CONGRUENCES FOR 3-REGULAR PARTITIONS WITH DESIGNATED SUMMANDS

M. S. Mahadeva Naika
Department of Mathematics, Bangalore University, Central College Campus, Bengaluru, Karnataka, India
msmnaika@rediffmail.com

D. S. Gireesh
Department of Mathematics, Bangalore University, Central College Campus, Bengaluru, Karnataka, India
gireeshdap@gmail.com

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Abstract

Andrews, Lewis and Lovejoy introduced the partition functions $PD(n)$ and $PDO(n)$ defined by the number of partitions of n with designated summands and the number of partitions of n with designated summands in which all parts are odd, respectively. They found several generating function identities and congruences modulo 3, 4, and 24 satisfied by the functions. In this paper, we find generating function identities and congruences modulo 4, 9, 12, 36, 48, and 144 for $PD_3(n)$, which represents the number of partitions of n with designated summands, whose parts are not divisible by 3.

1. Introduction

Andrews, Lewis and Lovejoy [1] introduced a new class of partitions, partitions with designated summands which are constructed by taking ordinary partitions and tagging exactly one part among parts with equal size. Fifteen partitions of 5 with designated summands are

$$5', \quad 4'+1', \quad 3'+2', \quad 3'+1'+1, \quad 3'+1+1', \quad 2'+2+1', \quad 2+2'+1',$$
$$2'+1'+1+1, \quad 2'+1+1'+1, \quad 2'+1+1+1', \quad 1'+1+1+1+1,$$
$$1+1'+1+1+1, \quad 1+1+1'+1+1, \quad 1+1+1+1'+1, \quad 1+1+1+1+1'.$$

The concept of partitions with designated summands goes back to MacMahon [10]. He considered partitions with designated summands and with exactly ℓ different sizes, see also Andrews and Rose [2]. The authors [1] derived the following generating function of $PD(n)$:

$$\sum_{n=0}^{\infty} PD(n)q^n = \frac{f_6}{f_1 f_2 f_3}.$$

Throughout the paper, we use the standard q-series notation, and f_k is defined as

$$f_k := (q^k;q^k)_\infty = \lim_{n\to\infty} \prod_{m=1}^{n} (1-q^{mk}), \qquad |q|<1.$$

For $|ab|<1$, Ramanujan's general theta function $f(a,b)$ is defined as

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}. \tag{1}$$

Using Jacobi's triple product identity [5, Entry 19, p.35], (1) becomes

$$f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty .$$

The most important special cases of $f(a,b)$ are

$$\psi(q) := f\left(q,q^3\right) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{f_2^2}{f_1}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = f_1.$$

Andrews *et al.* [1] derived the generating functions for $PD(2n)$ and $PD(2n+1)$ and they proved Ramanujan type congruences modulo 3 and powers of 2 for $PD(n)$. In particular, they proved that for $n \geq 0$,

$$PD(3n+2) \equiv 0 \pmod{3}. \tag{2}$$

Chen, Ji, Jin and Shen [7] established a Ramanujan type identity for the partition function $PD(3n+2)$ which implies the congruence (2) and they also gave a combinatorial interpretation of (2) by introducing a rank for partitions with designated summands. Xia [11] extended the work of deriving congruence properties of $PD(n)$ by employing the generating functions of $PD(3n)$ and $PD(3n+2)$ due to Chen *et al.* [7].

The authors [1] also studied $PDO(n)$, the total number of partitions of n with designated summands in which all parts are odd, and the generating function is given by

$$\sum_{n=0}^{\infty} PDO(n)q^n = \frac{f_4 f_6^2}{f_1 f_3 f_{12}}.$$

Thus $PDO(5) = 8$ are

$$5', \quad 3'+1'+1, \quad 3'+1+1', \quad 1'+1+1+1+1, \quad 1+1'+1+1+1,$$
$$1+1+1'+1+1, \quad 1+1+1+1'+1, \quad 1+1+1+1+1'.$$

They also established generating functions for $PDO(2n)$, $PDO(2n+1)$, $PDO(3n)$, $PD(3n+1)$, $PD(3n+2)$ by using q-series and modular forms and later Baruah and Ojah [4] proved the same by using theta function dissections. Baruah and Ojah also obtained generating functions for $PDO(4n)$, $PDO(4n+1)$, $PDO(4n+2)$, $PDO(4n+3)$, $PDO(6n)$, $PDO(6n+2)$, $PDO(6n+3)$, $PDO(6n+5)$, $PDO(9n+3)$, $PDO(9n+6)$, $PDO(12n)$, $PDO(12n+2)$, $PDO(12n+3)$, $PDO(12n+6)$, $PDO(12n+9)$, $PDO(12n+10)$ and Ramanujan like congruences for $PDO(n)$.

Motivated by the above work, in this paper, we study $PD_3(n)$, the total number of partitions of n with designated summands, whose parts are not divisible by 3 and the generating function is given by

$$\sum_{n=0}^{\infty} PD_3(n)q^n = \frac{f_6^2 f_9}{f_1 f_2 f_{18}}. \tag{3}$$

Thus $PD_3(5) = 12$ are

$$5', \quad 4'+1', \quad 2'+2+1', \quad 2+2'+1', \quad 2'+1'+1+1, \quad 2'+1+1'+1,$$
$$2'+1+1+1', \quad 1'+1+1+1+1, \quad 1+1'+1+1+1, \quad 1+1+1'+1+1,$$
$$1+1+1+1'+1, \quad 1+1+1+1+1'.$$

In Section 3, we prove the following theorems.

Theorem 1. *We have*

$$\sum_{n=0}^{\infty} PD_3(2n)q^n = \frac{f_3 f_6^3}{f_1^3 f_{18}}, \tag{4}$$

$$\sum_{n=0}^{\infty} PD_3(2n+1)q^n = \frac{f_2^2 f_3^3 f_{18}}{f_1^4 f_6 f_9}, \tag{5}$$

$$\sum_{n=0}^{\infty} PD_3(3n)q^n = \frac{f_3^4 f_6^2}{f_1^4 f_2^2}\left(\frac{1}{\nu^2(q)} - 2q\nu(q)\right), \tag{6}$$

$$\sum_{n=0}^{\infty} PD_3(3n+1)q^n = \frac{f_3^4 f_6^2}{f_1^4 f_2^2}\left(\frac{1}{\nu(q)} + 4q\nu^2(q)\right), \tag{7}$$

$$\sum_{n=0}^{\infty} PD_3(3n+2)q^n = 3\frac{f_3^4 f_6^2}{f_1^4 f_2^2}, \tag{8}$$

$$\sum_{n=0}^{\infty} PD_3(4n)q^n = \frac{f_2^6 f_3^6}{f_1^9 f_6^2 f_9}, \tag{9}$$

$$\sum_{n=0}^{\infty} PD_3(4n+2)q^n = 3\frac{f_2^2 f_3^4 f_6^2}{f_1^7 f_9}, \tag{10}$$

$$\sum_{n=0}^{\infty} PD_3(6n+2)q^n = 3\frac{f_2^9 f_3^7}{f_1^{13} f_6^3} + 9q\frac{f_2 f_3^3 f_6^5}{f_1^9}, \tag{11}$$

$$\sum_{n=0}^{\infty} PD_3(6n+5)q^n = 12\frac{f_2^5 f_3^5 f_6}{f_1^{11}}, \tag{12}$$

where

$$\nu(q) = \frac{f_1 f_6^3}{f_2 f_3^3}. \tag{13}$$

Theorem 2. *For each $n \geq 0$,*

$$PD_3(6n+3) \equiv 0 \pmod{4}, \tag{14}$$

$$PD_3(6n+5) \equiv 0 \pmod{12}, \tag{15}$$

$$PD_3(12n+8) \equiv 0 \pmod{48}, \tag{16}$$

$$PD_3(24n+4) \equiv 0 \pmod{9}, \tag{17}$$

$$PD_3(24n+20) \equiv 0 \pmod{144}, \tag{18}$$

$$PD_3(24n+22) \equiv 0 \pmod{36}, \tag{19}$$

$$PD_3(48n+38) \equiv 0 \pmod{12}. \tag{20}$$

Theorem 3. *For each nonnegative integer n and $\alpha \geq 0$, we have*

$$PD_3\left(4 \times 3^{\alpha} n + 2 \times 3^{\alpha}\right) \equiv PD_3(4n+2) \pmod{9}, \tag{21}$$

$$PD_3\left(4 \times 3^{\alpha+1} n + 10 \times 3^{\alpha}\right) \equiv 0 \pmod{9}. \tag{22}$$

In the process of proving the above results, we find the congruence modulo 9 connecting $PD_3(n)$ and $a_3(n)$,

$$PD_3(6n+2) \equiv 3a_3(n) \pmod{9}, \tag{23}$$

where $a_3(n)$ denotes the number of partitions of n that are 3-cores. The generating function for $a_3(n)$ is given by

$$\sum_{n=0}^{\infty} a_3(n)q^n = \frac{f_3^3}{f_1}.$$

Theorem 4. *Let p be a prime with $\left(\frac{-3}{p}\right) = -1$. Then for any nonnegative integer α,*

$$\sum_{n=0}^{\infty} PD_3\left(4p^{2\alpha} n + 2p^{2\alpha}\right) q^n \equiv 3\psi(q)\psi(q^3) \pmod{9}, \tag{24}$$

and for $n \geq 0$, $1 \leq j \leq p-1$,

$$PD_3\left(4p^{2\alpha+1}(pn+j) + 2p^{2\alpha+2}\right) \equiv 0 \pmod{9}. \tag{25}$$

Theorem 5. *If $p \geq 5$ is a prime such that $\left(\frac{-6}{p}\right) = -1$, then for all integers $\alpha \geq 0$,*

$$\sum_{n=0}^{\infty} PD_3\left(48p^{2\alpha}n + 14p^{2\alpha}\right) q^n \equiv 6f_1f_6 \pmod{12}. \tag{26}$$

Theorem 6. *Let $p \geq 5$ be prime and $\left(\frac{-6}{p}\right) = -1$. Then for all integers $n \geq 0$ and $\alpha \geq 1$,*

$$PD_3\left(48p^{2\alpha}n + p^{2\alpha-1}(14p + 48j)\right) \equiv 0 \pmod{12}, \tag{27}$$

where $j = 1, 2, \ldots, p-1$.

2. Preliminaries

We recall 2-dissection identities for certain quotients of theta functions and p-dissection identities of $f(-q)$ and $\psi(q)$ which play key roles in proving our main results.

Lemma 1. *[5, Corollary, p. 49] We have*

$$\psi(q) = f(q^3, q^6) + q\psi(q^9). \tag{28}$$

Lemma 2. *The following 2-dissections hold:*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{29}$$

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}, \tag{30}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}. \tag{31}$$

Lemma 2 is a consequence of Ramanujan's dissection formulas, collected by Berndt [5, Entry 25, p. 40].

Lemma 3. *The following 2-dissection holds:*

$$\frac{f_3}{f_1} = \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q\frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}. \tag{32}$$

Xia and Yao [14] proved (32) by employing Jacobi triple product identity.

Lemma 4. *The following 2-dissections hold:*

$$\frac{1}{f_1f_3}=\frac{f_8^2f_{12}^5}{f_2^2f_4f_6^4f_{24}^2}+q\frac{f_4^5f_{24}^2}{f_2^4f_6^2f_8^2f_{12}}, \tag{33}$$

$$\frac{f_1^3}{f_3}=\frac{f_4^3}{f_{12}}-3q\frac{f_2^2f_{12}^3}{f_4f_6^2}, \tag{34}$$

$$\frac{f_3^3}{f_1}=\frac{f_4^3f_6^2}{f_2^2f_{12}}+q\frac{f_{12}^3}{f_4}, \tag{35}$$

$$\frac{f_3}{f_1^3}=\frac{f_4^6f_6^3}{f_2^9f_{12}^2}+3q\frac{f_4^2f_6f_{12}^2}{f_2^7}. \tag{36}$$

Equation (33) was proved by Baruah and Ojah [3], and (34) and (35) was proved by Hirschhorn, Garvan, and Borwein [9]. Replacing q by $-q$ in (34) and using the relation

$$(-q;-q)_\infty=\frac{f_2^3}{f_1f_4},$$

we obtain (36).

Lemma 5. *[12, p. 93, Eq. (2.14)] The following 2-dissection holds:*

$$\frac{f_3^2}{f_1^2}=\frac{f_4^4f_6f_{12}^2}{f_2^5f_8f_{24}}+2q\frac{f_4f_6^2f_8f_{24}}{f_2^4f_{12}}. \tag{37}$$

Lemma 6. *The following 2-dissection holds:*

$$\frac{f_9}{f_1}=\frac{f_{12}^3f_{18}}{f_2^2f_6f_{36}}+q\frac{f_4^2f_6f_{36}}{f_2^3f_{12}}. \tag{38}$$

Lemma 6 was proved by Xia and Yao [13].

Lemma 7. *[6, Eq. (13)] We have*

$$\frac{1}{f_1f_2}=\frac{f_9^3f_{18}^3}{f_3^4f_6^4}\left\{\frac{1}{\nu^2(q^3)}-2q^3\nu(q^3)+q\left(\frac{1}{\nu(q^3)}+4q^3\nu^2(q^3)\right)+3q^2\right\}. \tag{39}$$

Lemma 8. *[8, Theorem 2.1] For any odd prime p,*

$$\psi(q)=\sum_{m=0}^{\frac{p-3}{2}}q^{\frac{m^2+m}{2}}f\left(q^{\frac{p^2+(2m+1)p}{2}},q^{\frac{p^2-(2m+1)p}{2}}\right)+q^{\frac{p^2-1}{8}}\psi(q^{p^2}). \tag{40}$$

Furthermore, $\frac{m^2+m}{2}\not\equiv\frac{p^2-1}{8} \pmod p$ *for* $0\le m\le\frac{p-3}{2}$.

Lemma 9. *[8, Theorem 2.2] For any prime $p\ge 5$,*

$$f_1=\sum_{\substack{k=-\frac{p-1}{2}\\ k\neq(\pm p-1)/6}}^{\frac{p-1}{2}}(-1)^kq^{\frac{3k^2+k}{2}}f\left(-q^{\frac{3p^2+(6k+1)p}{2}},-q^{\frac{3p^2-(6k+1)p}{2}}\right)+(-1)^{\frac{\pm p-1}{6}}q^{\frac{p^2-1}{24}}f_{p^2}. \tag{41}$$

Furthermore, for $-(p-1)/2 \le k \le (p-1)/2$ *and* $k \ne (\pm p - 1)/6$,

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod{p}.$$

3. Proofs of Main Results

Proof of Theorem 1. Substituting (38) into (3), we find that

$$\sum_{n=0}^{\infty} PD_3(n)q^n = \frac{f_6^2}{f_2 f_{18}}\left(\frac{f_{12}^3 f_{18}}{f_2^2 f_6 f_{36}} + q\frac{f_4^2 f_6 f_{36}}{f_2^3 f_{12}}\right).$$

Extracting the terms involving even and odd powers of q from the above equation, we obtain (4) and (5), respectively.

Invoking (3) and (39),

$$\sum_{n=0}^{\infty} PD_3(n)q^n = \frac{f_9^4 f_{18}^2}{f_3^4 f_6^2}\left\{\frac{1}{\nu^2(q^3)} - 2q^3\nu(q^3) + q\left(\frac{1}{\nu(q^3)} + 4q^3\nu^2(q^3)\right) + 3q^2\right\}. \tag{42}$$

Extracting the terms involving q^{3n}, q^{3n+1}, and q^{3n+2} from (42), we arrive at (6), (7), and (8), respectively.

Substituting (36) into (4),

$$\sum_{n=0}^{\infty} PD_3(2n)q^n = \frac{f_6^3}{f_{18}}\left(\frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}\right),$$

which yields (9) and (10).

Applying (35) and (36) in (8), we obtain

$$\sum_{n=0}^{\infty} PD_3(3n+2)q^n = 3\frac{f_6^2}{f_2^2}\left(\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}\right)\left(\frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}\right).$$

Extracting the terms involving q^{2n} and q^{2n+1} from both sides of the above equation, we arrive at (11) and (12), respectively.

Proof of Theorem 2. From the Binomial Theorem, for any positive integer, k,

$$f_k^2 \equiv f_{2k} \pmod{2}, \tag{43}$$
$$f_k^4 \equiv f_{2k}^2 \pmod{4}, \tag{44}$$
$$f_k^3 \equiv f_{3k} \pmod{3}, \tag{45}$$
$$f_k^9 \equiv f_{3k}^3 \pmod{9}. \tag{46}$$

Using (44), (5) can expressed as

$$\sum_{n=0}^{\infty} PD_3(2n+1)q^n \equiv \frac{f_3^3 f_{18}}{f_6 f_9} \pmod{4}. \tag{47}$$

Equating the coefficients of q^{3n+1} from both sides of (47), we obtain (14).

From (12), we easily arrive at (15).

Substituting (30) and (31) into (8), we find that

$$\sum_{n=0}^{\infty} PD_3(3n+2)q^n = 3\frac{f_6^2}{f_2^2}\left(\frac{f_{12}^{10}}{f_6^2 f_{24}^4} - 4q^3\frac{f_6^2 f_{24}^4}{f_{12}^2}\right)\left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\right),$$

which yields

$$\sum_{n=0}^{\infty} PD_3(6n+2)q^n \equiv 3\frac{f_2^{14} f_6^{10}}{f_1^{16} f_4^4 f_{12}^4} \pmod{48}. \tag{48}$$

Using (31) in (48),

$$\sum_{n=0}^{\infty} PD_3(6n+2)q^n \equiv 3\frac{f_2^{14} f_6^{10}}{f_4^4 f_{12}^4}\left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\right)^4 \pmod{48},$$

which implies

$$\sum_{n=0}^{\infty} PD_3(6n+2)q^n \equiv 3\frac{f_4^{52} f_6^{10}}{f_2^{42} f_8^{16} f_{12}^4} \pmod{48}. \tag{49}$$

Equating the coefficients of q^{2n+1} from (49), we arrive at (16).

However, from (46),

$$\frac{f_2^6 f_3^6}{f_1^9 f_6^2 f_9} \equiv \frac{f_2^6 f_3^3}{f_6^2 f_9} \pmod{9}. \tag{50}$$

Using (50) in (9), we obtain

$$\sum_{n=0}^{\infty} PD_3(4n)q^n \equiv \frac{f_2^6 f_3^3}{f_6^2 f_9} \pmod{9}. \tag{51}$$

Replacing q by q^3 in (34) and then substituting the resulting equation into (51),

$$\sum_{n=0}^{\infty} PD_3(4n)q^n \equiv \frac{f_2^6}{f_6^2}\left(\frac{f_{12}^3}{f_{36}} - 3q^3\frac{f_6^2 f_{36}^3}{f_{12} f_{18}^2}\right) \pmod{9},$$

which implies

$$\sum_{n=0}^{\infty} PD_3(8n+4)q^n \equiv -3q\frac{f_1^6 f_{18}^3}{f_6 f_9^2} \pmod{9}. \tag{52}$$

From (45),

$$\frac{f_1^6 f_{18}^3}{f_6 f_9^2} \equiv \frac{f_3^2 f_{18}^3}{f_6 f_9^2} \pmod 3. \tag{53}$$

Substituting (53) into (52), we find that

$$\sum_{n=0}^{\infty} PD_3(8n+4)q^n \equiv -3q\frac{f_3^2 f_{18}^3}{f_6 f_9^2} \pmod 9. \tag{54}$$

Equating the coefficients of q^{3n} from (54), we arrive at (17).

From (54),

$$PD_3(24n+20) \equiv 0 \pmod 9. \tag{55}$$

Replacing n by $2n+1$ in (16), we find that

$$PD_3(24n+20) \equiv 0 \pmod{48}. \tag{56}$$

Combining (55) and (56), we arrive at (18).

From (7), it is easy to see that

$$\sum_{n=0}^{\infty} PD_3(3n+1)q^n \equiv \frac{f_3^4 f_6^2}{f_1^4 f_2^2}\frac{1}{\nu(q)} \pmod 4. \tag{57}$$

However, from (44),

$$\frac{f_3^4 f_6^2}{f_1^4 f_2^2} \equiv \frac{f_{12}^2}{f_4^2} \pmod 4. \tag{58}$$

Substituting (13) and (58) into (57), we obtain

$$\sum_{n=0}^{\infty} PD_3(3n+1)q^n \equiv \frac{f_2 f_3^3 f_{12}^2}{f_1 f_6^3 f_4^2} \pmod 4. \tag{59}$$

Invoking (35) and (59),

$$\sum_{n=0}^{\infty} PD_3(3n+1)q^n \equiv \frac{f_2 f_{12}^2}{f_6^3 f_4^2}\left(\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}\right) \pmod 4,$$

which yields

$$\sum_{n=0}^{\infty} PD_3(6n+4)q^n \equiv \frac{f_1 f_6^5}{f_2^3 f_3^3} \pmod 4. \tag{60}$$

From (44),

$$\frac{f_1 f_6^5}{f_2^3 f_3^3} \equiv \frac{f_6^3 f_3}{f_2 f_1^3} \pmod 4. \tag{61}$$

Following (61), we can express (60) as

$$\sum_{n=0}^{\infty} PD_3(6n+4)q^n \equiv \frac{f_6^3 f_3}{f_2 f_1^3} \pmod 4. \tag{62}$$

Substituting (36) into (62), we find that

$$\sum_{n=0}^{\infty} PD_3(6n+4)q^n \equiv \frac{f_6^3}{f_2}\left(\frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}\right) \pmod 4.$$

Extracting those terms containing q^{2n+1}, dividing throughout by q and then replacing q^2 by q from the above

$$\begin{aligned}\sum_{n=0}^{\infty} PD_3(12n+10)q^n &\equiv 3\frac{f_2^2 f_3^4 f_6^2}{f_1^8} \pmod 4\\ &\equiv 3\frac{f_6^4}{f_2^2} \pmod 4. \end{aligned}\tag{63}$$

Equating the coefficients of q^{2n+1} from (63), we obtain

$$PD_3(24n+22) \equiv 0 \pmod 4. \tag{64}$$

From (45),

$$\frac{f_2^2 f_3^4 f_6^2}{f_1^7 f_9} \equiv \psi(q)\psi(q^3) \pmod 3. \tag{65}$$

Invoking (10) and (65),

$$\sum_{n=0}^{\infty} PD_3(4n+2)q^n \equiv 3\psi(q)\psi(q^3) \pmod 9. \tag{66}$$

Using (28) in (66), we find that

$$\sum_{n=0}^{\infty} PD_3(4n+2)q^n \equiv 3\psi(q^3)\left(f(q^3,q^6) + q\psi(q^9)\right) \pmod 9, \tag{67}$$

which yields

$$PD_3(12n+10) \equiv 0 \pmod 9. \tag{68}$$

From (64) and (68), we arrive at (19).

Using (44), (8) can be expressed as

$$\sum_{n=0}^{\infty} PD_3(3n+2)q^n \equiv 3\frac{f_{12}^2}{f_4^2} \pmod{12}. \tag{69}$$

Replacing q by q^4 in (37) and then using the resulting equation in (69), we obtain

$$\sum_{n=0}^{\infty} PD_3(3n+2)q^n \equiv 3\frac{f_{16}^4 f_{24} f_{48}^2}{f_8^5 f_{32} f_{96}} + 6q^4 \frac{f_{16} f_{24}^2 f_{32} f_{96}}{f_8^4 f_{48}} \pmod{12},$$

which implies

$$\sum_{n=0}^{\infty} PD_3(24n+14)q^n \equiv 6\frac{f_2 f_3^2 f_4 f_{12}}{f_1^4 f_6} \pmod{12}. \tag{70}$$

From (43),

$$\frac{f_2 f_3^2 f_4 f_{12}}{f_1^4 f_6} \equiv f_2 f_{12} \pmod{2}. \tag{71}$$

Following (71), we can express (70) as

$$\sum_{n=0}^{\infty} PD_3(24n+14)q^n \equiv 6 f_2 f_{12} \pmod{12}. \tag{72}$$

Equating the coefficients of q^{2n+1} from the above, we arrive at (20).

Proof of Theorem 3. Extracting the terms involving q^{3n+1} from (67),

$$\sum_{n=0}^{\infty} PD_3(12n+6)q^n \equiv 3\psi(q)\psi(q^3) \pmod{9}. \tag{73}$$

Invoking (66) and (73), we find that

$$\sum_{n=0}^{\infty} PD_3(12n+6)q^n \equiv \sum_{n=0}^{\infty} PD_3(4n+2)q^n \pmod{9},$$

which yields, for each $n \geq 0$,

$$PD_3(12n+6) \equiv PD_3(4n+2) \pmod{9}. \tag{74}$$

The congruence (21) follows from (74) and by mathematical induction.

Replacing n by $3n+2$ in (21) and then using (68), we arrive at (22).

From (11), it is easy to see that

$$\sum_{n=0}^{\infty} PD_3(6n+2)q^n = 3\frac{f_2^9 f_3^7}{f_1^{13} f_6^3} \pmod{9}. \tag{75}$$

However, from (45),

$$\frac{f_2^9 f_3^7}{f_1^{13} f_6^3} \equiv \frac{f_3^3}{f_1} \pmod{3}. \tag{76}$$

Substituting (76) into (75), we find that

$$\sum_{n=0}^{\infty} PD_3(6n+2)q^n \equiv 3\frac{f_3^3}{f_1} \equiv 3\sum_{n=0}^{\infty} a_3(n)q^n \pmod 9. \tag{77}$$

Equating the coefficients of q^n from (77), we arrive at (23).

Proof of Theorem 4. Equation (66) is the $\alpha = 0$ case of (24). If we assume that (24) holds for some $\alpha \geq 0$, then, substituting (40) in (24),

$$\begin{aligned}
&\sum_{n=0}^{\infty} PD_3\left(4p^{2\alpha}n + 2p^{2\alpha}\right)q^n \\
&\equiv 3\left(\sum_{m=0}^{\frac{p-3}{2}} q^{\frac{m^2+m}{2}} f\left(q^{\frac{p^2+(2m+1)p}{2}}, q^{\frac{p^2-(2m+1)p}{2}}\right) + q^{\frac{p^2-1}{8}}\psi(q^{p^2})\right) \\
&\times\left(\sum_{m=0}^{\frac{p-3}{2}} q^{3\frac{m^2+m}{2}} f\left(q^{3\frac{p^2+(2m+1)p}{2}}, q^{3\frac{p^2-(2m+1)p}{2}}\right) + q^{3\frac{p^2-1}{8}}\psi(q^{3p^2})\right) \pmod 9.
\end{aligned} \tag{78}$$

For any odd prime, p, and $0 \leq m_1, m_2 \leq (p-3)/2$, consider the congruence

$$\frac{m_1^2+m_1}{2} + 3\frac{m_2^2+m_2}{2} \equiv \frac{4p^2-4}{8} \pmod p,$$

which implies that

$$(2m_1+1)^2 + 3(2m_2+1)^2 \equiv 0 \pmod p. \tag{79}$$

Since $\left(\frac{-3}{p}\right) = -1$, the only solution of the congruence (79) is $m_1 = m_2 = \dfrac{p-1}{2}$. Therefore, equating the coefficients of $q^{pn+\frac{4p^2-4}{8}}$ from both sides of (78), dividing throughout by $q^{\frac{4p^2-4}{8}}$ and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} PD_3\left(4p^{2\alpha}\left(pn + \frac{4p^2-4}{8}\right) + 2p^{2\alpha}\right)q^n \equiv 3\psi(q^p)\psi(q^{3p}) \pmod 9. \tag{80}$$

Equating the coefficients of q^{pn} on both sides of (80) and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} PD_3\left(4p^{2\alpha+2}n + 2p^{2\alpha+2}\right)q^n \equiv 3\psi(q)\psi(q^3) \pmod 9,$$

which is the $\alpha + 1$ case of (24).

Extracting the terms involving q^{pn+j} for $1 \leq j \leq p-1$ in (80), we arrive at (25).

Proof of Theorem 5. Extracting the terms involving q^{2n} from (72) and then replacing q^2 by q,

$$\sum_{n=0}^{\infty} PD_3(48n+14)q^n \equiv 6f_1f_6 \pmod{12}. \tag{81}$$

For a prime $p \geq 5$ and $-(p-1)/2 \leq k, m \leq (p-1)/2$, consider

$$\frac{3k^2+k}{2} + 6 \times \frac{3m^2+m}{2} \equiv \frac{7p^2-7}{24} \ (\mathrm{mod}\ p),$$

therefore,

$$(6k+1)^2 + 6(6m+1)^2 \equiv 0 \ (\mathrm{mod}\ p).$$

Since $\left(\frac{-6}{p}\right) = -1$, the only solution of the above congruence is $k = m = (\pm p - 1)/6$. Therefore, from Lemma 9,

$$\sum_{n=0}^{\infty} PD_3\left(48\left(p^2 n + 7 \times \frac{p^2-1}{24}\right) + 14\right) q^n \equiv 6f_1f_6 \pmod{12}. \tag{82}$$

Using (81), (82), and induction on α, we arrive at (26).

Proof of Theorem 6. From Lemma 9 and Theorem 5, for each $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} PD_3\left(48p^{2\alpha}\left(pn + 7 \times \frac{p^2-1}{24}\right) + 14p^{2\alpha}\right) q^n \equiv 6f_pf_{6p} \pmod{12}.$$

That is,

$$\sum_{n=0}^{\infty} PD_3\left(48p^{2\alpha+1}n + 14p^{2\alpha+2}\right) q^n \equiv 6f_pf_{6p} \pmod{12}. \tag{83}$$

Since there are no terms on the right of (83) where the powers of q are congruent to $1, 2, \ldots, p-1$ modulo p,

$$PD_3\left(48p^{2\alpha+1}(pn+j) + 14p^{2\alpha+2}\right) \equiv 0 \pmod{12},$$

for $j = 1, 2, \ldots, p-1$. Therefore, for $j = 1, 2, \ldots, p-1$ and $\alpha \geq 1$, we arrive at (27).

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