RELATION BETWEEN TWO WEIGHTED ZERO-SUM CONSTANTS

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Abstract

Let G be a finite abelian group (written additively), with exponent $\exp(G) = m$ and let A be a non-empty subset of $\{1, 2, \ldots, m-1\}$. The constant $\eta_A(G)$ (respectively $\mathsf{s}_A(G)$) is defined to be the smallest positive integer t such that any sequence of length t of elements of G contains a non-empty A-weighted zero-sum subsequence of length at most m (respectively, of length equal to m). These generalize the constants $\eta(G)$ and $\mathsf{s}(G)$, which correspond to the case $A = \{1\}$. In 2007, Gao et al. conjectured that $s(G) = \eta(G) + \exp(G) - 1$ for any finite abelian group G; here we shall discuss a similar relation corresponding to the weight $A = \{1, -1\}$.

1. Introduction

Let G be a finite abelian group written additively. By a sequence over G, we mean a finite sequence of terms from G which is unordered and repetition of terms is allowed and we view sequences over G as elements of the free abelian monoid $\mathcal{F}(G)$ and use multiplicative notation. Our notation is consistent with [3], [4], and [6].

For $S \in \mathcal{F}(G)$, if

$$S = x_1 x_2 \cdot \ldots \cdot x_t = \prod_{g \in G} g^{\mathsf{v}_g(S)},$$

where $\mathsf{v}_g(S) \geq 0$ is the *multiplicity* of g in S,

$$|S| = t = \sum_{g \in G} \mathsf{v}_g(S)$$

is the *length* of S. The sequence S *contains* some $g \in G$, if $\mathsf{v}_g(S) \geq 1$.

If S and T are sequences over G, then T is said to be a *subsequence* of S if $\mathsf{v}_g(T) \leq \mathsf{v}_g(S)$ for every $g \in G$ and we write $T|S$; ST^{-1} denotes the sequence L where $\mathsf{v}_g(L) = \mathsf{v}_g(S) - \mathsf{v}_g(T)$ for every $g \in G$.

For a non-empty subset A of $\{1, 2, \ldots, m-1\}$, where m is the exponent of G (denoted by $\exp(G)$), a sequence $S = x_1 x_2 \cdot \ldots \cdot x_t$ over G is said to be an *A-weighted zero-sum sequence,* if there exists $\bar{a} = (a_1, a_2, \cdots, a_t) \in A^t$ such that $\sum_{i=1}^{t} a_i x_i = 0$.

The constant $\eta_A(G)$ (respectively, $\mathsf{s}_A(G)$) is defined to be the smallest positive integer t such that any sequence of length t of elements of G contains a non-empty A-weighted zero-sum subsequence of length at most m (respectively, of length equal to m). These generalize the constants $\eta(G)$ and $\mathsf{s}(G)$, which correspond to the case $A = \{1\}$.

The conjecture of Gao et al. [5] that $\mathsf{s}(G) = \eta(G) + \exp(G) - 1$ holds for an abelian group G, was established in the case of rank at most two by Geroldinger and Halter-Koch [4]. Regarding the weighted analogue

$$\mathsf{s}_A(G) = \eta_A(G) + \exp(G) - 1, \tag{1}$$

for a finite cyclic group $G = \mathbb{Z}_n$, it coincides with a result established by Grynkiewicz et al. [7].

Since by definition, there is a sequence S of length $\eta_A(G) - 1$ which does not have any non-empty A-weighted zero-sum subsequence of length not exceeding $\exp(G)$, by appending a sequence of 0's of length $\exp(G) - 1$, we observe that

$$\mathsf{s}_A(G) \geq \eta_A(G) + \exp(G) - 1. \tag{2}$$

In the case of the weight $A = \{1, -1\}$, we write $\eta_{\pm}(G)$ for $\eta_{\{1,-1\}}(G)$ and similarly for $\mathsf{s}_{\{1,-1\}}(G)$.

When G is an elementary 2-group, then $\mathsf{s}_{\pm}(G) = \mathsf{s}(G)$ and $\eta_{\pm}(G) = \eta(G)$.

One has the following trivial bounds for the problem of Harborth [8]

$$1 + 2^d(n-1) \leq \mathsf{s}(\mathbb{Z}_n^d) \leq 1 + n^d(n-1). \tag{3}$$

Since the number of elements of $\mathbb{Z}_n^d$ having coordinates 0 or 1 is 2^d, considering a sequence where each of these elements are repeated $(n-1)$ times, one obtains the lower bound, while the upper bound follows from the fact that in any sequence of $1 + n^d(n-1)$ elements of $\mathbb{Z}_n^d$, at least one element will appear at least n times.

When $n = 2$, from (3), we have

$$\mathsf{s}_{\pm}(\mathbb{Z}_2^d) = \mathsf{s}(\mathbb{Z}_2^d) = 1 + 2^d.$$

Since a sequence of length $2^d - 1$ of all distinct non-zero elements of $\mathbb{Z}_2^d$ does not have any zero-sum subsequence of length ≤ 2, $\eta_{\pm}(\mathbb{Z}_2^d) = \eta(\mathbb{Z}_2^d) \ge 2^d$.

Therefore, by (2),

$$1 + 2^d = \mathsf{s}_{\pm}(\mathbb{Z}_2^d) \ge \eta_{\pm}(\mathbb{Z}_2^d) + 1 \ge 2^d + 1,$$

and hence

$$\mathsf{s}_{\pm}(\mathbb{Z}_2^d) = 1 + 2^d = \eta_{\pm}(\mathbb{Z}_2^d) + 1.$$

Recently, it has been shown in [9] that in the case of the weight $A = \{1, -1\}$, the relation (1) does not hold for $G = \mathbb{Z}_n \oplus \mathbb{Z}_n$, for an odd integer $n > 7$. However, in the same paper, it was shown that the relation holds with $A = \{1, -1\}$, for any abelian group G of order 8 and 16.

Here, in Section 2, as a corollary to Theorem 3, we shall observe that in the case of the weight $A = \{1, -1\}$, the relation (1) holds for the groups $\mathbb{Z}_2 \oplus \mathbb{Z}_{2n}$, when n is a power of 2. In Section 3, in Theorem 4, we show that the relation (1) holds for any abelian group G of order 32. We shall also make some related observations.

Regarding the relation (1) with $A = \{1, -1\}$, we are tempted to make the following conjecture.

Conjecture 1. *The relation* $\mathsf{s}_{\pm}(G) = \eta_{\pm}(G) + \exp(G) - 1$ *holds for any finite abelian* 2*-group* G*.*

2. General Results and Lemmas

We shall need a recent result on a weighted analogue of the Harborth constant of a class of finite abelian groups. *The Harborth constant* $\mathsf{g}(G)$, of a finite abelian group G, is the smallest positive integer l such that any subset of G of cardinality l has a subset of cardinality equal to $\exp(G)$ whose elements sum to the identity element. For the plus-minus weighted analogue $\mathsf{g}_{\pm}(G)$, one requires a subset of cardinality $\exp(G)$, a $\{\pm 1\}$-weighted sum of whose elements is equal to the identity element.

We shall require the following theorem of Marchan et al. (Theorem 1.3, [10]).

Theorem 1. *Let* $n \in \mathbb{N}$*. For* $n \ge 3$ *we have* $\mathsf{g}_{\pm}(\mathbb{Z}_2 \oplus \mathbb{Z}_{2n}) = 2n + 2$*. Moreover,* $\mathsf{g}_{\pm}(\mathbb{Z}_2 \oplus \mathbb{Z}_2) = \mathsf{g}_{\pm}(\mathbb{Z}_2 \oplus \mathbb{Z}_4) = 5$.

We shall also need the following theorem (Theorem 4.1, [2]).

Theorem 2. *Let G be a finite and nontrivial abelian group and let $S \in \mathcal{F}(G)$ be a sequence.*

1. *If $|S| \geq \log_2 |G| + 1$ and G is not an elementary 2-group, then S contains a proper nontrivial $\{\pm 1\}$-weighted zero-sum subsequence.*

2. *If $|S| \geq \log_2 |G| + 2$ and G is not an elementary 2-group of even rank, then S contains a proper nontrivial $\{\pm 1\}$-weighted zero-sum subsequence of even length.*

3. *If $|S| > \log_2 |G|$, then S contains a nontrivial $\{\pm 1\}$-weighted zero-sum subsequence, and if $|S| > \log_2 |G| + 1$, then such a subsequence may be found with even length.*

Now we state our first theorem.

Theorem 3. *We have $\mathsf{s}_{\pm}(\mathbb{Z}_2 \oplus \mathbb{Z}_{2n}) \leq 2n + \lceil \log_2 2n \rceil + 1$.*

Proof. Let S be a sequence over $G = \mathbb{Z}_2 \oplus \mathbb{Z}_{2n}$ of length $|S| = 2n + \lceil \log_2 2n \rceil + 1$. We write $S = T^2U$, where T and U are subsequences of S and U is square-free. If $2|T| \leq \lceil \log_2 2n \rceil - 1$, then $|U| = |S| - 2|T| \geq 2n + 2$ and by Theorem 1, U has a $\{\pm 1\}$-weighted zero-sum subsequence of length $\exp(G) = 2n$ and we are through. So we assume that

$$2|T| \geq \lceil \log_2 2n \rceil,$$

and therefore we may write, $S = V^2W$ where $2|V| = \lceil \log_2 2n \rceil + \delta$, where $\delta = 0$ or 1 according as $\lceil \log_2 2n \rceil$ is even or odd, W being the remaining part of the sequence S. We have, $|W| = |S| - \lceil \log_2 2n \rceil - \delta = 2n + 1 - \delta$.

Since by Part (2) of Theorem 2, any sequence of length at least $\lceil \log_2 2n \rceil + 3$ has a proper non-trivial $\{\pm 1\}$-weighted zero-sum subsequence of even length, we get pairwise disjoint $\{\pm 1\}$ zero-sum subsequences $A_1, \ldots, A_l$ of W, each of even length, such that $|W| - \sum_{i=1}^{l} |A_i| \leq \lceil \log_2 2n \rceil + 2$, and hence

$$2n - 1 - \delta - \lceil \log_2 2n \rceil \leq \sum_{i=1}^{l} |A_i| \leq |W|.$$

Each $|A_i|$ being even, when $\lceil \log_2 2n \rceil$ is even (and so $\delta = 0$), we have

$$2n - \lceil \log_2 2n \rceil \leq \sum_{i=1}^{l} |A_i| \leq 2n,$$

and since $2|V| = \lceil \log_2 2n \rceil$, there exists a subsequence V_1 of V such that $V_1^2 \prod_{i=1}^{l} A_i$ is a $\{\pm 1\}$-weighted zero-sum subsequence of length $2n$.

Similarly, when $\lceil \log_2 2n \rceil$ is odd and hence $\delta = 1$, we have

$$2n - \lceil \log_2 2n \rceil - 1 \leq \sum_{i=1}^{l} |A_i| \leq 2n,$$

and since $2|V| = \lceil \log_2 2n \rceil + 1$, there exists a subsequence V_1 of V such that $V_1^2 \prod_{i=1}^{l} A_i$ is a desired subsequence of S. $\square$

Corollary 1. *If n is a power of 2, then*

$$\mathsf{s}_{\pm}(\mathbb{Z}_2 \oplus \mathbb{Z}_{2n}) = 2n + \lceil \log_2 2n \rceil + 1 = \eta_{\pm}(\mathbb{Z}_2 \oplus \mathbb{Z}_{2n}) + 2n - 1.$$

Proof. We have

$$\begin{aligned} 2n + \lceil \log_2 2n \rceil + 1 &\geq \mathsf{s}_{\pm}(\mathbb{Z}_2 \oplus \mathbb{Z}_{2n}) \quad \text{(by Theorem 3)} \\ &\geq \eta_{\pm}(\mathbb{Z}_2 \oplus \mathbb{Z}_{2n}) + 2n - 1 \quad \text{(by (2))} \\ &\geq \lfloor \log_2 2n \rfloor + 2n + 1. \end{aligned}$$

Considering the sequence $(1,0)(0,1)(0,2) \cdot \ldots \cdot (0,2^r)$, where r is defined by $2^{r+1} \leq 2n < 2^{r+2}$, we have $\eta_{\pm}(\mathbb{Z}_2 \oplus \mathbb{Z}_{2n}) \geq \lfloor \log_2 2n \rfloor + 2$, giving the last inequality.

If n is a power of 2, then $\lceil \log_2 2n \rceil = \lfloor \log_2 2n \rfloor$ and hence the corollary. $\square$

Next we establish lower bounds of $\eta_{\pm}(G)$ for a class of finite abelian groups G; these bounds will be useful in the next section.

Lemma 1. *For positive integers r and n, we have*

$$\eta_{\pm}(\mathbb{Z}_2^r \oplus \mathbb{Z}_{2n}) \geq \max\left\{ \lfloor \log_2 2n \rfloor + r + \left\lfloor \frac{r}{2n-1} \right\rfloor, r + A(r,n) \right\} + 1,$$

where

$$A(r,n) = \begin{cases} 1 & \text{if } r \leq n, \\ \left\lfloor \frac{r}{n} \right\rfloor & \text{if } r > n. \end{cases}$$

Proof. For $n = 1$, as observed in the introduction, $\eta_{\pm}(\mathbb{Z}_2^{r+1}) = 2^{r+1}$, and the lower bound in the lemma holds.

Now, we assume $n > 1$ and consider the sequence

$$S = \prod_{i=1}^{r} e_i \prod_{t=0}^{s} f_t \prod_{j=1}^{k} g_j,$$

where $s = \lfloor \log_2 2n \rfloor - 1$, $k = \left\lfloor \frac{r}{2n-1} \right\rfloor$ and e_i, f_t, and g_j are defined as follows:

$$e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0),$$

having 1 at the i-th position, for $1 \leq i \leq r$,

$$f_t = (0, 0, \ldots, 0, 2^t), \quad \text{for } 0 \leq t \leq s,$$

$$\text{and} \quad g_{j+1} = (0, 0, \ldots 0, 1, 1, \ldots, 1, 0, \ldots, 0, 1),$$

having 1 at the $(r+1)$-th position and positions $(2n-1)j+1, (2n-1)j+2, \ldots, (2n-1)j+2n-1$, for $0 \leq j \leq k-1$.

Now, $\prod_{i=1}^r e_i \prod_{t=1}^s f_t$ is a zero-sum free sequence with respect to weight $\{1, -1\}$, and therefore any $\{\pm 1\}$-weighted zero-sum subsequence of S must involve one or more g_i's. However, any $\{\pm 1\}$-weighted zero-sum subsequence containing one of the g_i's, must have at least $2n-1$ elements among the e_i's and an element from the f_i's. Thus the length of any $\{\pm 1\}$-weighted zero-sum subsequence of S will be at least $2n+1$, thereby implying that

$$\eta_{\pm}(\mathbb{Z}_2^r \oplus \mathbb{Z}_{2n}) \geq r + \lfloor \log_2 2n \rfloor + \left\lfloor \frac{r}{2n-1} \right\rfloor + 1. \tag{4}$$

We proceed to observe that

$$\eta_{\pm}(\mathbb{Z}_2^r \oplus \mathbb{Z}_{2n}) \geq r + A(r, n) + 1. \tag{5}$$

If $r \leq n$, then since the sequence $S_1 = f_1 \prod_{i=1}^r e_i$ has no $\{\pm 1\}$-weighted zero-sum subsequence, we are done in this case.

If $r > n$, we consider the sequence $S = \prod_{i=1}^r e_i \prod_{j=1}^u h_j$, where $u = \left\lfloor \frac{r}{n} \right\rfloor$ and

$$h_{j+1} = (0, 0, \ldots 0, 1, 1, \ldots, 1, 0, \ldots, 0, 1),$$

having 1 at the $(r+1)$-th position and the positions $nj+1, nj+2, \ldots, nj+n$, for $0 \leq j \leq u-1$.

Observing that $\prod_{i=1}^r e_i$ has no non-empty $\{\pm 1\}$-weighted zero-sum subsequence, any $\{\pm 1\}$-weighted zero-sum subsequence of S must involve one or more h_i's. However, any $\{\pm 1\}$-weighted zero-sum subsequence containing one of the h_i's must have at least two h_i's, considering the last position, and hence has to be at least of length $2n+2$; and hence we have (5).

From (4) and (5), the lemma follows. $\square$

3. The Case of an Abelian Group of Order 32

After making a couple of remarks applicable to general abelian groups, we shall have some lemmas dealing with different cases of abelian groups of order 32.

Remark I. If $S = a_1 \cdot \ldots \cdot a_n$ is a square-free sequence (that is, the elements in S are distinct) over an abelian group and $\binom{n}{2} > M$, where M is the cardinality of a set containing the sums $a_i + a_j$ $1 \leq i < j \leq n$, then $a_i + a_j = a_k + a_l$ for some $i, j, k, l \in \{1, 2, \ldots, n\}$ with $\{i, j\} \neq \{k, l\}$, and the assumption that S is square-free forces that $\{i, j\} \cap \{k, l\} = \emptyset$ and hence one obtains a $\{\pm 1\}$-weighted zero-sum subsequence of length 4.

Remark II. If $S = a_1 \cdot \ldots \cdot a_n$ with $n \geq 4$ is a sequence over an abelian group, and $a_1 = a_2$, so that $a_1 a_2$ is a $\{\pm 1\}$-weighted zero-sum subsequence of length 2, writing $T = a_3 a_4 \cdot \ldots \cdot a_n$, if we want to avoid having a $\{\pm 1\}$-weighted zero-sum subsequence of S of length 4, we must assume that the elements in T are distinct.

Lemma 2. *We have*

$$\mathsf{s}_{\pm}(\mathbb{Z}_2^3 \oplus \mathbb{Z}_4) = 10 \text{ and } \eta_{\pm}(\mathbb{Z}_2^3 \oplus \mathbb{Z}_4) = 7.$$

Proof. We first observe that the sum of two distinct elements of order at most two in the group $\mathbb{Z}_2^3 \oplus \mathbb{Z}_4$ is of order two, and the sum of two elements of order four in $\mathbb{Z}_2^3 \oplus \mathbb{Z}_4$ is of order at most two.

Let $S = a_1 \cdot \ldots \cdot a_{10} \in \mathcal{F}(\mathbb{Z}_2^3 \oplus \mathbb{Z}_4)$.

Suppose that S is square-free. Since $\binom{10}{2} = 45 > 32$, by Remark I above, we obtain a $\{\pm 1\}$-weighted zero-sum subsequence of length 4.

Without loss of generality, we now assume that $a_1 = a_2$ and writing $T = a_3 a_4 \cdot \ldots \cdot a_{10}$, by Remark II, we can assume that the elements in T are distinct.

If T has at most one element of order 4, we have at least seven elements, say $a_3, a_4, a_5, a_6, a_7, a_8, a_9$, of order at most two. Since $\binom{7}{2} = 21 > 15$, and the number of elements of order 2 in the group $\mathbb{Z}_2^3 \oplus \mathbb{Z}_4$ is 15, by the observation made in the beginning of this proof and Remark I, we get a $\{\pm 1\}$-weighted zero-sum subsequence of length 4.

So, we assume that T has at least two elements of order 4 and consider the collection of weighted sums:

$$\begin{aligned} D &= \{a_i \pm a_j, 3 \leq i < j \leq 10, \quad a_i, a_j \text{ are of order } 4\} \\ &\quad \cup \{a_r + a_s, 3 \leq r < s \leq 10, \quad a_r, a_s \text{ are of order } 2\}. \end{aligned}$$

Consider $2 \cdot \binom{c}{2} + \binom{d}{2}$, where c and d are the numbers of a_i's with $3 \leq i \leq 10$, of order 4 and of order 2, respectively (so that $c + d = 8$). Observing that $2 \cdot \binom{c}{2} + \binom{d}{2}$ is not less than 16, as c varies from 2 to 8, two weighted sums in D must be equal.

If one of the $\{\pm 1\}$-weighted sums $a_i \pm a_j$, where a_i, a_j are of order 4, is equal to some $a_r + a_s$, where a_r, a_s are of order 2, we get a $\{\pm 1\}$-weighted zero-sum subsequence of length 4.

If two distinct sums $a_r + a_s$, $a_u + a_v$ are equal, we are through by Remark I.

The following observation is being noted for future use.

Observation I. If some $a_i \pm a_j$ is equal to some $a_p \pm a_q$, where a_i, a_j, a_p, a_q are of order 4, there must be a $\{\pm 1\}$-weighted zero-sum subsequence of length 4 or 2.

If $\{i,j\} \cap \{p,q\} = \emptyset$, then we get a $\{\pm 1\}$-weighted zero-sum subsequence of length 4.

Since a_j is of order 4, $a_i + a_j \neq a_i - a_j$. Hence, the other possibilities are

$$a_i + \epsilon a_j = a_p + \delta a_q,$$

for some $i, j, p, q,\ i < j,\ p < q,\ |\{i,j\} \cap \{p,q\}| = 1$ and $\epsilon, \delta \in \{1, -1\}$.

If $i = p$, one gets a $\{\pm 1\}$-weighted zero-sum subsequence of length 2. If $i = q$ (by symmetry, the case $p = j$ is similar), and $\delta = 1$, then once again we get a $\{\pm 1\}$-weighted zero-sum subsequence of length 2. If $i = q$, and $\delta = -1$, we shall get an expression of the form

$$a_s + \lambda a_t = 2a_u, \text{ where } \{s,t,u\} = \{i,j\} \cup \{p,q\},\ \lambda \in \{1,-1\}. \tag{6}$$

If $j = q$, then since $i \neq p$ and hence $a_i \neq a_p$ by our assumption, it is forced that $\epsilon \neq \delta$ and once again we get an expression of the form (6).

Now, (6) implies that $a_s + (\lambda + 2)a_t = 2(a_u + a_t) = 0$, since a sum of two order 4 elements here is of order 2.

Since $\lambda + 2 \in \{1, 3\}$ and $3a_t = -a_t$ we get $a_s a_t$ to be a $\{\pm 1\}$-weighted zero-sum subsequence of length 2.

Hence, the observation.

Therefore, we have proved that $\mathsf{s}_{\pm}(\mathbb{Z}_2^3 \oplus \mathbb{Z}_4) \leq 10$. Again, by Lemma 1,

$$\eta_{\pm}(\mathbb{Z}_2^3 \oplus \mathbb{Z}_4) \geq \lfloor \log_2 4 \rfloor + 3 + \left\lfloor \frac{3}{4-1} \right\rfloor + 1 = 7.$$

From these and (2), we have

$$10 \geq \mathsf{s}_{\pm}(\mathbb{Z}_2^3 \oplus \mathbb{Z}_4) \geq \eta_{\pm}(\mathbb{Z}_2^3 \oplus \mathbb{Z}_4) + 3 \geq 10,$$

and we are done. □

Lemma 3. *We have*

$$\mathsf{s}_{\pm}(\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4) = 9 \text{ and } \eta_{\pm}(\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4) = 6.$$

Proof. Let $S = a_1 \cdot \ldots \cdot a_9 \in \mathcal{F}(\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4)$.

If S is square-free, then since $\binom{9}{2} = 36 > 32$, by Remark I, one obtains a $\{\pm 1\}$-weighted zero-sum subsequence of length 4.

Without loss of generality, we now assume that $a_1 = a_2$ giving a $\{\pm 1\}$-weighted zero-sum subsequence $S_1 = a_1 a_2$. By Remark II, the elements in $T = a_3 a_4 \cdot \ldots \cdot a_9$ are assumed to be distinct.

We shall now show that T has a $\{\pm 1\}$-weighted zero-sum subsequence of length 2 or 4. Noting that the group $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4$ consists of the identity element, seven elements of order 2 and twenty-four elements of order 4, we shall proceed to take care of various cases depending on the number of elements of order 4 in the sequence T.

If $(x, y, z) \in \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4$ is an element of order 4, depending on whether only y or only z is of order 4 in $\mathbb{Z}_4$, we call it respectively of Type 1 and Type 2; if both y and z are of order 4 in $\mathbb{Z}_4$, we call it of Type 3. One observes that the sum of two order 4 elements of the same type is of order 2, the sum of two order 4 elements of different types is of order 4 and the sum of three elements of order 4, one from each type, is of order 2. From the fact that the sum of two order 4 elements of different types is of order 4, we have the following.

Observation II. If a_i, a_j, a_k are order 4 elements of distinct types, then the four sums $a_i \pm a_j \pm a_k$ are distinct.

`Case (i).` If T does not have more than two elements of order 4, so that it has at least five elements, say a_3, a_4, a_5, a_6, a_7, of order at most 2, since $\binom{5}{2} = 10 > 8$, as before by Remark I, we have a $\{\pm 1\}$-weighted zero-sum subsequence of length 4.

`Case (ii).` Suppose T has exactly three elements, say a_7, a_8, a_9, of order 4, so that a_3, a_4, a_5, a_6 are of order 2.

The number of subsequences of length two of $a_3a_4a_5a_6$ is $\binom{4}{2} = 6$ and corresponding to each such subsequence a_ia_j, $3 \le i < j \le 6$, we have an order 2 element $a_i + a_j$.

Now, if among a_7, a_8, a_9, at least two elements, say a_7, a_8, are of the same type, then consider the elements $a_7 \pm a_8$. Since a_8 is of order 4, we have $a_7 + a_8 \ne a_7 - a_8$.

Therefore, if none of the elements $a_i + a_j$, $3 \le i < j \le 6$, and $a_7 \pm a_8$ is 0, then either two among the distinct sums $a_i + a_j$, $3 \le i < j \le 6$, will be equal, or one of the sums $a_i + a_j$, $3 \le i < j \le 6$, will be equal to one of the sums $a_7 \pm a_8$. Thus in any case we shall get a $\{\pm 1\}$-weighted zero-sum subsequence S_2 of length 4 or 2. If $|S_2| = 2$, then S_1S_2 is a $\{\pm 1\}$-weighted zero-sum subsequence of length 4.

If the three elements a_7, a_8, a_9 are of three distinct types, then by Observation II above, the four sums $a_7 \pm a_8 \pm a_9$ are distinct elements of the subgroup $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. If one of the elements a_3, a_4, a_5, a_6 is equal to one of the sums $a_7 \pm a_8 \pm a_9$, it gives us a $\{\pm 1\}$-weighted zero-sum subsequence of length 4. Otherwise, the elements a_3, a_4, a_5, a_6 together with the four sums $a_7 \pm a_8 \pm a_9$ will be all the distinct elements of the subgroup $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, and since, for $k > 1$, the sum of all the distinct elements of $\mathbb{Z}_2^k$ is zero, observing that $4a_7 = 0$ (a_7 being an element in a group of exponent 4), here we have $a_3 + a_4 + a_5 + a_6 = 0$, and we are through.

`Case (iii).` Suppose T has four elements, say a_6, a_7, a_8, a_9, of order 4, so that a_3, a_4, a_5 are of order 2.

Now, if among a_6, a_7, a_8, a_9, at least three elements, say a_6, a_7, a_8, are of the same type, then considering the elements $a_6 \pm a_7, a_6 \pm a_8, a_7 \pm a_8$, along with three sums $a_i + a_j$, $3 \leq i < j \leq 5$, two of them must be equal, and hence by Observation I in the proof of Lemma 2, we have a $\{\pm 1\}$-weighted zero-sum subsequence of length 4.

If there are two elements, say a_6, a_7, of a particular type, and a_8, a_9, of another, we consider the elements $a_6 \pm a_7, a_8 \pm a_9$, along with three sums $a_i + a_j$, $3 \leq i < j \leq 5$. If any two of these are equal, or any one of them is zero, then we are through. If it is not the case, then being all the non-zero elements of order 2, as was observed in the previous case, their sum is 0; however, since the sum is $2(a_6 + a_8)$, and $(a_6 + a_8)$ is of order 4, it is not possible.

Finally, if two elements, say a_6, a_7, are of a particular type, and among the remaining elements a_8, a_9, one element each is in the remaining types, consider the collection

$$a_5 + a_6 \pm a_8 \pm a_9, a_6 + a_7, a_6 + a_7 + a_3 + a_5, a_6 + a_7 + a_4 + a_5.$$

Once again, if any two of these are equal, or any one of them is zero, then we are through. Otherwise, their sum $3a_6 + 3a_7 + a_3 + a_4 = -a_6 - a_7 + a_3 + a_4$ is zero, providing us with a $\{\pm 1\}$-weighted zero-sum subsequence of length 4.

`Case (iv).` Suppose there are at least six elements, say $a_4, a_5, a_6, a_7, a_8, a_9$, which are of order 4.

If there are four elements, say a_4, a_5, a_6, a_7, which are of the same type, then consider

$$a_4 \pm a_j, \quad j \in \{5, 6, 7\}, \text{ and } a_5 \pm a_6.$$

If all of them are distinct, then one of them is 0, thus providing us with a $\{\pm 1\}$-weighted zero-sum subsequence of length 2. If two of them are equal, then once again we are through by the argument in Observation I made during the proof of Lemma 2.

If there are not more than three elements of a particular type, then we have the following possibilities.

If it happens that there are three elements of a particular type, so that at least two are of another type, without loss of generality, let a_4, a_5, a_6 be of one type, and a_7, a_8 of another, and we are through by considering the elements $a_4 \pm a_5, a_4 \pm a_6, a_5 \pm a_6, a_7 \pm a_8$.

Otherwise, there are two elements of order 4 of each type. Let a_4 and a_5 be of type 1, a_6 and a_7 be of type 2, and a_8 and a_9 be of type 3.

By Observation II, the elements $a_4 \pm a_6 \pm a_8$ are four distinct elements and similarly, $a_4 \pm a_7 \pm a_9$ are four distinct elements.

If one among the first group is equal to one of the second group, we get a $\{\pm 1\}$-weighted zero-sum subsequence of length 4. Otherwise, it gives the complete list of eight distinct elements of order 2. If a_3 is of order 2, then it is equal to one of these, and once again, we get a $\{\pm 1\}$-weighted zero-sum subsequence of length 4. If a_3 is of order 4, then there are three elements of a particular type and two of another, a case which has been already covered.

Case (v). The last case to deal with is the one where T has five elements, say a_5, a_6, a_7, a_8, a_9, of order 4, so that a_3, a_4 are of order 2.

Among the elements of order 4, if there are four elements of the same type, or there are three elements of a particular type and two of another, it has been taken care of while dealing with Case (iv).

If none of these happens, then there are order 4 elements of all the three types. In fact, the following two situations will arise.

Suppose there is one element, say a_5, of a particular type, two elements, say a_6, a_7, are of another type, and a_8, a_9 are of the remaining type. In this situation, if one among the four distinct elements $a_5 \pm a_6 \pm a_8$ is equal to one among the distinct elements $a_5 \pm a_7 \pm a_9$, we get a $\{\pm 1\}$-weighted zero-sum subsequence of length 4. If all these eight elements are distinct, a_3 must be equal to one of these, thus giving us a $\{\pm 1\}$-weighted zero-sum subsequence of length 4.

In the second possible situation, there will be three elements, say a_5, a_6, a_7, of a particular type and from the remaining elements a_8, a_9, one element each in the remaining types. Consider the elements

$$a_5 \pm a_8 \pm a_9, a_5 \pm a_6 + a_3, a_5 \pm a_7 + a_3,$$

where, as seen before, equality of two of these will make us through. If they are distinct, as argued before, a_4 is equal to one of these and we get a $\{\pm 1\}$-weighted zero-sum subsequence of length 4.

Therefore,

$$\mathsf{s}_{\pm}(\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4) \le 9. \tag{7}$$

Since the sequence $g_1 \cdot \ldots \cdot g_5$, where g_i's are defined by

$$g_1 = (1,0,0), g_2 = (0,1,0), g_3 = (0,2,0), g_4 = (0,0,1), g_5 = (0,0,2),$$

does not have a non-empty $\{\pm 1\}$-weighted zero-sum subsequence, we have

$$\eta_{\pm}(\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4) \ge 6. \tag{8}$$

From (7), (8) and (2), we have

$$9 \ge \mathsf{s}_{\pm}(\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4) \ge \eta_A(\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4) + \exp(\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4) - 1 \ge 6 + 4 - 1 = 9,$$

and hence the lemma. $\square$

Lemma 4. *If G be an abelian group of order 32 with* $\exp(G) = 8$, *then*

$$\mathsf{s}_{\pm}(G) = 13 \text{ and } \eta_{\pm}(G) = 6.$$

Proof. Let $S = a_1 \cdot \ldots \cdot a_{13} \in \mathcal{F}(G)$. We proceed to show that S has a $\{\pm 1\}$-weighted zero-sum subsequence of length 8.

`Case (A)`. If S is square-free, then observing that $\binom{13}{2} = 78 > 32$, by Remark I, we obtain a $\{\pm 1\}$-weighted zero-sum subsequence $S_1 = a_i a_j a_k a_l$ of length 4. Since $\binom{9}{2} = 36 > 32$, the sequence SS_1^{-1} being of length 9, it will have another $\{\pm 1\}$-weighted zero-sum subsequence S_2 of length 4. This shows that S has a $\{\pm 1\}$-weighted zero-sum subsequence $S_1 S_2$ of length 8, in this case.

`Case (B)`. If S is not square-free, let $a_1 = a_2$, so that $T = a_1 a_2$ is a $\{\pm 1\}$-weighted zero-sum subsequence of length 2.

`Subcase (B-1)`. If ST^{-1} is square-free, then observing that $\binom{11}{2} > 32$, we have a $\{\pm 1\}$-weighted zero-sum subsequence T_1 of ST^{-1} of length 4.

Since $|ST^{-1}T_1^{-1}| = 7$, by Part (3) of Theorem 2, $ST^{-1}T_1^{-1}$ has a $\{\pm 1\}$-weighted zero-sum subsequence T_2 with $|T_2| \in \{2, 4, 6\}$.

If $|T_2| = 2$, then TT_1T_2 is a $\{\pm 1\}$-weighted zero-sum subsequence of length 8. Considering, T_1T_2 when $|T_2| = 4$, and TT_2 when $|T_2| = 6$, we get the required $\{\pm 1\}$-weighted zero-sum subsequence of length 8.

`Subcase (B-2)`. If ST^{-1} is not square-free, let $a_3 = a_4$ so that $U_1 = a_3 a_4$ is a $\{\pm 1\}$-weighted zero-sum subsequence of length 2. Since $|ST^{-1}U_1^{-1}| = 9$, and $\binom{9}{2} > 32$, $ST^{-1}U_1^{-1}$ will have a $\{\pm 1\}$-weighted zero-sum subsequence U_2 of length 2 or 4. If $|U_2| = 4$, we are through; if $|U_2| = 2$, so that $|ST^{-1}U_1^{-1}U_2^{-1}| = 7$, we invoke Part (3) of Theorem 2, as in the above subcase, and we are through.

Thus we have established that

$$\mathsf{s}_{\pm}(G) \leq 13. \tag{9}$$

Now, G can be $\mathbb{Z}_4 \oplus \mathbb{Z}_8$ or $\mathbb{Z}_2^2 \oplus \mathbb{Z}_8$. If $G = \mathbb{Z}_4 \oplus \mathbb{Z}_8$, then the sequence $b_1 \cdot \ldots \cdot b_5$, where $b_i \in G$ are defined by

$$b_1 = (0,1), b_2 = (0,2), b_3 = (0,4), b_4 = (1,0), b_5 = (2,0),$$

does not have a non-empty $\{\pm 1\}$-weighted zero-sum subsequence. If $G = \mathbb{Z}_2^2 \oplus \mathbb{Z}_8$, then the sequence $c_1 \cdot \ldots \cdot c_5$, where $c_i \in G$ are defined by

$$c_1 = (0,0,1), c_2 = (0,0,2), c_3 = (0,0,4), c_4 = (0,1,0), c_5 = (1,0,0),$$

does not have a non-empty $\{\pm 1\}$-weighted zero-sum subsequence. Therefore,

$$\eta_{\pm}(G) \geq 6. \tag{10}$$

From (9), (10) and (2) we have

$$13 \geq \mathsf{s}_{\pm}(G) \geq \eta_{\pm}(G) + \exp(G) - 1 \geq 6 + 8 - 1 = 13,$$

and hence the lemma. □

Theorem 4. *If G is an abelian group of order 32, the following relation holds:*

$$s_{\pm}(G) = \eta_{\pm}(G) + \exp(G) - 1.$$

Proof. In the case of any finite cyclic group G, the result was established in [1]; and as mentioned in the introduction, in this case the corresponding result for general weights was established by Grynkiewicz et al. [7].

Also, as mentioned in the introduction, when the finite group G is an elementary abelian 2-group, the relation stated in the theorem holds.

If $G = \mathbb{Z}_2 \oplus \mathbb{Z}_{16}$, the theorem follows from our corollary to Theorem 3 in Section 2, and if $\exp(G) = 8$, the theorem follows from Lemma 4.

Finally, Lemmas 2, 3, take care of the remaining case $\exp(G) = 4$. □

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