TWO EXPLICIT FORMULAS OF THE SCHRÖDER NUMBERS

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Abstract

In the paper, by several methods and approaches, the authors establish two explicit formulas for the large and little Schröder numbers.

1. Introduction

In combinatorics and number theory, there are two kinds of Schröder numbers, the large Schröder numbers S_n and the little Schröder numbers s_n. They are named after the German mathematician Ernst Schröder.

A large Schröder number S_n describes the number of paths from the southwest corner $(0,0)$ of an $n \times n$ grid to the northeast corner (n,n), using only single steps north, northeast, or east, that do not rise above the southwest-northeast diagonal. The first eleven large Schröder numbers S_n for $0 \le n \le 10$ are

$$1, \quad 2, \quad 6, \quad 22, \quad 90, \quad 394, \quad 1806, \quad 8558, \quad 41586, \quad 206098, \quad 1037718.$$

In [3, Theorem 8.5.7], it was proved that the large Schröder numbers S_n have the

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generating function

$$G(x) = \frac{1 - x - \sqrt{x^2 - 6x + 1}}{2x} = \sum_{n=0}^{\infty} S_n x^n, \tag{1.1}$$

which can also be rearranged as

$$\mathcal{G}(x) = G(-x) = \frac{\sqrt{x^2 + 6x + 1} - 1 - x}{2x} = \sum_{n=0}^{\infty} (-1)^n S_n x^n. \tag{1.2}$$

The little Schröder numbers s_n form an integer sequence that can be used to count the number of plane trees with a given set of leaves, the number of ways of inserting parentheses into a sequence, and the number of ways of dissecting a convex polygon into smaller polygons by inserting diagonals. The first eleven little Schröder numbers s_n for $1 \le n \le 11$ are

$$1, \quad 1, \quad 3, \quad 11, \quad 45, \quad 197, \quad 903, \quad 4279, \quad 20793, \quad 103049, \quad 518859.$$

They are also called the small Schröder numbers, the Schröder-Hipparchus numbers, or the Schröder numbers, after Ernst Schröder and the ancient Greek mathematician Hipparchus who appears from evidence in Plutarch to have known of these numbers. They are also called the super-Catalan numbers, after Eugéne Charles Catalan, but are different from a generalization of the Catalan numbers [8, 18]. In [3, Theorem 8.5.6], it was proved that the little Schröder numbers s_n have the generating function

$$g(x) = \frac{1 + x - \sqrt{x^2 - 6x + 1}}{4} = \sum_{n=1}^{\infty} s_n x^n. \tag{1.3}$$

For more information on the large Schröder numbers S_n and the little Schröder numbers s_n, please refer to [3, 13, 14, 15, 19, 20] and plenty of references therein.

Comparing (1.1) with (1.3), we can reveal

$$\sqrt{x^2 - 6x + 1} = 1 + x - 4\sum_{n=1}^{\infty} s_n x^n = 1 - x - 2\sum_{n=0}^{\infty} S_n x^{n+1},$$

that is,

$$1 - 2\sum_{n=1}^{\infty} s_n x^{n-1} = 1 - 2\sum_{n=0}^{\infty} s_{n+1} x^n = -\sum_{n=0}^{\infty} S_n x^n.$$

Accordingly, we acquire

$$S_n = 2s_{n+1}, \quad n \in \mathbb{N}.$$

See also [3, Corollary 8.5.8]. This relation tells us that it is sufficient to analytically study the large Schröder numbers S_n.

The main aim of this paper is, by several methods and approaches, to establish two explicit formulas for the large and little Schröder numbers S_n and s_n.

Theorem 1. *For $n \in \mathbb{N}$, the large and little Schröder numbers S_n and s_{n+1} can be computed by*

$$S_n = 2s_{n+1} = \frac{(-1)^n}{12}\frac{1}{6^n}\sum_{k=\lceil (n+1)/2\rceil}^{n+1}\frac{6^{2k}}{k!}\left\langle\frac{1}{2}\right\rangle_k\binom{k}{n-k+1} \tag{1.4}$$

and

$$\begin{aligned}S_n = 2s_{n+1} = \frac{1}{12}\frac{n!}{6^n}\sum_{k=0}^{n}\sum_{r+s=k}\sum_{\ell+m=n}\sum_{q=0}^{s}\sum_{j=0}^{n-r-1}(-1)^{s-q}6^{2(r+j)}\left\langle\frac{1}{2}\right\rangle_k\left\langle\frac{1}{2}-k\right\rangle_j \\ \times\binom{\ell-1}{r-1}\binom{s}{q}\binom{j}{n-r-j+1}\binom{m+2q-1}{2q-1}\frac{1}{r!s!j!},\end{aligned} \tag{1.5}$$

where $\lceil x\rceil$ stands for the ceiling function which gives the smallest integer not less than x and $\langle x\rangle_n$ is the falling factorial defined by

$$\langle x\rangle_n = \prod_{k=0}^{n-1}(x-k) = \begin{cases} x(x-1)\cdots(x-n+1), & n \ge 1, \\ 1, & n = 0. \end{cases}$$

2. Lemmas

In order to prove our main results, we need the following notions and lemmas.

In combinatorial mathematics, the Bell polynomials of the second kind $\mathrm{B}_{n,k}$ are defined by

$$\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{\ell_i\in\{0\}\cup\mathbb{N} \\ \sum_{i=1}^{n} i\ell_i = n \\ \sum_{i=1}^{n}\ell_i = k}}\frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}$$

for $n \ge k \ge 0$. See [4, p. 134, Theorem A]. In combinatorial analysis, the Faà di Bruno formula plays an important role and can be described by

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}[f\circ h(t)] = \sum_{k=0}^{n} f^{(k)}(h(t))\mathrm{B}_{n,k}\big(h'(t), h''(t), \dots, h^{(n-k+1)}(t)\big) \tag{2.1}$$

in terms of the Bell polynomials of the second kind $\mathrm{B}_{n,k}$. See [4, p. 139, Theorem C].

Lemma 1 ([4, p. 135])**.** *For $n \ge k \ge 0$, we have*

$$\mathrm{B}_{n,k}\big(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}\big) = a^k b^n \mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}), \tag{2.2}$$

where a and b are any complex numbers.

Lemma 2 ([5, Theoem 4.1], [12, Theorem 3.1], and [17, Lemma 2.5])**.** *For $n \ge k \ge 0$, the Bell polynomials of the second kind $\mathrm{B}_{n,k}$ satisfy*

$$\mathrm{B}_{n,k}(x,1,0,\dots,0) = \frac{(n-k)!}{2^{n-k}}\binom{n}{k}\binom{k}{n-k}x^{2k-n}. \tag{2.3}$$

Lemma 3 ([2, p. 40, Exercise 5)], [6, Section 2.2, p. 849], [9, p. 94], and [17, Lemma 2.1])**.** *Let $u(x)$ and $v(x) \ne 0$ be two differentiable functions. Let $U_{(n+1)\times 1}(x)$ be an $(n+1)\times 1$ matrix whose elements $u_{k,1}(x) = u^{(k-1)}(x)$ for $1 \le k \le n+1$, let $V_{(n+1)\times n}(x)$ be an $(n+1)\times n$ matrix whose elements*

$$v_{i,j}(x) = \begin{cases} \dbinom{i-1}{j-1} v^{(i-j)}(x), & i-j \ge 0 \\ 0, & i-j < 0 \end{cases}$$

for $1 \le i \le n+1$ and $1 \le j \le n$, and let $|W_{(n+1)\times(n+1)}(x)|$ denote the determinant of the $(n+1)\times(n+1)$ matrix

$$W_{(n+1)\times(n+1)}(x) = \begin{pmatrix} U_{(n+1)\times 1}(x) & V_{(n+1)\times n}(x) \end{pmatrix}.$$

Then the nth derivative of the ratio $\frac{u(x)}{v(x)}$ can be computed by

$$\frac{\mathrm{d}^n}{\mathrm{d}\, x^n}\left[\frac{u(x)}{v(x)}\right] = (-1)^n \frac{\left|W_{(n+1)\times(n+1)}(x)\right|}{v^{n+1}(x)}.$$

Lemma 4 ([16, Theorem 2.1] and [21])**.** *Let M be a square matrix of order $n \times n$ and partitioned as*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Let K^{-1} denote the inverse of an invertible matrix K.

1. *If B is a $k \times k$ matrix and C nonsingular, then the determinant of M can be computed by*
$$|M| = (-1)^{(n+1)k}|C|\left|B - AC^{-1}D\right|. \tag{2.4}$$
2. *If A, B, C, D are respectively $p\times p$, $p \times q$, $q \times p$, and $q \times q$ matrices and if D is invertible, then the determinant of M can be computed by*
$$|M| = |D|\left|A - BD^{-1}C\right|. \tag{2.5}$$

Lemma 5 ([1, Example 2.6] and [4, p. 136, Eq. [3n]])**.** *The Bell polynomials of the second kind $\mathrm{B}_{n,k}$ satisfy*

$$\begin{aligned} &\mathrm{B}_{n,k}(x_1+y_1, x_2+y_2, \dots, x_{n-k+1}+y_{n-k+1}) \\ &\quad = \sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}\mathrm{B}_{\ell,r}(x_1,x_2,\dots,x_{\ell-r+1})\mathrm{B}_{m,s}(y_1,y_2,\dots,y_{m-s+1}). \end{aligned} \tag{2.6}$$

Lemma 6. *For $n \ge k \ge 0$, we have*

$$\mathrm{B}_{n,k}(1!,2!,\dots,(n-k+1)!) = \binom{n}{k}\binom{n-1}{k-1}(n-k)! \tag{2.7}$$

and

$$\mathrm{B}_{n,k}(2!,3!,\dots,(n-k+2)!) = \frac{1}{k!}\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{k}{\ell}\frac{(n+2\ell-1)!}{(2\ell-1)!}. \tag{2.8}$$

Proof. The formula (2.7) can be found in [4, p. 135] or [7, Theorem 1].

In [4, p. 133], it was stated that

$$\frac{1}{k!}\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right)^k = \sum_{n=k}^{\infty}\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})\frac{t^n}{n!} \tag{2.9}$$

for $k \ge 0$. Hence, it follows from (2.9) that

$$\begin{aligned}\sum_{n=k}^{\infty}\mathrm{B}_{n,k}(2!,3!,\dots,(n-k+2)!)\frac{t^n}{n!} &= \frac{1}{k!}\left(\sum_{m=1}^{\infty}(m+1)!\frac{t^m}{m!}\right)^k \\ &= \frac{1}{k!}\left[\frac{1}{(t-1)^2}-1\right]^k = \frac{1}{k!}\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{k}{\ell}\frac{1}{(t-1)^{2\ell}}\end{aligned}$$

and, by differentiating with respect to t,

$$\begin{aligned}&\sum_{n=m}^{\infty}\mathrm{B}_{n,k}(2!,3!,\dots,(n-k+2)!)\frac{t^{n-m}}{(n-m)!} \\ &\qquad = \frac{1}{k!}\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{k}{\ell}(-1)^m\frac{(2\ell+m-1)!}{(2\ell-1)!}\frac{1}{(t-1)^{2\ell+m}}, \quad m \ge k \ge 0.\end{aligned}$$

Further letting $t \to 0$ in the above equation yields the formula (2.8). □

3. Proofs of Theorem 1

We are now in a position to prove our main results.

First proof of the formula (1.4). It is clear that the generating function $G(x)$ is naturally defined on $(-\infty,0)\cup\big(0,3-2\sqrt{2}\,\big]\cup\big[3+2\sqrt{2}\,,\infty\big)$. It is easy to calculate that $\lim_{x\to0}\mathcal{G}(x)=\lim_{x\to0}G(x)=2$. Therefore, at the point $x=0$, both of the functions $G(x)$ and $\mathcal{G}(x)$ are removably discontinuous. Hence, the function $\mathcal{G}(x)$ can be regarded to be defined on $\big(-\infty,-3-2\sqrt{2}\,\big]\cup\big[2\sqrt{2}\,-3,\infty\big)$. Since

$$\sqrt{x^2+6x+1} = \sum_{k=0}^{\infty}\frac{\langle 1/2\rangle_k}{k!}[x(x+6)]^k, \quad |x(x+6)|<1, \tag{3.1}$$

we obtain

$$\mathcal{G}(x) = -\frac{1}{2} + \frac{1}{2}\sum_{k=1}^{\infty} \frac{\langle 1/2\rangle_k}{k!} x^{k-1}(x+6)^k, \quad |x(x+6)| < 1. \tag{3.2}$$

This implies that the function $\mathcal{G}(x) = G(-x)$ is infinitely differentiable at $x = 0$.

Utilizing the expansion (3.2) and differentiating give

$$\begin{aligned}
2\mathcal{G}^{(m)}(0) &= \sum_{k=1}^{\infty} \frac{\langle 1/2\rangle_k}{k!} \lim_{x\to 0}\bigl[x^{k-1}(x+6)^k\bigr]^{(m)} \\
&= \sum_{k=1}^{\infty} \frac{\langle 1/2\rangle_k}{k!} \lim_{x\to 0}\sum_{\ell=0}^{m}\binom{m}{\ell}\bigl(x^{k-1}\bigr)^{(\ell)}\bigl[(x+6)^k\bigr]^{(m-\ell)} \\
&= \sum_{k=1}^{\infty} \frac{\langle 1/2\rangle_k}{k!} \sum_{\ell=0}^{m}\binom{m}{\ell}\lim_{x\to 0}\bigl(x^{k-1}\bigr)^{(\ell)}\lim_{x\to 0}\bigl[(x+6)^k\bigr]^{(m-\ell)} \\
&= \sum_{k=1}^{m+1} \frac{\langle 1/2\rangle_k}{k!}\binom{m}{k-1}(k-1)!\lim_{x\to 0}\bigl[(x+6)^k\bigr]^{(m-k+1)} \\
&= \sum_{k=\lceil (m+1)/2\rceil}^{m+1} \frac{\langle 1/2\rangle_k}{k}\binom{m}{k-1}\lim_{x\to 0}\bigl[(x+6)^k\bigr]^{(m-k+1)} \\
&= \sum_{k=\lceil (m+1)/2\rceil}^{m+1} \frac{\langle 1/2\rangle_k}{k}\binom{m}{k-1}\langle k\rangle_{m-k+1}\lim_{x\to 0}(x+6)^{2k-m-1} \\
&= \sum_{k=\lceil (m+1)/2\rceil}^{m+1} \frac{\langle 1/2\rangle_k}{k}\binom{m}{k-1}\langle k\rangle_{m-k+1}6^{2k-m-1}
\end{aligned}$$

for $m \ge 1$. From (1.2), it follows that

$$\mathcal{G}^{(m)}(0) = (-1)^m m! S_m, \quad m \ge 1. \tag{3.3}$$

Consequently,

$$S_m = (-1)^m \frac{\mathcal{G}^{(m)}(0)}{m!} = \frac{1}{2}\frac{(-1)^m}{m!}\sum_{k=\lceil (m+1)/2\rceil}^{m+1} \frac{\langle 1/2\rangle_k}{k}\binom{m}{k-1}\langle k\rangle_{m-k+1}6^{2k-m-1}$$

for $m \ge 1$, which can be rewritten as the formula (1.4). The first proof of the formula (1.4) is complete. □

Second proof of the formula (1.4). By Faà di Bruno's formula (2.1), the mth derivative of the function $\sqrt{x^2+6x+1}$ can be computed by

$$\begin{aligned}\left(\sqrt{x^2+6x+1}\right)^{(m)} &= \sum_{k=0}^{m}\left(\sqrt{u}\right)^{(k)}\mathrm{B}_{m,k}(2x+6,2,0,\dots,0)\\ &= \sum_{k=0}^{m}\left\langle\frac{1}{2}\right\rangle_k u^{1/2-k}\mathrm{B}_{m,k}(2x+6,2,0,\dots,0)\\ &= \sum_{k=0}^{m}\left\langle\frac{1}{2}\right\rangle_k \left(x^2+6x+1\right)^{1/2-k}\mathrm{B}_{m,k}(2x+6,2,0,\dots,0)\\ &\to \sum_{k=0}^{m}\left\langle\frac{1}{2}\right\rangle_k \mathrm{B}_{m,k}(6,2,0,\dots,0)\end{aligned}$$

as $x\to 0$, where $u=u(x)=x^2+6x+1$ and $m\ge 0$. By (2.2) and (2.3), we obtain

$$\begin{aligned}\left(\sqrt{x^2+6x+1}\right)^{(m)} &= \sum_{k=0}^{m}\left\langle\frac{1}{2}\right\rangle_k \left(x^2+6x+1\right)^{1/2-k}(m-k)!\\ &\quad\times\binom{m}{k}\binom{k}{m-k}[2(x+3)]^{2k-m}\\ &\to \sum_{k=0}^{m}\left\langle\frac{1}{2}\right\rangle_k (m-k)!\binom{m}{k}\binom{k}{m-k}6^{2k-m}\end{aligned} \tag{3.4}$$

as $x\to 0$ for $m\ge 0$. The equation (1.2) can be rewritten as

$$\sqrt{x^2+6x+1}-1-x=2\sum_{n=0}^{\infty}(-1)^n S_n x^{n+1}$$

which implies that

$$2(-1)^n(n+1)!S_n=\lim_{x\to 0}\left(\sqrt{x^2+6x+1}-1-x\right)^{(n+1)},\quad n\ge 0.$$

Consequently, we obtain

$$\begin{aligned}S_n &= \frac{1}{2}\frac{(-1)^n}{(n+1)!}\lim_{x\to 0}\left(\sqrt{x^2+6x+1}\right)^{(n+1)}\\ &= \frac{1}{2}\frac{(-1)^n}{(n+1)!}\sum_{k=0}^{n+1}\left\langle\frac{1}{2}\right\rangle_k (n-k+1)!\binom{n+1}{k}\binom{k}{n-k+1}6^{2k-n-1}\end{aligned} \tag{3.5}$$

for $n\ge 1$. The formula (1.4) is thus proved again. □

Third proof of the formula (1.4). Applying Lemma 3 to the functions

$$u(x)=\sqrt{x^2+6x+1}-1-x$$

and $v(x)=x$ yields

$$\mathcal{G}^{(n)}(x)=\frac{(-1)^n}{2x^{n+1}}\begin{vmatrix} u(x) & x & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ u'(x) & 1 & x & 0 & \cdots & 0 & 0 & 0 & 0\\ u''(x) & 0 & 2 & x & \cdots & 0 & 0 & 0 & 0\\ u^{(3)}(x) & 0 & 0 & 3 & \cdots & 0 & 0 & 0 & 0\\ \hdotsfor{9}\\ u^{(n-3)}(x) & 0 & 0 & 0 & \cdots & n-3 & x & 0 & 0\\ u^{(n-2)}(x) & 0 & 0 & 0 & \cdots & 0 & n-2 & x & 0\\ u^{(n-1)}(x) & 0 & 0 & 0 & \cdots & 0 & 0 & n-1 & x\\ u^{(n)}(x) & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & n \end{vmatrix}. \tag{3.6}$$

By the formulas (2.4) and (2.5) and by induction, we have

$$\begin{aligned}\mathcal{G}^{(n)}(x)&=\frac{(-1)^{n+(n+2)n}}{2x^{n+1}}u^{(n)}(x)\left|\begin{pmatrix} x & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 1 & x & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 2 & x & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & 3 & \cdots & 0 & 0 & 0 & 0\\ \hdotsfor{8}\\ 0 & 0 & 0 & \cdots & n-3 & x & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & n-2 & x & 0\\ 0 & 0 & 0 & \cdots & 0 & 0 & n-1 & x \end{pmatrix}\right.\\ &\qquad\left.-\frac{1}{u^{(n)}(x)}\begin{pmatrix} u(x)\\ u'(x)\\ u''(x)\\ u^{(3)}(x)\\ \cdots\cdots\cdots\\ u^{(n-3)}(x)\\ u^{(n-2)}(x)\\ u^{(n-1)}(x) \end{pmatrix}\begin{pmatrix}0 & 0 & 0 & \cdots & 0 & 0 & 0 & n\end{pmatrix}\right|\\ &=\frac{1}{2x^{n+1}}\begin{vmatrix} x & 0 & 0 & \cdots & 0 & 0 & 0 & -nu(x)\\ 1 & x & 0 & \cdots & 0 & 0 & 0 & -nu'(x)\\ 0 & 2 & x & \cdots & 0 & 0 & 0 & -nu''(x)\\ 0 & 0 & 3 & \cdots & 0 & 0 & 0 & -nu^{(3)}(x)\\ \hdotsfor{8}\\ 0 & 0 & 0 & \cdots & n-3 & x & 0 & -nu^{(n-3)}(x)\\ 0 & 0 & 0 & \cdots & 0 & n-2 & x & -nu^{(n-2)}(x)\\ 0 & 0 & 0 & \cdots & 0 & 0 & n-1 & xu^{(n)}(x)-nu^{(n-1)}(x) \end{vmatrix}\\ &=\frac{1}{2x^{n+1}}\end{aligned}$$

$$\begin{vmatrix} x & 0 & 0 & \cdots & 0 & (-1)^m\langle n\rangle_m u(x) \\ 1 & x & 0 & \cdots & 0 & (-1)^m\langle n\rangle_m u'(x) \\ 0 & 2 & x & \cdots & 0 & (-1)^m\langle n\rangle_m u''(x) \\ 0 & 0 & 3 & \cdots & 0 & (-1)^m\langle n\rangle_m u^{(3)}(x) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & x & (-1)^m\langle n\rangle_m u^{(n-m-1)}(x) \\ 0 & 0 & 0 & \cdots & n-m & \sum_{k=0}^{m}(-1)^k\langle n\rangle_k x^{m-k}u^{(n-k)}(x) \end{vmatrix}$$

$$\begin{aligned}
&= \frac{1}{2x^{n+1}}\begin{vmatrix} x & (-1)^{n-1}\langle n\rangle_{n-1}u(x) \\ 1 & \sum_{k=0}^{n-1}(-1)^k\langle n\rangle_k x^{n-k-1}u^{(n-k)}(x) \end{vmatrix} \\
&= \frac{1}{2x^{n+1}}\left[\sum_{k=0}^{n-1}(-1)^k\langle n\rangle_k x^{n-k}u^{(n-k)}(x) - (-1)^{n-1}\langle n\rangle_{n-1}u(x)\right] \\
&\to \frac{1}{2(n+1)!}\lim_{x\to 0}\left[\sum_{k=0}^{n}(-1)^k\langle n\rangle_k x^{n-k}u^{(n-k)}(x)\right]^{(n+1)} \\
&= \frac{1}{2(n+1)!}\lim_{x\to 0}\sum_{k=0}^{n}(-1)^{n-k}\langle n\rangle_{n-k}\left[x^k u^{(k)}(x)\right]^{(n+1)} \\
&= \frac{1}{2(n+1)!}\sum_{k=0}^{n}(-1)^{n-k}\langle n\rangle_{n-k}\sum_{\ell=0}^{n+1}\binom{n+1}{\ell}\lim_{x\to 0}\left[\left(x^k\right)^{(\ell)}u^{(n+k-\ell+1)}(x)\right] \\
&= \frac{1}{2(n+1)!}\sum_{k=0}^{n}(-1)^{n-k}\langle n\rangle_{n-k}\binom{n+1}{k}k!u^{(n+1)}(0) \\
&= \frac{1}{2(n+1)!}u^{(n+1)}(0)\sum_{k=0}^{n}(-1)^{n-k}\langle n\rangle_{n-k}\binom{n+1}{k}k! \\
&= \frac{1}{2(n+1)}u^{(n+1)}(0)
\end{aligned}$$

as $x \to 0$, where $3 \le m \le n-2$. This means, by considering (3.3), that

$$S_n = \frac{1}{2}\frac{(-1)^n}{(n+1)!}u^{(n+1)}(0),$$

which is same as the first line in (3.5). Combining this with (3.4) recovers the formula (1.4). □

Fourth proof of the formula (1.4). Employing (2.5) to compute (3.6) gives

$$
\begin{aligned}
\mathcal{G}^{(n)}(x) &= \frac{(-1)^n}{2x^{n+1}}
\begin{vmatrix}
1 & x & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & 2 & x & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 3 & \cdots & 0 & 0 & 0 & 0\\
\multicolumn{8}{c}{\dotfill}\\
0 & 0 & 0 & \cdots & 0 & n-2 & x & 0\\
0 & 0 & 0 & \cdots & 0 & 0 & n-1 & x\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & n
\end{vmatrix}\\
&\times \left(u(x) - \begin{pmatrix} x\\ 0\\ 0\\ \cdots\\ 0\\ 0\\ 0\\ 0 \end{pmatrix}^T
\begin{pmatrix}
1 & x & 0 & \cdots & 0 & 0 & 0\\
0 & 2 & x & \cdots & 0 & 0 & 0\\
0 & 0 & 3 & \cdots & 0 & 0 & 0\\
\multicolumn{7}{c}{\dotfill}\\
0 & 0 & 0 & \cdots & x & 0 & 0\\
0 & 0 & 0 & \cdots & n-2 & x & 0\\
0 & 0 & 0 & \cdots & 0 & n-1 & x\\
0 & 0 & 0 & \cdots & 0 & 0 & n
\end{pmatrix}^{-1}
\begin{pmatrix} u'(x)\\ u''(x)\\ u^{(3)}(x)\\ \cdots\cdots\cdots\\ u^{(n-3)}(x)\\ u^{(n-2)}(x)\\ u^{(n-1)}(x)\\ u^{(n)}(x) \end{pmatrix} \right)\\
&= \frac{(-1)^n n!}{2x^{n+1}} \left(u(x) - \begin{pmatrix} x & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix} P
\begin{pmatrix} u'(x)\\ u''(x)\\ u^{(3)}(x)\\ \cdots\cdots\cdots\\ u^{(n-3)}(x)\\ u^{(n-2)}(x)\\ u^{(n-1)}(x)\\ u^{(n)}(x) \end{pmatrix} \right)\\
&= \frac{(-1)^n n!}{2x^{n+1}} \left(u(x) - \begin{pmatrix} \frac{x}{\langle 1\rangle_1} & -\frac{x^2}{\langle 2\rangle_2} & \frac{x^3}{\langle 3\rangle_3} & \cdots & \frac{(-1)^{n-1}x^n}{\langle n\rangle_n} \end{pmatrix}
\begin{pmatrix} u'(x)\\ u''(x)\\ u^{(3)}(x)\\ \cdots\cdots\cdots\\ u^{(n-1)}(x)\\ u^{(n)}(x) \end{pmatrix} \right)\\
&= \frac{(-1)^n n!}{2x^{n+1}} \left[u(x) - \sum_{k=1}^{n} (-1)^{k+1} \frac{x^k}{\langle k\rangle_k} u^{(k)}(x) \right]\\
&\to \frac{(-1)^{n+1}}{2(n+1)} \sum_{k=0}^{n} \frac{(-1)^{k+1}}{k!} \lim_{x\to 0} \left[x^k u^{(k)}(x) \right]^{(n+1)}\\
&= \frac{(-1)^{n+1}}{2(n+1)} \sum_{k=0}^{n} \frac{(-1)^{k+1}}{k!} \lim_{x\to 0} \sum_{\ell=0}^{n+1} \binom{n+1}{\ell} \left(x^k\right)^{(\ell)} u^{(n-\ell+k+1)}(x)\\
&= \frac{(-1)^{n+1}}{2(n+1)} \sum_{k=0}^{n} \frac{(-1)^{k+1}}{k!} \binom{n+1}{k} k! u^{(n+1)}(0)\\
&= \frac{(-1)^{n+1}}{2(n+1)} u^{(n+1)}(0) \sum_{k=0}^{n} (-1)^{k+1} \binom{n+1}{k}\\
&= \frac{1}{2(n+1)} u^{(n+1)}(0)
\end{aligned}
$$

as $x \to 0$, where K^T stands for the transpose of a matrix K and

$$P = \begin{pmatrix} \frac{1}{\langle 1\rangle_1} & -\frac{x}{\langle 2\rangle_2} & \frac{x^2}{\langle 3\rangle_3} & \cdots & \frac{(-1)^{n-3}x^{n-3}}{\langle n-2\rangle_{n-2}} & \frac{(-1)^{n-2}x^{n-2}}{\langle n-1\rangle_{n-1}} & \frac{(-1)^{n-1}x^{n-1}}{\langle n\rangle_n} \\ 0 & \frac{1}{\langle 2\rangle_1} & -\frac{x}{\langle 3\rangle_2} & \cdots & \frac{(-1)^{n-4}x^{n-4}}{\langle n-2\rangle_{n-3}} & \frac{(-1)^{n-3}x^{n-3}}{\langle n-1\rangle_{n-2}} & \frac{(-1)^{n-2}x^{n-2}}{\langle n\rangle_{n-1}} \\ 0 & 0 & \frac{1}{\langle 3\rangle_1} & \cdots & \frac{(-1)^{n-5}x^{n-5}}{\langle n-2\rangle_{n-4}} & \frac{(-1)^{n-4}x^{n-4}}{\langle n-1\rangle_{n-3}} & \frac{(-1)^{n-3}x^{n-3}}{\langle n\rangle_{n-2}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \frac{x}{\langle n-2\rangle_2} & \frac{x^2}{\langle n-1\rangle_3} & -\frac{x^3}{\langle n\rangle_4} \\ 0 & 0 & 0 & \cdots & \frac{1}{\langle n-2\rangle_1} & -\frac{x}{\langle n-1\rangle_2} & \frac{x^2}{\langle n\rangle_3} \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{\langle n-1\rangle_1} & -\frac{x}{\langle n\rangle_2} \\ 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{\langle n\rangle_1} \end{pmatrix}.$$

Similar to the above argument, we verify the formula (1.4) once again. □

Fifth proof of the formula (1.4). Combining the equation (1.2) with (3.1) leads to

$$\sum_{k=0}^{\infty} \frac{\langle 1/2\rangle_k}{k!}[x(x+6)]^k - 1 - x = 2\sum_{k=0}^{\infty}(-1)^k S_k x^{k+1}.$$

This equality can be reformulated as

$$\sum_{k=1}^{\infty} \frac{\langle 1/2\rangle_k}{k!} x^k \sum_{\ell=0}^{k} 6^{k-\ell}\binom{k}{\ell} x^\ell - x = 2\sum_{k=1}^{\infty}(-1)^{k-1} S_{k-1} x^k,$$

$$\sum_{k=1}^{\infty}\sum_{\ell=0}^{k} \frac{\langle 1/2\rangle_k}{k!} 6^{k-\ell}\binom{k}{\ell} x^{k+\ell} - x = 2\sum_{k=1}^{\infty}(-1)^{k-1} S_{k-1} x^k,$$

$$\sum_{k=1}^{\infty}\sum_{m=k}^{2k} \frac{\langle 1/2\rangle_k}{k!} 6^{2k-m}\binom{k}{m-k} x^m - x = 2\sum_{k=1}^{\infty}(-1)^{k-1} S_{k-1} x^k,$$

$$\sum_{m=1}^{\infty}\sum_{k=\lceil m/2\rceil}^{m} \frac{\langle 1/2\rangle_k}{k!} 6^{2k-m}\binom{k}{m-k} x^m - x = 2\sum_{m=1}^{\infty}(-1)^{m-1} S_{m-1} x^m,$$

$$\sum_{k=\lceil m/2\rceil}^{m} \frac{\langle 1/2\rangle_k}{k!} 6^{2k-m}\binom{k}{m-k} = 2(-1)^{m-1} S_{m-1},$$

$$S_{m-1} = \frac{(-1)^{m-1}}{2}\sum_{k=\lceil m/2\rceil}^{m} \frac{\langle 1/2\rangle_k}{k!} 6^{2k-m}\binom{k}{m-k},$$

$$S_m = \frac{(-1)^m}{2}\sum_{k=\lceil (m+1)/2\rceil}^{m+1} \frac{\langle 1/2\rangle_k}{k!} 6^{2k-m-1}\binom{k}{m-k+1}, \quad m \ge 1.$$

The explicit formula (1.4) is thus proved once again. □

Proof of the formula (1.5). The generating function (1.1) can be rearranged as

$$G(x)=\frac{1}{2}\biggl(\frac{1}{x}-1-\sqrt{1-\frac{6}{x}+\frac{1}{x^2}}\biggr),\quad x>0.$$

Then, by virtue of the formulas (2.1) and (2.6), we see that the nth derivative of $G(x)$ for $n\in\mathbb{N}$ equals

$$\begin{aligned}
2G^{(n)}(x)&=\frac{(-1)^n n!}{x^{n+1}}-\sum_{k=0}^{n}\bigl(\sqrt{u}\bigr)^{(k)}\mathrm{B}_{n,k}\bigl(u'(x),u''(x),\dots,u^{(n-k+1)}(x)\bigr)\\
&=\frac{(-1)^n n!}{x^{n+1}}-\sum_{k=0}^{n}\biggl\langle\frac{1}{2}\biggr\rangle_k\biggl(1-\frac{6}{x}+\frac{1}{x^2}\biggr)^{1/2-k}\\
&\quad\times\mathrm{B}_{n,k}\biggl(\frac{6}{x^2}-\frac{2}{x^3},-\frac{12}{x^3}+\frac{6}{x^4},\dots,(-1)^{n-k}\frac{6(n-k+1)!}{x^{n-k+2}}\\
&\quad+(-1)^{n-k+1}\frac{(n-k+2)!}{x^{n-k+3}}\biggr)\\
&=\frac{(-1)^n n!}{x^{n+1}}-\sum_{k=0}^{n}\biggl\langle\frac{1}{2}\biggr\rangle_k\biggl(1-\frac{6}{x}+\frac{1}{x^2}\biggr)^{1/2-k}\\
&\quad\times\sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}\mathrm{B}_{\ell,r}\biggl(\frac{6}{x^2},-\frac{12}{x^3},\dots,(-1)^{\ell-r}\frac{6(\ell-r+1)!}{x^{\ell-r+2}}\biggr)\\
&\quad\times\mathrm{B}_{m,s}\biggl(-\frac{2}{x^3},\frac{6}{x^4},\dots,(-1)^{m-s+1}\frac{(m-s+2)!}{x^{m-s+3}}\biggr),
\end{aligned}$$

where $u(x)=1-\frac{6}{x}+\frac{1}{x^2}$. Further making use of the formulas (2.2), (2.7), and (2.8) results in

$$
\begin{aligned}
2G^{(n)}(x) &= \frac{(-1)^n n!}{x^{n+1}} - \sum_{k=0}^{n}\left\langle \frac{1}{2}\right\rangle_k \left(1-\frac{6}{x}+\frac{1}{x^2}\right)^{1/2-k}\\
&\quad\times \sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}6^r(-1)^{\ell+r}\frac{1}{x^{\ell+r}}\mathrm{B}_{\ell,r}(1!,2!,\dots,(\ell-r+1)!)\\
&\quad\times(-1)^m\frac{1}{x^{m+2s}}\mathrm{B}_{m,s}(2!,3!,\dots,(m-s+2)!)\\
&= \frac{(-1)^n n!}{x^{n+1}} - \sum_{k=0}^{n}\left\langle \frac{1}{2}\right\rangle_k \left(1-\frac{6}{x}+\frac{1}{x^2}\right)^{1/2-k}\sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}6^r(-1)^{m+\ell+r}\\
&\quad\times\frac{1}{x^{m+2s+\ell+r}}\binom{\ell}{r}\binom{\ell-1}{r-1}(\ell-r)!\frac{1}{s!}\sum_{q=0}^{s}(-1)^{s-q}\binom{s}{q}\frac{(m+2q-1)!}{(2q-1)!}\\
&= \frac{(-1)^n n!}{x^{n+1}} - \frac{(-1)^n}{x^n}\sum_{k=0}^{n}\left\langle \frac{1}{2}\right\rangle_k \left(1-\frac{6}{x}+\frac{1}{x^2}\right)^{1/2-k}\frac{1}{x^k}\sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}6^r\\
&\quad\times\frac{(-1)^r}{x^s}\binom{\ell}{r}\binom{\ell-1}{r-1}(\ell-r)!\frac{1}{s!}\sum_{q=0}^{s}(-1)^{s-q}\binom{s}{q}\frac{(m+2q-1)!}{(2q-1)!}\\
&= \frac{(-1)^n}{x^{n+1}}\Bigg[n! - \sum_{k=0}^{n}\left\langle \frac{1}{2}\right\rangle_k \left(x^2-6x+1\right)^{1/2-k}x^k\sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}6^r(-1)^r\\
&\quad\times\frac{1}{x^s}\binom{\ell}{r}\binom{\ell-1}{r-1}(\ell-r)!\frac{1}{s!}\sum_{q=0}^{s}(-1)^{s-q}\binom{s}{q}\frac{(m+2q-1)!}{(2q-1)!}\Bigg]\\
&\to \frac{(-1)^n}{(n+1)!}\lim_{x\to0^+}\Bigg[n! - \sum_{k=0}^{n}\left\langle \frac{1}{2}\right\rangle_k \sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}6^r(-1)^r\binom{\ell}{r}\binom{\ell-1}{r-1}\\
&\quad\times\frac{(\ell-r)!}{s!}\sum_{q=0}^{s}(-1)^{s-q}\binom{s}{q}\frac{(m+2q-1)!}{(2q-1)!}\left(x^2-6x+1\right)^{1/2-k}x^{k-s}\Bigg]^{(n+1)}\\
&= \frac{(-1)^{n+1}}{(n+1)!}\sum_{k=0}^{n}\left\langle \frac{1}{2}\right\rangle_k \sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}6^r(-1)^r\binom{\ell}{r}\binom{\ell-1}{r-1}\frac{(\ell-r)!}{s!}\\
&\quad\times\sum_{q=0}^{s}(-1)^{s-q}\binom{s}{q}\frac{(m+2q-1)!}{(2q-1)!}\lim_{x\to0^+}\left[\left(x^2-6x+1\right)^{1/2-k}x^{k-s}\right]^{(n+1)}\\
&= \frac{(-1)^{n+1}}{(n+1)!}\sum_{k=0}^{n}\left\langle \frac{1}{2}\right\rangle_k \sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}6^r(-1)^r\binom{\ell}{r}\binom{\ell-1}{r-1}\frac{(\ell-r)!}{s!}\sum_{q=0}^{s}(-1)^{s-q}\\
&\quad\times\binom{s}{q}\frac{(m+2q-1)!}{(2q-1)!}\lim_{x\to0^+}\sum_{p=0}^{n+1}\binom{n+1}{p}\left[\left(x^2-6x+1\right)^{1/2-k}\right]^{(n-p+1)}\left(x^{k-s}\right)^{(p)}\\
&= \frac{(-1)^{n+1}}{(n+1)!}\sum_{k=0}^{n}\left\langle \frac{1}{2}\right\rangle_k \sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}6^r(-1)^r\binom{\ell}{r}\binom{\ell-1}{r-1}\frac{(\ell-r)!}{s!}\sum_{q=0}^{s}(-1)^{s-q}\\
&\quad\times\binom{s}{q}\frac{(m+2q-1)!}{(2q-1)!}(k-s)!\binom{n+1}{k-s}\lim_{x\to0^+}\left[\left(x^2-6x+1\right)^{1/2-k}\right]^{(n-k+s+1)}
\end{aligned}
$$

as $t \to 0^+$. By the formulas (2.1), (2.2), and (2.3), we have

$$\begin{aligned}
&\lim_{x\to0^+}\bigl[\bigl(x^2-6x+1\bigr)^{1/2-k}\bigr]^{(n-k+s+1)}\\
&=\lim_{x\to0^+}\sum_{j=0}^{n-k+s-1}\bigl(u^{1/2-k}\bigr)^{(j)}\mathrm{B}_{n-k+s+1,j}(2x-6,2,0\dots,0)\\
&=\lim_{x\to0^+}\sum_{j=0}^{n-k+s-1}\biggl\langle\frac12-k\biggr\rangle_j u^{1/2-k-j}\mathrm{B}_{n-k+s+1,j}(2x-6,2,0\dots,0)\\
&=\sum_{j=0}^{n-k+s-1}\biggl\langle\frac12-k\biggr\rangle_j \mathrm{B}_{n-k+s+1,j}(-6,2,0\dots,0)\\
&=\sum_{j=0}^{n-k+s-1}(-1)^{2j-n+k-s-1}\biggl\langle\frac12-k\biggr\rangle_j\frac{(n-k-j+s+1)!}{6^{n-k-2j+s+1}}\\
&\quad\times\binom{n-k+s+1}{j}\binom{j}{n-k-j+s+1},
\end{aligned}$$

where $u=u(x)=x^2-6x+1$. As a result, from the generating function (1.1), it follows that

$$\begin{aligned}
S_n&=\frac{1}{n!}G^{(n)}(0)\\
&=\frac12\frac{(-1)^{n+1}}{(n+1)!}\sum_{k=0}^{n}\biggl\langle\frac12\biggr\rangle_k\sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}6^r(-1)^r\binom{\ell}{r}\binom{\ell-1}{r-1}\frac{(\ell-r)!}{s!}\\
&\quad\times\sum_{q=0}^{s}(-1)^{s-q}\binom{s}{q}\frac{(m+2q-1)!}{(2q-1)!}(k-s)!\binom{n+1}{k-s}\sum_{j=0}^{n-k+s-1}(-1)^{2j-n+k-s-1}\\
&\quad\times\biggl\langle\frac12-k\biggr\rangle_j\frac{(n-k-j+s+1)!}{6^{n-k-2j+s+1}}\binom{n-k+s+1}{j}\binom{j}{n-k-j+s+1}
\end{aligned}$$

which can be simplified as the formula (1.5). The proof of Theorem 1 is complete. □

Remark 1. This paper is a slightly revised version of the preprint [11] and a companion of the preprint [10].

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