ON GENERALIZED FIBONACCI AND LUCAS NUMBERS OF THE FORM $wkx^2 \pm 1$

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Abstract

Let P be an odd integer, (U_n) and (V_n) denote generalized Fibonacci and Lucas sequences defined by $U_0 = 0$, $U_1 = 1$, and $U_{n+1} = PU_n + U_{n-1}$, $V_0 = 2$, $V_1 = P$, and $V_{n+1} = PV_n + V_{n-1}$ for $n \geq 1$. In this paper, we solve the equations $U_n = kx^2 \pm 1$ under some conditions on n. Moreover, we determine all indices n such that the equations $V_n = wkx^2 \pm 1$, where $w \in \{1, 2, 3, 6\}$, $k|P$ with $k > 1$, have solutions.

1. Introduction

Let P and Q be nonzero integers, let $D = P^2 + 4Q$ be called the discriminant, and assume that $D > 0$ (to exclude degenerate cases). Consider the polynomial $X^2 - PX - Q$, called characteristic polynomial, which has the roots

$$\alpha = \frac{P + \sqrt{D}}{2} \quad \text{and} \quad \beta = \frac{P - \sqrt{D}}{2}.$$

For each $n \geq 0$, define $U_n = U_n(P, Q)$ and $V_n = V_n(P, Q)$ as follows:

$$U_0 = 0, \quad U_1 = 1, \quad U_{n+1} = PU_n + QU_{n-1} \quad (\text{for } n \geq 1),$$

$$V_0 = 2, \quad V_1 = P, \quad V_{n+1} = PV_n + QV_{n-1}, \quad (\text{for } n \geq 1).$$

We shall consider special cases of the generalized Fibonacci and Lucas sequences. For $(P, Q) = (1, 1)$, (U_n) is the sequence of Fibonacci numbers and (V_n) is the sequence of Lucas numbers. For $(P, Q) = (2, 1)$, (U_n) and (V_n) are the sequences of Pell numbers, respectively Pell–Lucas numbers.

It is convenient to extend these sequences also for negative indices:

$$U_{-n} = -\frac{U_n}{(-Q)^n}, \qquad V_{-n} = \frac{V_n}{(-Q)^n}$$

for $n \geq 1$. With this definition, the two relations above hold for all integers n.

Binet's formulas express the numbers U_n and V_n in terms of α and β :

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad V_n = \alpha^n + \beta^n.$$

Note that by Binet's formulas

$$\begin{aligned} U_n(-P,Q) &= (-1)^{n-1}U_n(P,Q), \\ V_n(-P,Q) &= (-1)^n V_n(P,Q). \end{aligned}$$

So, it will be assume that $P \geq 1$. For more information about generalized Fibonacci and Lucas sequences, the reader can follow [8, 16, 21, 18].

Elementary treatement of F_n is equal to polynomial in x is quite old. For example, Cohn [4] determined the squares in the Fibonacci sequence by an elementary method in 1964. Various authors worked on extending this approach to other quadratic polynomials. The starting impulse for this paper is the determination by Steiner [24], and more simply by Williams [27], of the Fibonacci numbers of the form x^2+1. Finkelstein determined Fibonacci and Lucas numbers of the form x^2+1 in [6], respectively in [7]. Ribenboim [20] introduced a general method which allows to identify numbers in generalized Fibonacci and Lucas numbers of the form $x^2 \pm 1$. Robbins [22] considered the Fibonacci numbers of the form $x^2 - 1$ and $x^3 \pm 1$. The same author [23] also determined Fibonacci numbers of the form $px^2 \pm 1$ and $px^3 \pm 1$. Antoniadis [2] generalized the result of Finkelstein [6], [7], and Robbins [22] about the Fibonacci and Lucas numbers of the form $x^2 \pm 1$. The problem of determining the terms of the linear recurrence sequence which can be represented by quadratic or cubic polynomials has been also of interest to many mathematicians. We recall that London and Finkelstein [13], as well as Pethő [17], Langarias and Weissel [12] showed that the only Fibonacci numbers which are cubes are $U_1 = U_2 = 1$, $U_6 = 8$. On the other hand, $V_1 = 1$ is the only Lucas number which is a cube. Luo [15] interested in Fibonacci numbers of the form $\frac{x(x+1)}{2}$ in 1989. Luca [14] determined Fibonacci numbers of the form $x^2 + x + 2$. In [3], Bugeaud, Mignotte, Luca, and Siksek showed that the only Fibonacci numbers which are at distance 1 from a perfect power are $1, 2, 3, 5$, and 8.

In [1], the authors showed that when $a \neq 0$ and $b \neq \pm 2$ are integers, the equation $V_n(P, \pm 1) = ax^2 + b$ has only a finite number of solutions n. Moreover, the same authors showed that when $a \neq 0$ and b are integers, the equation $U_n(P, \pm 1) = ax^2 + b$ has only a finite number of solutions n. Keskin [11] solved the equations $U_n(P, -1) = kx^2 \pm 1$ and $V_n(P, -1) = kx^2 \pm 1$ and $V_n(P, -1) = 2kx^2 \pm 1$ when $k|P$ with $k > 1$. After, Karaatlı and Keskin [9] solved the equations $V_n(P, -1) = 5kx^2 \pm 1$ and $V_n(P, -1) = 7kx^2 \pm 1$ when $k|P$ with $k > 1$. In [10], Karaatlı added to the above list the values of n for which $V_n(P, 1)$ is of the form $5kx^2 \pm 1$ and $7kx^2 \pm 1$ when $k|P$ with $k > 1$.

The object of this paper is to determine the indices n such that the equations $U_n(P,1) = kx^2 \pm 1$ and $V_n(P,1) = wkx^2 \pm 1$, where $w \in \{1,2,3,6\}$, $k|P$ with $k > 1$, have solutions.

Section 2 consists of preliminaries where all the required facts are gathered for convenience of the reader. And in Section 3, we give our main theorems. Throughout the paper $\left(\frac{*}{*}\right)$ denotes the Jacobi symbol.

2. Preliminaries

We can give the following lemma without proof since its proof can be done by induction.

Lemma 1. *Let n be a positive integer. Then*

$$V_n \equiv \begin{cases} 2 \pmod{P^2} \text{ if } n \text{ is even,} \\ nP \pmod{P^2} \text{ if } n \text{ is odd,} \end{cases} \tag{1}$$

and

$$U_n \equiv \begin{cases} \frac{n}{2}P \pmod{P^2} \text{ if } n \text{ is even,} \\ 1 \pmod{P^2} \text{ if } n \text{ is odd.} \end{cases} \tag{2}$$

The following two lemmas can be found in [25].

Lemma 2. *Let $n \in \mathbb{N} \cup \{0\}$, $m, r \in \mathbb{Z}$ and m be a nonzero integer. Then*

$$U_{2mn+r} \equiv (-1)^{mn} U_r \pmod{U_m}, \tag{3}$$

$$V_{2mn+r} \equiv (-1)^{mn} V_r \pmod{U_m}. \tag{4}$$

Lemma 3. *Let $n \in \mathbb{N} \cup \{0\}$ and $m, r \in \mathbb{Z}$. Then*

$$U_{2mn+r} \equiv (-1)^{(m+1)n} U_r \pmod{V_m}, \tag{5}$$

$$V_{2mn+r} \equiv (-1)^{(m+1)n} V_r \pmod{V_m}. \tag{6}$$

When P is odd, since $8|U_6$, using (3) we get

$$U_{12q+r} \equiv U_r \pmod 8. \tag{7}$$

Lemma 4. [Şiar and Keskin, [26], Theorem 3.4] *Let $k > 1$ be a squarefree positive divisor of odd P. If $V_n = kx^2$ for some integer x, then $n = 1$ or $n = 3$.*

We have the following lemma from [5] and [19].

Lemma 5. *If P is odd, then the equation $V_n = x^2$ has the solutions $n = 1$, $P = \square$(=perfect square), and $P \neq 1$ or $n = 1, 3$ and $P = 1$ or $n = 3$ and $P = 3$.*

Among the numerous identities and divisibility properties satisfied by the generalized Fibonacci and Lucas numbers we list below which will be used in this paper.

$$U_{2m+1} - 1 = U_m V_{m+1} \text{ if } m \text{ is even,} \tag{8}$$

$$U_{2m+1} + 1 = U_m V_{m+1} \text{ if } m \text{ is odd,} \tag{9}$$

$$V_{-n} = (-1)^n V_n, \tag{10}$$

$$U_{-n} = (-1)^{n+1} U_n, \tag{11}$$

$$V_{2n} = V_n^2 - 2(-1)^n. \tag{12}$$

If $r \geq 1$, then by (12),

$$V_{2^r} \equiv 2 \pmod{P}. \tag{13}$$

The following follows from (13);

$$(V_{2^r}, P) = 1, \tag{14}$$

if $3|P$, then

$$\left(\frac{3}{V_{2^r}}\right) = 1. \tag{15}$$

Moreover, by using induction, it can be seen that

$$V_{2^r} \equiv \begin{cases} 3 \pmod{8} \text{ if } r = 1, \\ 7 \pmod{8} \text{ if } r \geq 2, \end{cases}$$

and thus

$$\left(\frac{2}{V_{2^r}}\right) = \begin{cases} -1 \text{ if } r = 1, \\ 1 \text{ if } r \geq 2, \end{cases} \tag{16}$$

and

$$\left(\frac{-1}{V_{2^r}}\right) = -1 \tag{17}$$

for all $r \geq 1$.

If P is odd, then

$$\left(\frac{U_3}{V_{2^r}}\right) = 1 \tag{18}$$

for all $r \geq 2$, and

$$\left(\frac{-V_4 + 1}{V_{2^r}}\right) = -1 \tag{19}$$

for all $r \geq 3$.

Let $m = 2^a k$, $n = 2^b l$, k and l odd, $a, b \geq 0$, and $d = (a, b)$. Then

$$(U_m, V_n) = \begin{cases} V_d \text{ if } a > b, \\ 1 \text{ or } 2 \text{ if } a \leq b. \end{cases} \tag{20}$$

If $r \geq 1$, then

$$\left(\frac{P}{V_{2^r}}\right) = (-1)^{\frac{P-1}{2}} (-1)^{\frac{P^2-1}{8}}, \tag{21}$$

and if k is any positive divisor of P, then (21) implies that

$$\left(\frac{k}{V_{2^r}}\right) = (-1)^{\frac{k-1}{2}} (-1)^{\frac{k^2-1}{8}}. \tag{22}$$

Moreover,

$$\left(\frac{V_2}{V_{2^r}}\right) = 1 \tag{23}$$

for all $r \geq 3$.

3. Main Theorems

We assume from this point on that n is a positive integer and unless otherwise stated, P is odd and $Q = 1$. We also assume that $k|P$ with $k > 1$ in all of the stataments of theorems below.

In [2], Antoniadis solved the equation $U_m = kx^2 + l$ under the conditions that $k \equiv 1, 3 \pmod 8$, $l \equiv 1 \pmod 8$ and $m \equiv 1 \pmod 2$. Here, we will solve the equation by taking $k \equiv 5, 7 \pmod 8$ and $l = 1$.

Theorem 1. *If $k \equiv 1, 3 \pmod 8$, then the equation $U_n = kx^2 + 1$, $n = 2m + 1$, $m \equiv \pm 1, 3 \pmod 8$, has no solutions. If $k \equiv 5, 7 \pmod 8$, then the equation $U_n = kx^2 + 1$, $n = 2m + 1$, $m \equiv 1 \pmod 4$, has no solutions. If $U_n = kx^2 + 1$ for some integer x, then $n = 1$ or $n = 5$, where $n = 2m + 1$ with m even.*

Proof. Assume that $U_n = kx^2 + 1$ for some integer x. If n were even, then by (2), $U_n \equiv \frac{n}{2} P \pmod{P^2}$, which would imply that $k|U_n$, because $k|P$. This follows from our assumptions that $k|1$, contradicting $k > 1$. Therefore, n is odd, say $n = 2m + 1$ for some $m > 0$. The remainder of the proof is split into two cases.

`Case I:` m odd. Assume that $m \equiv 1 \pmod 8$ and $k \equiv 1, 3 \pmod 8$. We can write $m = 8q + 1$ and therefore $n = 16q + 3 = 2 \cdot 2^r a + 3$ with $2 \nmid a$ and $r \geq 3$. Then by (5), we have

$$kx^2 + 1 = U_n = U_{2 \cdot 2^r a + 3} \equiv -U_3 \pmod{V_{2^r}},$$

implying that
$$kx^2 \equiv -(P^2+2) \equiv -V_2 \pmod{V_{2^r}}.$$
This shows that
$$1 = \left(\frac{-kV_2}{V_{2^r}}\right).$$
However, this is impossible since $\left(\frac{-1}{V_{2^r}}\right) = -1$ by (17), $\left(\frac{k}{V_{2^r}}\right) = 1$ by (22), and $\left(\frac{V_2}{V_{2^r}}\right) = 1$ by (23).

Assume that $m \equiv -1 \pmod 8$. Then we immediately have $n = 16q-1 = 2\cdot 2^r a - 1$ with $2 \nmid a$ and $r \geq 3$. By (5) and (11) , we get
$$kx^2 + 1 = U_n = U_{2\cdot 2^r a-1} \equiv -U_{-1} \pmod{V_{2^r}},$$
implying that
$$kx^2 \equiv -2 \pmod{V_{2^r}}.$$
This shows that
$$1 = \left(\frac{-2}{V_{2^r}}\right),$$
which is impossible since $\left(\frac{-2}{V_{2^r}}\right) = -1$ by (16) and (17).

Assume that $m \equiv 3 \pmod 8$. Then we can write $n = 16q + 7 = 8(2c+1) - 1 = 8c - 1$ with odd c. The detail of the proof is exactly the same to that of above case. So, we again get a contradiction.

Now assume that $m \equiv 1 \pmod 4$ and $k \equiv 5, 7 \pmod 8$. If we write $m = 4q + 1$, then we get $n = 8q + 3$. Firstly, assume that q is even. And thus,
$$kx^2 + 1 = U_n = U_{2\cdot 4q+3} \equiv U_3 \pmod{V_4}$$
by (5). This means that
$$kx^2 \equiv P^2 \pmod{V_4},$$
which is impossible since $\left(\frac{k}{V_4}\right) = -1$ by (22). Hence, q is odd.

Let $3|q$. Then $n = 24t + 3$. Applying (7), we have
$$kx^2 + 1 = U_n \equiv U_3 \pmod 8,$$
i.e.,
$$kx^2 \equiv 1 \pmod 8.$$
However, this is impossible since $k \equiv 5, 7 \pmod 8$.

Let $3 \nmid q$, say $q = 3t \pm 1$. Then we immediately have $n = 24t + 11$ or $n = 24t - 5$. In both cases, we can easily write $n = 4c - 1$ with odd c. So, by using (5) and (17), we get
$$kx^2 + 1 = U_n = U_{2\cdot 2c-1} \equiv -U_{-1} \equiv -1 \pmod{V_2},$$

implying that

$$kx^2 \equiv -2 \pmod{V_2}.$$

But this is also impossible since $\left(\frac{k}{V_2}\right) = -1$ by (22), $\left(\frac{-1}{V_2}\right) = -1$ by (17), and $\left(\frac{2}{V_2}\right) = -1$ by (16).

`Case II:` m even. By using (8), we get $kx^2 = U_m V_{m+1}$. Then $(U_m, V_{m+1}) = P$ by (20). Thus, it follows that $U_m = k_1 P u^2$ and $V_{m+1} = k_2 P v^2$ with $k_1 k_2 = k$. By Lemmas 4 and 5, we have $m+1 = 1$ or $m+1 = 3$. That is to say, $m = 0$ or $m = 2$. Hence, $n = 1$ or $n = 5$. This completes the proof. □

Theorem 2. *The equation $U_n = kx^2 - 1$ has no solutions.*

Proof. Assume that $U_n = kx^2 - 1$ for some integer x. Then by (2), n is odd. Let $n = 4q \pm 1$ for some $q > 0$. By (3) and (11), we have

$$kx^2 - 1 = U_n = U_{2 \cdot 2q-1} \equiv U_{\pm 1} \equiv 1 \pmod{U_2}.$$

This implies that

$$kx^2 \equiv 2 \pmod{k}.$$

But this is impossible since $k > 1$ is odd. □

Theorem 3. *The equations $V_n = wkx^2 + 1$, $w \in \{1, 2, 3, 6\}$, have no solutions.*

Proof. Assume that $V_n = wkx^2 + 1$ for some integer x. Then n is even. For, otherwise we would have $k|V_n$ by (1), because $k|P$. Then this would imply that $k|1$, contradicting $k > 1$. Hence, $n = 2m$, say.

Assume that m is even. Then by (12) and (1) , we have $V_n = V_m^2 - 2 \equiv 2 \pmod{P^2}$. Since $k > 1$ and $k|P$, it follows that $wkx^2 \equiv 1 \pmod{k}$, a contradiction.

Assume now that m is odd. Then by (12), we have $wkx^2 + 1 = V_n = V_m^2 + 2$, implying that $wkx^2 = V_m^2 + 1$. Since m is odd, we see from (1) that $k|V_m$, because $k|P$. This shows that $k|1$, which contradicts the fact that $k > 1$. This completes the proof. □

Theorem 4. *The equation $V_n = kx^2 - 1$ has only the solution $(n, P, k, x) = \left(2, 3U_t(4,-1), 3, \frac{V_t(4,-1)}{2}\right)$ with t odd.*

Proof. Assume that $V_n = kx^2 - 1$ for some integer x. Then by (1), n is even, $n = 2m$, say. If m is even, then $kx^2 - 1 = V_n = V_m^2 - 2$. Since $V_m^2 \equiv 4 \pmod{k}$ by (1), it follows that $kx^2 \equiv 3 \pmod{k}$. This shows that $k = 3$ and therefore $3|P$. We can write $m = 2^r a$ with a odd and $r \geq 1$, and therefore $n = 2 \cdot 2^r a$. Thus, we have

$$3x^2 - 1 \equiv -2 \pmod{V_{2^r}}$$

by (6). This shows that
$$1 = \left(\frac{-3}{V_{2^r}}\right).$$
However, this is impossible since $\left(\frac{-1}{V_{2^r}}\right) = -1$ by (17) and $\left(\frac{3}{V_{2^r}}\right) = 1$ by (15). Hence, m is odd. Then we have $kx^2 - 1 = V_n = V_m^2 + 2$ by (12). By (1), $P|V_m$, which implies that $k|3$, i.e., $k = 3$ and therefore $3|P$. Let $m > 1$. Then we can write $m = 4q \pm 1$ and therefore $n = 8q \pm 2 = 2 \cdot 2^r a \pm 2$ with odd a and $r \geq 2$. By (6) and (10), we obtain
$$3x^2 - 1 = V_n = V_{2 \cdot 2^r a \pm 2} \equiv -V_{\pm 2} \equiv -(P^2 + 2) \pmod{V_{2^r}}.$$
This shows that
$$3x^2 \equiv -U_3 \pmod{V_{2^r}},$$
which implies that
$$1 = \left(\frac{-3U_3}{V_{2^r}}\right).$$
However, this is impossible since $\left(\frac{-1}{V_{2^r}}\right) = -1$ by (17), $\left(\frac{3}{V_{2^r}}\right) = 1$ by (15), and $\left(\frac{U_3}{V_{2^r}}\right) = 1$ by (18). Therefore $m = 1$ and so $n = 2$. Thus, $3x^2 - 1 = V_2 = P^2 + 2$ and this implies that $P^2 - 3x^2 = -3$. Since $3|P$, it follows that $P = 3a$ with a odd. Substituting this into $P^2 - 3x^2 = -3$, we have the equation $x^2 - 3a^2 = 1$. It is well known that all positive integer solutions of this equation are given by $(x, a) = (\frac{V_t(4,-1)}{2}, U_t(4,-1))$. Since a is odd, we must have t is odd. As a consequence, we get the solution $(n, P, k, x) = (2, 3U_t(4,-1), 3, \frac{V_t(4,-1)}{2})$ with t odd. This completes the proof. □

Theorem 5. *The equations $V_n = wkx^2 - 1$, $w \in \{2, 3, 6\}$, have no solutions.*

Proof. Assume that $V_n = wkx^2 - 1$ for some integer x. If n were odd, then by (1), $V_n \equiv nP \pmod{P^2}$, which would imply that $k|V_n$, because $k|P$. So, we would have that $k|1$, which would contradict our assumption that $k > 1$. Therefore n is even, $n = 2m$, say.

If m is even, then $V_n = V_{2m} = V_m^2 - 2$ by (12). This implies that $wkx^2 \equiv 3 \pmod{k}$ by (1) and the fact that $k|P$. This shows that $k|3$.

If m is odd, then $V_n = V_{2m} = V_m^2 + 2$ by (12), implying that $wkx^2 = V_m^2 + 3$. Since m is odd, it follows from (1) that $P|V_m$. And so $k|3$ since $k|P$. We see from the explanations above that $k = 3$ (independently of the parity of m). Therefore we are interested in finding the solutions of the equations $V_n = 3wx^2 - 1$, where $w \in \{2, 3, 6\}$. We distinguish three cases.

`Case I:` $w = 2$. Assume that m is even. If $m \equiv 0 \pmod 8$, then $m = 8q$ and therefore $n = 16q = 2 \cdot 2^r a$ with odd a and $r \geq 3$. This shows that

$$6x^2 - 1 = V_n = V_{2 \cdot 2^r a} \equiv -2 \pmod{V_{2^r}},$$

i.e.,

$$6x^2 \equiv -1 \pmod{V_{2^r}}$$

by (6). However, this is impossible since $\left(\frac{2}{V_{2^r}}\right)\left(\frac{3}{V_{2^r}}\right) = 1$ by (16) and (15), and $\left(\frac{-1}{V_{2^r}}\right) = -1$ by (17).

If $m \equiv \pm 2 \pmod 8$, then $n = 16q \pm 4 = 2 \cdot 2^r a \pm 4$ with a odd and $r \geq 3$. By (6) and (10), we readily obtain

$$6x^2 \equiv -V_4 + 1 \pmod{V_{2^r}}.$$

This implies that

$$\left(\frac{6}{V_{2^r}}\right) = \left(\frac{-V_4 + 1}{V_{2^r}}\right).$$

However, this is impossible since $\left(\frac{2}{V_{2^r}}\right)\left(\frac{3}{V_{2^r}}\right) = 1$ by (16) and (15), and $\left(\frac{-V_4+1}{V_{2^r}}\right) = -1$ by (19).

Lastly, if $m \equiv 4 \pmod 8$, then $n = 16q + 8 = 8c$ with odd c. Then by (6), we get

$$6x^2 \equiv -1 \pmod{V_4},$$

which is also impossible since $\left(\frac{2}{V_{2^r}}\right)\left(\frac{3}{V_{2^r}}\right) = 1$ by (16) and (15), and $\left(\frac{-1}{V_{2^r}}\right) = -1$ by (17).

Assume now that $m > 1$ is odd. Therefore, writing $m = 4q \pm 1$, we have $n = 8q \pm 2 = 2 \cdot 2^r a \pm 2$ with a odd and $r \geq 2$. This shows that

$$6x^2 - 1 = V_n = V_{2 \cdot 2^r a \pm 2} \equiv -V_{\pm 2} \equiv -(P^2 + 2) \pmod{V_{2^r}}$$

by (6) and (10). Rearranging the congruence above gives

$$6x^2 \equiv -U_3 \pmod{V_{2^r}},$$

which is also impossible since $\left(\frac{2}{V_{2^r}}\right)\left(\frac{3}{V_{2^r}}\right) = 1$ by (16) and (15), $\left(\frac{-1}{V_{2^r}}\right) = -1$ by (17), and $\left(\frac{U_3}{V_{2^r}}\right) = 1$ by (18). Hence, $m = 1$ and so $n = 2$. This gives $6x^2 - 1 = V_2 = P^2 + 2$. Using the fact that $3|P$, say $P = 3a$. Then we readily obtain $2x^2 = 3a^2 + 1$, which is impossible since $x^2 \equiv 2 \pmod 3$ in that case.

`Case II:` $w = 3$. Assume that m is even. Putting $m = 2^r a$ with a odd and $r \geq 1$ gives $n = 2 \cdot 2^r a$. Therefore we have

$$9x^2 - 1 = V_n = V_{2 \cdot 2^r a} \equiv -2 \pmod{V_{2^r}}$$

by (5). This implies that

$$(3x)^2 \equiv -1 \pmod{V_{2^r}}.$$

However, this is impossible by (17).

Assume now that $m > 1$ is odd, $m = 4q \pm 1$, say. Therefore $n = 8q \pm 2 = 2 \cdot 2^r a \pm 2$ with a odd and $r \geq 1$. By (5), we readily obtain

$$(3x)^2 \equiv -U_3 \pmod{V_{2^r}}.$$

However, this is impossible since $\left(\frac{-1}{V_{2^r}}\right) = -1$ by (17) and $\left(\frac{U_3}{V_{2^r}}\right) = 1$ by (18). Hence, $m = 1$ and so $n = 2$. Thus, we have $9x^2 - 1 = V_2 = P^2 + 2$, i.e., $P^2 + 3 = (3x)^2$, gives $3 = (3x - P)(3x + P)$ and the only positive solution is $(x, P) = \left(\frac{2}{3}, 2\right)$, which is not convenient since we must have P is odd and x is integer.

`Case III:` $w = 6$. Assume that m is even. Let $m \equiv 0 \pmod 4$. Putting $m = 4q$, we therefore have $n = 8q = 2 \cdot 2^r a$ with odd a and $r \geq 2$. Applying (6), we have

$$18x^2 - 1 = V_n = V_{2 \cdot 2^r a} \equiv -2 \pmod{V_{2^r}},$$

i.e.,

$$2(3x)^2 \equiv -1 \pmod{V_{2^r}}.$$

However, this is impossible since $\left(\frac{2}{V_{2^r}}\right) = 1$ by (16) and $\left(\frac{-1}{V_{2^r}}\right) = -1$ by (17).

Let $m \equiv 2 \pmod 4$. Then $n = 8q+4$. Actually, we can easily write $n = 16s \pm 4 = 2 \cdot 2^r a \pm 4$ with a odd and $r \geq 3$ (dependently the parity of q). And so by (6) and (10), we readily obtain

$$2(3x)^2 \equiv -V_4 + 1 \pmod{V_{2^r}}.$$

However, this is impossible since $\left(\frac{2}{V_{2^r}}\right) = 1$ by (16) and $\left(\frac{-V_4+1}{V_{2^r}}\right) = -1$ by (19).

Assume now that $m > 1$ is odd. Therefore, writing $m = 4q \pm 1$, we have $n = 8q \pm 2 = 2 \cdot 2^r a \pm 2$ with a odd and $r \geq 2$. This shows that

$$18x^2 - 1 = V_n = V_{2 \cdot 2^r a \pm 2} \equiv -V_{\pm 2} \equiv -(P^2 + 2) \pmod{V_{2^r}}$$

by (6) and (10). It immediately follows from the congruence above that

$$2(3x^2) \equiv -U_3 \pmod{V_{2^r}},$$

which is also impossible since $\left(\frac{2}{V_{2^r}}\right) = 1$ by (16), $\left(\frac{-1}{V_{2^r}}\right) = -1$ by (17), and $\left(\frac{U_3}{V_{2^r}}\right) = 1$ by (18). Hence, $m = 1$ and so $n = 2$. This shows that $18x^2 - 1 = V_2 = P^2 + 2$. Using the fact that $3|P$, say $P = 3a$, gives $3a^2 + 1 = 6x^2$, which is impossible. This completes the proof. $\square$

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