. The sequences in (8) are

$$\left(\left\lceil n\sqrt{2} - \sqrt{2} - \sqrt{1/2} \right\rceil\right)_{n\in\mathbb{N}} = (1, 2, 4, 5, 7, 8, 9, \ldots),$$
$$\left(\left\lceil n(2+\sqrt{2}) + 1 + \sqrt{1/2} \right\rceil\right)_{n\in\mathbb{N}} = (3, 6, 10, 13, 17, \ldots),$$

these being essentially A258833 and A258834. The sets B_1 and B_2 of Theorem 7 are represented at the end of Section 1; viz., changing the signs of the negative numbers in B_1 and B_2 gives the sequences in Example 16. The trigonometric connection is as follows. Let

$$g(x) = f(-x) = \sin\left(\frac{\pi}{4} - \frac{x\pi}{\sqrt{2}}\right) \sin\left(\frac{\pi}{4} - \frac{(x-1)\pi}{\sqrt{2}}\right).$$

Then the positive integers n such that $g(n) < 0$ are given by A258833, and those such that $g(n) > 0$, by A258834.

Example 9. Taking $\alpha = \sqrt{2}$ and $\gamma = 1/2$ gives $\beta = 2+\sqrt{2}$ and $\delta = -1/2-\sqrt{1/2}$. The inequality (10) fails, so that $0 \in B_1$, as in the second case in Theorem 7. We have

$$B_1 = \{\ldots, -7, -6, -4, -3, -1, 0, 1, 3, 4, 6, 7, 8, \ldots\}$$
$$B_2 = \{\ldots, -15, -9, -5, -2, 2, 5, 9, 12, 15, 19, \ldots\},$$

and the sequences in (9) are

$$\left(\left\lceil n\sqrt{2} - 1/2 \right\rceil\right)_{n\in\mathbb{N}} = (1, 3, 4, 6, 7, 8, 10, \ldots);$$
$$\left(\left\lceil n(2+\sqrt{2}) - 3/2 - \sqrt{2} + \sqrt{1/2} \right\rceil\right)_{n\in\mathbb{N}} = (2, 5, 9, 12, 15, \ldots),$$

these being A022846 and A063957.

Note that for fixed irrational $\alpha > 1$, the integers γ that satisfy (10) comprise the nonhomogeneous Beatty sequence $(\lceil n\alpha/(\alpha-1) \rceil)_{n\in\mathbb{N}}$.

6. Concluding Remarks

The author thanks the referee for suggesting a general setting for the results in Sections 2 and 3, as follows. Suppose that f is a function such that asymptotically

$$f(h) - ah^m = o(h^m) \text{ as } h \to 0 \tag{11}$$

for some $a > 0$ and $m \in \mathbb{N}$. Let $k(n)$ be the least $k \in \mathbb{N}$ such that

$$f(\frac{1}{x}) < \frac{1}{n^m}$$

for every $x \geq k$.

Problem 10. Under what conditions does $k(n)$ ultimately identify with the Beatty sequence $(\lfloor na^{1/m} \rfloor)_{n \in \mathbb{N}}$?

In Section 2,

$$\frac{\tan(h)}{h} - 1 - \frac{1}{3}h^2 = O(h^4),$$

and in Section 3,

$$1 - \frac{\sin(h)}{h} - \frac{1}{6}h^2 = O(h^4).$$

In both cases, the identification with corresponding Beatty sequences starts from $n = 1$. In both cases, the coefficient $a^{1/m}$, which is $1/\sqrt{3}$ or $1/\sqrt{6}$, is algebraic of degree 2, and the residue grows asymptotically as $h^4 = h^{2m}$. In these special cases, perhaps Theorems 2 and 9 can be proved using Liouville's Approximation Theorem.

Finally, (11) can serve as a recipe for finding a wide variety of examples in answer to the Question, using trigonometric, exponential, logarithmic, and other functions.

References

[1] A. S. Fraenkel, The bracket function and complementary sets of integers, *Canadian J. of Mathematics* 21 (1969) 6-27.

[2] Online Encyclopedia of Integer Sequences: https://oeis.org/.