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q-MULTIPARAMETER-BERNOULLI POLYNOMIALS AND q-MULTIPARAMETER-CAUCHY POLYNOMIALS BY JACKSON'S INTEGRALS

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Abstract

We define q-multiparameter-Bernoulli polynomials and q-multiparameter-Cauchy polynomials by using Jackson's integrals, which generalize the previously known numbers, including poly-Bernoulli $B_n^{(k)}$ and the poly-Cauchy numbers of the first kind $c_n^{(k)}$ and of the second kind $\widehat{c}_n^{(k)}$. We investigate their properties connected with multiparameter Stirling numbers which generalize the original Stirling numbers. We also give the relations between q-multiparameter-Bernoulli polynomials and q-multiparameter-Cauchy polynomials.

1. Introduction

Let n and k be integers with $n \ge 0$, and let $L = (l_1, \dots, l_k)$ be a k-tuple of real numbers with $\ell := l_1 \cdots l_k \ne 0$ and $A = (\alpha_0, \alpha_1, \dots, \alpha_{n-1})$ be a n-tuple of real numbers. Let q be a real number with $0 \le q < 1$.

Jackson's q-derivative with $0 < q < 1$ (see e.g., [1, (10.2.3)], [12]) is defined by

$$D_q f = \frac{d_q f}{d_q x} = \frac{f(x) - f(qx)}{(1-q)x}$$

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and *Jackson's q-integral* ([1, (10.1.3)], [12]) is defined by

$$\int_0^x f(t)d_q t = (1-q)x\sum_{n=0}^\infty f(q^n x)q^n\,.$$

The Jackson integral gives a unique q-antiderivative within a certain class of functions. In particular, when $f(x)=x^m$ for some nonnegative integer m, then

$$D_q f = \frac{x^m - q^m x^m}{(1-q)x} = [m]_q x^{m-1}$$

and

$$\begin{aligned}\int_0^x t^m d_q t &= (1-q)x\sum_{n=0}^\infty q^{mn}x^m q^n\\ &= (1-q)x^{m+1}\sum_{n=0}^\infty q^{n(m+1)} = \frac{x^{m+1}}{[m+1]_q}\,.\end{aligned}$$

Here,

$$[x]_q = \frac{1-q^x}{1-q}$$

is the q-number with $[0]_q=0$ (see e.g. [1, (10.2.3)], [12]). Note that $\lim_{q\to 1}[x]_q = x$.

Define *poly-Bernoulli polynomials* $B_{n,\rho,q}^{(k)}(z)$ *with a parameter* ρ by

$$\frac{\rho}{1-e^{-\rho t}}\mathrm{Li}_{k,q}\left(\frac{1-e^{-\rho t}}{\rho}\right)e^{-tz} = \sum_{n=0}^\infty B_{n,\rho,q}^{(k)}(z)\frac{t^n}{n!}\,, \tag{1}$$

where $\mathrm{Li}_{k,q}(z)$ is the *q-polylogarithm function* (see [16]) defined by

$$\mathrm{Li}_{k,q}(z) = \sum_{n=1}^\infty \frac{z^n}{[n]_q^k}\,.$$

Notice that

$$\lim_{q\to 1} B_{n,\rho,q}^{(k)}(z) = B_{n,\rho}^{(k)}(z)\,,$$

which is the poly-Bernoulli polynomial with a ρ parameter (see [6]), and

$$\lim_{q\to 1}\mathrm{Li}_{k,q}(z) = \mathrm{Li}_k(z)\,,$$

which is the ordinary polylogarithm function, defined by

$$\mathrm{Li}_k(z) = \sum_{m=1}^\infty \frac{z^m}{m^k}\,. \tag{2}$$

In addition, when $z=0$, $B_{n,\rho}^{(k)}(0)=B_{n,\rho}^{(k)}$ is the poly-Bernoulli number with a ρ parameter. When $z=0$ and $\rho=1$, $B_{n,1}^{(k)}(0)=B_n^{(k)}$ is the *poly-Bernoulli number* (see [15]) defined by

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}=\sum_{n=0}^\infty B_n^{(k)}\frac{t^n}{n!}\,. \tag{3}$$

The poly-Bernoulli numbers are expressed as special values at negative arguments of certain combinations of multiple zeta values. The poly-Bernoulli numbers can be expressed in terms of the Stirling numbers of the second kind.

$$B_n^{(k)}=\sum_{m=0}^n\frac{(-1)^{n-m}m!S_2(n,m)}{(m+1)^k}\quad (n\ge 0,\ k\ge 1)$$

([15, Theorem 1]), where $S_2(n,m)$ is the Stirling number of the second kind, see [7], determined by the falling factorial:

$$x^n=\sum_{m=0}^n S_2(n,m)x(x-1)\cdots(x-m+1)\,.$$

The poly-Bernoulli numbers are extended to the poly-Bernoulli polynomials (see [3, 8]) and to the special multi-poly-Bernoulli numbers (see [11]). The Bernoulli polynomials occur in the study of many special functions and in particular the Riemann zeta function and the Hurwitz zeta function. They are an Appell sequence, i.e., a Sheffer sequence for the ordinary derivative operator.

Define the *q-multiparameter-poly-Cauchy polynomials of the first kind* $c_{n,L,A,q}^{(k)}(z)$ by

$$c_{n,L,A,q}^{(k)}(z)=\int_0^{l_1}\cdots\int_0^{l_k}(x_1\cdots x_k-\alpha_0-z)\cdots(x_1\cdots x_k-\alpha_{n-1}-z)d_qx_1\cdots d_qx_k\,. \tag{4}$$

Notice that

$$\lim_{q\to 1}c_{n,L,A,q}^{(k)}(z)=c_{n,L,A}^{(k)}(z)\,,$$

which are the multiparameter-poly-Cauchy polynomials of the first kind. The idea of dealing with multiparameters $\alpha_0,\alpha_1,\dots,\alpha_{n-1}$ instead of $0,1,\dots,n-1$ has already been considered in [25]. Namely, If $l_1=\cdots=l_k=1$ and $z=0$, the number $c_{n,(1,\dots,1),A}^{(k)}=c_{n,A}^{(k)}$ has been studied to prove the convexity. It has been proven that $c_{n,A}^{(k)}$ is log-convex, satisfying $(c_{n,A}^{(k)})^2-c_{n-1,A}^{(k)}c_{n+1,A}^{(k)}\le 0$.

In addition, if $\alpha_i=i\rho$ $(i=0,1,\dots,n-1)$, then the number $c_{n,A}^{(k)}$ is reduced to the poly-Cauchy numbers of the first kind with a parameter ρ (see [19]). Furthermore, if $\rho=1$, then the number $c_{n,A}^{(k)}$ is reduced to the poly-Cauchy number $c_n^{(k)}$ (see [18]). If $k=1$, then $c_n^{(1)}=c_n$ is the classical Cauchy number (see [7, 27]). The number $c_n/n!$ is sometimes referred to as the Bernoulli number of the second kind (see [4, 13, 28]).

The poly-Cauchy numbers have been considered as analogues of the poly-Bernoulli numbers $B_n^{(k)}$. The poly-Cauchy numbers of the first kind, $c_n^{(k)}$, can be expressed in terms of the Stirling numbers of the first kind:

$$c_n^{(k)} = \sum_{m=0}^{n} \frac{(-1)^{n-m} S_1(n,m)}{(m+1)^k} \quad (n \geq 0,\ k \geq 1)$$

([18, Theorem 1]), where $S_1(n,m)$ is the (unsigned) Stirling number of the first kind (see [7]), determined by the rising factorial:

$$x(x+1)\cdots(x+n-1) = \sum_{m=0}^{n} S_1(n,m)x^m\,. \tag{5}$$

Similarly, define the *q-multiparameter-poly-Cauchy polynomials of the second kind* $\widehat{c}_{n,L,A,q}^{(k)}(z)$ by

$$\begin{aligned}&\widehat{c}_{n,L,A,q}^{(k)}(z)\\ &= \int_0^{l_1} \cdots \int_0^{l_k} (-x_1\cdots x_k - \alpha_0 + z)\cdots(-x_1\cdots x_k - \alpha_{n-1} + z) d_q x_1 \cdots d_q x_k\,.\end{aligned} \tag{6}$$

If $q \to 1$, $l_1 = \cdots = l_k = 1$, $\alpha_i = i\rho$ $(i = 0, 1, \ldots, n-1)$ and $z = 0$, the number $\widehat{c}_{n,A}^{(k)}$ is reduced to the poly-Cauchy numbers of the second kind with a parameter ρ (see [19]). Furthermore, if $\rho = 1$, then the number $\widehat{c}_{n,A}^{(k)}$ is reduced to the poly-Cauchy numbers of the second kind $\widehat{c}_n^{(k)}$ (see [18]). If $k = 1$, then $\widehat{c}_n^{(1)} = \widehat{c}_n$ is the classical Cauchy number (see [7, 27]). The poly-Cauchy numbers of the second kind $\widehat{c}_n^{(k)}$ can be expressed in terms of the Stirling numbers of the first kind by

$$\widehat{c}_n^{(k)} = (-1)^n \sum_{m=0}^{n} \frac{S_1(n,m)}{(m+1)^k} \quad (n \geq 0,\ k \geq 1)$$

([18, Theorem 4]). The generating function of the poly-Cauchy numbers of the second kind $\widehat{c}_n^{(k)}$ is given by

$$\mathrm{Lif}_k\big(-\ln(1+t)\big) = \sum_{n=0}^{\infty} \widehat{c}_n^{(k)} \frac{t^n}{n!} \tag{7}$$

([18, Theorem 5]).

The poly-Cauchy numbers (of the both kinds) are extended to the poly-Cauchy polynomials (see [14]), and to the poly-Cauchy numbers with a q parameter (see [19]). The corresponding poly-Bernoulli numbers with a q parameter can be obtained in [6]. A different direction of generalizations of Cauchy numbers is about hypergeometric Cauchy numbers (see [21]). Arithmetical and combinatorial properties including sums of products have been studied (see [20, 23, 24]).

Various kinds of q-analogues or extensions have been studied. In [17], as generalizations of the poly-Cauchy numbers of the first kind $c_n^{(k)}$ and of the second kind $\widehat{c}_n^{(k)}$, by using Jackson's q-integrals, q-analogues or extensions of the poly-Cauchy numbers of the first kind $c_{n,q}^{(k)}$ and of the second kind $\widehat{c}_{n,q}^{(k)}$ are introduced, and their properties are investigated. In [22], by using Jackson's q-integrals, the concept about q-analogues or extensions of the poly-Bernoulli polynomials $B_{n,q}^{(k)}(z)$ with a parameter were also introduced.

In this paper, by using Jackson's q-integrals, as essential generalizations of the previously known numbers and polynomials, including poly-Bernoulli numbers $B_n^{(k)}$, the poly-Cauchy numbers of the first kind $c_n^{(k)}$ and of the second kind $\widehat{c}_n^{(k)}$, we introduce the concept of q-analogues or extensions of the poly-Bernoulli polynomials $B_{n,\rho,q}^{(k)}(z)$ with a parameter, and the poly-Cauchy polynomials of the first kind $c_{n,\rho,q}^{(k)}$ and of the second kind $\widehat{c}_{n,\rho,q}^{(k)}$ with a parameter. We investigate their properties connected with the usual Stirling numbers and the weighted Stirling numbers. We also give the relations between generalized poly-Bernoulli polynomials and two kinds of generalized poly-Cauchy polynomials.

2. q-multiparameter-Cauchy Polynomials

For an n-tuple $A = (\alpha_0, \alpha_1, \dots, \alpha_{n-1})$ of real numbers, define *multiparameter Stirling numbers of the first kind* $S_1(n, m, A)$ and *of the second kind* $S_2(n, m, A)$ by

$$(t-\alpha_0)(t-\alpha_1)\cdots(t-\alpha_{n-1}) = \sum_{m=0}^{n} S_1(n,m,A)t^m \tag{8}$$

and

$$\sum_{m=0}^{n} S_2(n,m,A)(t-\alpha_0)(t-\alpha_1)\cdots(t-\alpha_{m-1}) = t^n\,, \tag{9}$$

respectively (cf. [7, 9, 26]). If $\alpha_i = i\rho$ $(i = 0, 1, \dots, n-1)$, then

$$S_1\big(n,m,(0,\rho,\dots,(n-1)\rho)\big) = (-\rho)^{n-m} S_1(n,m)\,,$$
$$S_2\big(n,m,(0,\rho,\dots,(n-1)\rho)\big) = \rho^{n-m} S_2(n,m)\,,$$

where $S_1(n,m)$ and $S_2(n,m)$ are the (unsigned) Stirling numbers of the first kind and the Stirling numbers of the second kind, respectively.

The q-multiparameter-poly-Cauchy polynomials of the first kind can be expressed explicitly in terms of the multiparameter Stirling numbers of the first kind.

Theorem 1. *For all integers n and k with $n \ge 0$ and a real number q with $0 < q < 1$, we have*

$$c_{n,L,A,q}^{(k)}(z) = \sum_{m=0}^{n} S_1(n,m,A) \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^i \ell^{m-i+1}}{[m-i+1]_q^k}.$$

Proof. By definitions of (4) and (8), we have

$$\begin{aligned} c_{n,L,A,q}^{(k)}(z) &= \int_0^{l_1} \cdots \int_0^{l_k} \sum_{m=0}^{n} S_1(n,m,A)(x_1 \cdots x_k - z)^m d_q x_1 \cdots d_q x_k \\ &= \sum_{m=0}^{n} S_1(n,m,A) \sum_{i=0}^{m} \binom{m}{i} (-z)^{m-i} \int_0^{l_1} \cdots \int_0^{l_k} x_1^i \cdots x_k^i d_q x_1 \cdots d_q x_k \\ &= \sum_{m=0}^{n} S_1(n,m,A) \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^{m-i}}{[i+1]_q^k} \ell^{i+1} \\ &= \sum_{m=0}^{n} S_1(n,m,A) \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^i}{[m-i+1]_q^k} \ell^{m-i+1}. \end{aligned}$$

□

If $z = 0$, then we have the expression of the q-multiparameter-poly-Cauchy numbers of the first kind.

Corollary 1. *For all integers n and k with $n \ge 0$ and a real number q with $0 < q < 1$, we have*

$$c_{n,L,A,q}^{(k)} = \sum_{m=0}^{n} \frac{S_1(n,m,A)\ell^{m+1}}{[m+1]_q^k}.$$

Similarly, the q-multiparameter-poly-Cauchy polynomials of the second kind can be expressed explicitly in terms of the multiparameter Stirling numbers of the first kind. The proof is similar to that of Theorem 1 and is omitted.

Theorem 2. *For all integers n and k with $n \ge 0$ and a real number q with $0 < q < 1$, we have*

$$\widehat{c}_{n,L,A,q}^{(k)}(z) = \sum_{m=0}^{n} (-1)^m S_1(n,m,A) \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^i \ell^{m-i+1}}{[m-i+1]_q^k}.$$

If $z = 0$, then we have the expression of the q-multiparameter-poly-Cauchy numbers of the second kind.

Corollary 2. *For all integers n and k with $n \ge 0$ and a real number q with $0 < q < 1$, we have*

$$\widehat{c}_{n,L,A,q}^{(k)} = \sum_{m=0}^{n} \frac{(-1)^m S_1(n,m,A)\ell^{m+1}}{[m+1]_q^k}\,.$$

There are simple relations between two kinds of q-multiparameter-poly-Cauchy polynomials.

Theorem 3. *For all integers n and k with $n \ge 1$ and a real number q with $0 < q < 1$, we have*

$$(-1)^n c_{n,L,A,q}^{(k)}(z) = \widehat{c}_{n,L,-A,q}^{(k)}(z)\,, \tag{10}$$

$$(-1)^n \widehat{c}_{n,L,A,q}^{(k)}(z) = c_{n,L,-A,q}^{(k)}(z)\,, \tag{11}$$

where $-A = (-\alpha_0, -\alpha_1, \dots, -\alpha_{n-1})$.

Proof. We shall prove identity (11). The identity (10) is proven similarly and omitted. By the definition of $\widehat{c}_{n,L,A,q}^{(k)}(z)$, we see that

$$\begin{aligned}
&(-1)^n \widehat{c}_{n,L,A,q}^{(k)}(z)\\
&= (-1)^n \int_0^{l_1} \cdots \int_0^{l_k} (-x_1 \cdots x_k + z - \alpha_0) \cdots (-x_1 \cdots x_k + z - \alpha_{n-1}) d_q x_1 \cdots d_q x_k\\
&= \int_0^{l_1} \cdots \int_0^{l_k} (x_1 \cdots x_k - z + \alpha_0) \cdots (x_1 \cdots x_k - z + \alpha_{n-1}) d_q x_1 \cdots d_q x_k\\
&= c_{n,L,-A,q}^{(k)}(z)\,.
\end{aligned}$$

□

3. *q*-multiparameter-poly-Bernoulli Polynomials

Define the *q-multiparameter-poly-Bernoulli polynomials* $B_{n,L,A,q}^{(k)}(z)$ by

$$B_{n,L,A,q}^{(k)}(z) = \sum_{m=0}^{n} S_2(n,m,A) m! \sum_{i=0}^{m} \binom{m}{i} \frac{(-z)^i \ell^{m-i+1}}{[m-i+1]_q^k}\,. \tag{12}$$

This is a generalization of poly-Bernoulli polynomials $B_n^{(k)}(z)$, defined in [24]. If $q \to 1$, $l_1 = \cdots = l_k = 1$ and $\alpha_i = i$ $(i = 0, 1, \dots, n-1)$, then the polynomial $B_{n,L,A,q}^{(k)}(z)$ are reduced to the polynomial $B_n^{(k)}(z)$ in [24].

By putting $z=0$ in (12), the q-multiparameter-poly-Bernoulli numbers $B_{n,L,A,q}^{(k)}$ are given by

$$B_{n,L,A,q}^{(k)}=\sum_{m=0}^n\frac{S_2(n,m,A)m!\ell^{m+1}}{[m+1]_q^k}\,. \tag{13}$$

Since the orthogonality relations

$$\sum_{k=i}^n S_1(n,k,A)S_2(k,i,A)=\sum_{k=i}^n S_1(k,i,A)S_2(n,k,A)=\delta_{n,i}\,, \tag{14}$$

where $\delta_{n,i}$ is the Kronecker's delta, we obtain the inverse relation

$$f_n=\sum_{m=0}^n S_1(n,m,A)g_m \quad\Longleftrightarrow\quad g_n=\sum_{m=0}^n S_2(n,m,A)f_m\,. \tag{15}$$

Theorem 4. *For q-multiparameter-poly-Bernoulli and q-multiparameter-poly-Cauchy polynomials, we have*

$$\sum_{m=0}^n S_1(n,m,A)B_{m,L,A,q}^{(k)}(z)=n!\sum_{i=0}^n\binom{n}{i}\frac{(-z)^i\ell^{n-i+1}}{[n-i+1]_q^k}\,, \tag{16}$$

$$\sum_{m=0}^n S_2(n,m,A)c_{m,L,A,q}^{(k)}(z)=\sum_{i=0}^n\binom{n}{i}\frac{(-z)^i\ell^{n-i+1}}{[n-i+1]_q^k}\,, \tag{17}$$

$$\sum_{m=0}^n S_2(n,m,A)\widehat{c}_{m,L,A,q}^{(k)}(z)=(-1)^n\sum_{i=0}^n\binom{n}{i}\frac{(-z)^i\ell^{n-i+1}}{[n-i+1]_q^k}\,. \tag{18}$$

Remark. If $q\to 1$ and $\alpha_i=i\rho$ $(i=0,1,\dots,n-1)$, then Theorem 4 is reduced to Theorem 3.2 in [6].

Proof. By (12), applying (15) with

$$f_m=m!\sum_{i=0}^m\binom{m}{i}\frac{(-z)^i\ell^{m-i+1}}{[m-i+1]_q^k}\quad\text{and}\quad g_n=B_{n,L,A,q}^{(k)}(z)\,,$$

we get the identity (16). Similarly, by Theorem 1 and Theorem 2 we have the identities (17) and (18), respectively. □

If we put $z=0$ in Theorem 4, we have the identities for appropriate numbers.

Corollary 3. *For q-multiparameter-poly-Bernoulli and q-multiparameter-poly-Cauchy* *numbers, we have*

$$\sum_{m=0}^{n} S_1(n,m,A) B_{m,L,A,q}^{(k)} = \frac{n!\ell^{n+1}}{[n+1]_q^k}\,, \tag{19}$$

$$\sum_{m=0}^{n} S_2(n,m,A) c_{m,L,A,q}^{(k)} = \frac{\ell^{n+1}}{[n+1]_q^k}\,, \tag{20}$$

$$\sum_{m=0}^{n} S_2(n,m,A) \widehat{c}_{m,L,A,q}^{(k)} = \frac{(-1)^n \ell^{n+1}}{[n+1]_q^k}\,. \tag{21}$$

4. Several Relations of q-poly-Bernoulli Polynomials and q-poly-Cauchy Polynomials

Theorem 5. *For any z we have*

$$B_{n,L,A,q}^{(k)}(z) = \sum_{\mu=0}^{n} \sum_{m=\mu}^{n} m! S_2(n,m,A) S_2(m,\mu,A) c_{\mu,L,A,q}^{(k)}(z)\,,$$

$$B_{n,L,A,q}^{(k)}(z) = \sum_{\mu=0}^{n} \sum_{m=\mu}^{n} (-1)^m m! S_2(n,m,A) S_2(m,\mu,A) \widehat{c}_{\mu,L,A,q}^{(k)}(z)\,,$$

$$c_{n,L,A,q}^{(k)}(z) = \sum_{\mu=0}^{n} \sum_{m=\mu}^{n} \frac{1}{m!} S_1(n,m,A) S_1(m,\mu,A) B_{\mu,L,A,q}^{(k)}(z)\,,$$

$$\widehat{c}_{n,L,A,q}^{(k)}(z) = \sum_{\mu=0}^{n} \sum_{m=\mu}^{n} \frac{(-1)^m}{m!} S_1(n,m,A) S_1(m,\mu,A) B_{\mu,L,A,q}^{(k)}(z)\,.$$

Remark. If $\rho = 1$ and $q \to 1$ and $\alpha_i = i\rho$ $(i = 0, 1, \ldots, n-1)$, then Theorem 5 is reduced to Theorem 4.1 in [24]. A different generalization without Jackson's integrals is discussed in [23].

Proof. We shall prove the first and the fourth identities. The other two are proven similarly and omitted. By (17) in Theorem 4 and (12), we have

$$\begin{aligned} B_{n,L,A,q}^{(k)}(z) &= \sum_{m=0}^{n} S_2(n,m,A) m! \sum_{\mu=0}^{m} S_2(m,\mu,A) c_{\mu,L,A,q}^{(k)}(z) \\ &= \sum_{\mu=0}^{n} \sum_{m=\mu}^{n} m! S_2(n,m,A) S_2(m,\mu,A) c_{\mu,L,A,q}^{(k)}(z)\,. \end{aligned}$$

By (16) in Theorem 4 and Theorem 2, we have

$$\widehat{c}_{n,L,A,q}^{(k)}(z) = \sum_{m=0}^{n} \frac{(-1)^m}{m!} S_1(n,m,A) \sum_{\mu=0}^{m} S_1(m,\mu,A) B_{\mu,L,A,q}^{(k)}(z)$$
$$= \sum_{\mu=0}^{n} \sum_{m=\mu}^{n} \frac{(-1)^m}{m!} S_1(n,m,A) S_1(m,\mu,A) B_{\mu,L,A,q}^{(k)}(z)\,.$$

□

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