# POLYGONAL-SIERPIŃSKI-RIESEL SEQUENCES WITH TERMS HAVING AT LEAST TWO DISTINCT PRIME DIVISORS

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**Abstract**

It is known that there are infinitely many Sierpiński numbers and Riesel numbers in the sequences of triangular numbers, hexagonal numbers, pentagonal numbers, and many other polygonal sequences. Let $T_k$ denote the $k^{\text{th}}$ triangular number. We prove an additional property: for infinitely many $k$, every integer in the sequence $T_k 2^n + 1$ with $n$ a positive integer always has at least two distinct prime divisors. Furthermore, there are infinitely many $k$ such that every integer in the sequence $T_k 2^n - 1$ with $n$ a positive integer always has at least two distinct prime divisors. Also, there are infinitely many $k$ such that every integer in both sequences $T_k 2^n + 1$ and $T_k 2^n - 1$ with $n$ a positive integer always has at least two distinct prime divisors. Moreover, the above results hold when replacing $T_k$ with infinitely many different $s$-gonal number sequences.

## 1. Introduction

Polygonal numbers are those that can be expressed geometrically by an arrangement of equally spaced points. Such examples include triangular, square, pentagonal, and hexagonal numbers which all can be expressed with the corresponding geometric shape. For example the triangular numbers, $T_k$ for some positive integer $k$, can be represented as a triangular grid of equally spaced points in rows where the first row has one point, the second row has two points,..., and the $k^{\text{th}}$ row has $k$ points. Thus, every triangular number is of the form $\sum_{1\le i\le k} i = \frac{k(k+1)}{2}$ for positive integers $k$. In general for an integer $s \ge 3$, the $k^{\text{th}}$ $s$-gonal number is given by

$$S_k(s) = \frac{1}{2}k\big((s-2)k - (s-4)\big).$$

In 1960 Sierpiński [13] proved that there are infinitely many odd positive integers $k$ with the property that $k \cdot 2^n + 1$ is composite for all positive integers $n$. Such an integer $k$ is called a Sierpiński number. Two years later, Selfridge showed 78557 is a Sierpiński number (unpublished). This is currently the smallest known Sierpiński number. As of this writing, there are six candidates smaller than 78557 to consider: 10223, 21181, 22699, 24737, 55459, 67607. See `http://www.seventeenorbust.com` for the most up-to-date information.

Investigated by Riesel in 1956 [12], Riesel numbers are defined in a similar way: an odd positive integer $k$ is a Riesel number if $k \cdot 2^n - 1$ is composite for all positive integers $n$. The smallest known Riesel number is 509203. As of this writing there are 62 remaining candidates smaller that 509203 to consider. See `http://www.prothsearch.net/rieselprob.html` for the most recent information.

Previous work has been done to show an intersection between Sierpiński or Riesel numbers with familiar integer sequences such as the Fibonacci numbers [11] and the Lucas numbers [2]. In addition, intersections of Sierpiński and/or Riesel numbers were discovered with various polygonal number sequences [1] and perfect powers [9, 10].

Filaseta, Finch, and Kozek [9] proved the following conjecture of Y.-G. Chen: for each positive integer $r$, there exist infinitely many $k$ such that $k^r 2^n + 1$ has at least two distinct prime divisors for each positive integer $n$ [7]. Such a result immediately implies the terms in the sequence $k^r 2^n + 1$ with $n$ a positive integer are composite and $k^r$ is a Sierpiński number (for every such $k$) by definition. Similarly our results produce a condition stronger than just being a Sierpiński number and/or Riesel number.

A *covering system of congruences* (or more simply called a *covering*) is a set

$$\{(r_1, m_1), (r_2, m_2), \ldots, (r_t, m_t)\}$$

for which each integer $n$ satisfies $n \equiv r_i \pmod{m_i}$ for some $1 \leq i \leq t$. First discovered by Erdős who later used the idea to show there are infinitely many odd integers that are not of the form $2^k + p$ where $p$ is a prime [8], the notion of coverings has been applied to many problems in number theory. In fact, all of the previously cited works used coverings. The known results of intersections between various polygonal number sequences, the Riesel numbers, and the Sierpiński numbers [1] will be used and displayed throughout this paper in tables (all tables come from this work). These known results, along with additional constraints, help us to produce the following theorems.

**Theorem 1.** *There exist infinitely many positive integers $k$ such that $T_k$ is a Sierpiński number and $T_k 2^n + 1$ has at least two distinct prime divisors for every positive integer $n$.*

**Theorem 2.** *There exist infinitely many positive integers $k$ such that $T_k$ is a Riesel number and $T_k 2^n - 1$ has at least two distinct prime divisors for every positive integer $n$.*

**Theorem 3.** *There exist infinitely many positive integers $k$ such that $T_k$ is both a Sierpiński number and Riesel number; moreover, both $T_k 2^n + 1$ and $T_k 2^n - 1$ have at least two distinct prime divisors for every positive integer $n$.*

It is clear that the latter theorem implies the first two theorems. Yet, the latter theorem involves more congruence restrictions on $k$; hence, there is some intrinsic value on proving all three theorems to display the ideas separately and somewhat more apparently. We remark that comparable results were found for $T_k$ replaced by $k^r$ with $r$ in a fixed set [6, 9].

After proving these theorems for triangular numbers $T_k$ (as well as the hexagonal and pentagonal numbers) we prove how one can extrapolate the same theorems hold for infinitely many $s$-gonal sequences $S_k(s)$ replacing the triangular numbers $T_k$.

## 2. Notation and Preliminaries

Throughout we let $p$ denote a prime and $N$ denote a positive integer. Let $P(N)$ denote the greatest prime factor of $N$, and let $\nu_2(N)$ denote the exponent of 2 in the prime factorization of $N$. We use the Vinogradov symbols $\ll$ and $\gg$ to denote their usual meaning synonymous with big $O$ notation; that is, $f \ll g$ is equivalent to $f = O(g)$. We use $\approx$ to denote both $\ll$ and $\gg$; that is, $f \approx g$ means $f \ll g$ and $f \gg g$. For relatively prime integers $a$ and $n$ we let $\mathrm{ord}_n(a)$ denote the order of $a$ modulo $n$.

Within this section we utilize some facts pertaining to cyclotomic polynomials. For any positive integer $N$ we denote by $\Phi_N(x)$ the $N$th cyclotomic polynomial defined by

$$\Phi_N(x) = \prod_{\substack{1 \le k \le N \\ \gcd(k,N)=1}} (x - e^{2\pi i k/N}).$$

It is well known that this is a monic polynomial with integer coefficients. We use the following cyclotomic polynomial identities:

$$\Phi_{pN}(x) = \begin{cases} \Phi_N(x^p) & \text{if } p \mid N \\ \Phi_N(x^p)/\Phi_N(x) & \text{if } p \nmid N \end{cases} \tag{1}$$

and

$$x^N - 1 = \prod_{d \mid N} \Phi_d(x). \tag{2}$$

Our proofs of the theorems use results pertaining to the number of times the prime 2 may divide a cyclotomic polynomial evaluated at an odd integer. We note that more general results are known (cf. the lemmas of [3] and the cyclotomic criterion section of [4]). In an effort to keep this paper self contained we present a shorter proof for this more specific result.

**Lemma 1.** *Let $a$ be an integer and $N > 2$ be a positive integer. One has $2 \mid \Phi_N(a)$ if and only if $a$ is odd and $N = 2^h$ for some $h \geq 2$. Moreover, $4 \nmid \Phi_N(a)$.*

*Proof.* Assume $N > 2$. Observe for every prime $p$, we have $\Phi_p(1) = p$. From (1), $\Phi_N(1) = p$ if $N$ is a power of a prime $p$, and $\Phi_N(1) = 1$ if $N$ is an integer with more than one distinct prime factor. In general, from the definition or (2), one can obtain $\Phi_N(0) = 1$ or $-1$. Hence, if $N$ is not a power of 2 or if $a$ is even, then $\Phi_N(a) \equiv 1 \pmod 2$. If $N = 2^h$ with $h \geq 2$ and if $a$ is odd, then (1) yields

$$\Phi_N(a) \equiv \Phi_{2^h}(a) \equiv a^{2^{h-1}} + 1 \equiv 2 \pmod 4.$$

Thus, in any case, $4 \nmid \Phi_N(a)$ for $N > 2$. $\square$

Notice that the condition $N > 2$ in the above result is important, as $\Phi_2(a) = a+1$ can clearly, for the right choice of $a$, be divisible by an arbitrarily large power of 2.

The next result, a consequence of Lemma 1, provides bounds for the exponent of 2 in the prime factorizations of $a^r + 1$ and $a^r - 1$. Let $\nu_2(N)$ denote the exponent of 2 in the prime factorization of a positive integer $N$.

**Corollary 1.** *If $r$ is a positive integer and $a$ is an odd integer, then*

$$\nu_2(a^r - 1) \leq \nu_2(r) + \nu_2\big(\Phi_1(a)\big) + \nu_2\big(\Phi_2(a)\big) - 1 \quad \text{and} \quad \nu_2(a^r + 1) \leq \nu_2(a + 1).$$

*Proof.* First, we deal with $a^r - 1$. By (2),

$$a^r - 1 = \prod_{d \mid r} \Phi_d(a).$$

Since $a$ is odd, note $\Phi_1(a) = a - 1$ and $\Phi_2(a) = a + 1$ are even. By Lemma 1, if $2 \nmid r$, then $\nu_2(a^r - 1) = \nu_2\big(\Phi_1(a)\big)$. By Lemma 1, if $2 \mid r$, then

$$\begin{aligned}
\nu_2(a^r - 1) &= \sum_{d \mid r} \nu_2\big(\Phi_d(a)\big) \\
&= \nu_2\big(\Phi_1(a)\big) + \nu_2\big(\Phi_2(a)\big) + \sum_{2 \leq h \leq \nu_2(r)} \nu_2\big(\Phi_{2^h}(a)\big) \\
&= \nu_2\big(\Phi_1(a)\big) + \nu_2\big(\Phi_2(a)\big) + \sum_{2 \leq h \leq \nu_2(r)} 1.
\end{aligned}$$

It follows that $\nu_2(a^r - 1) \leq \nu_2(r) + \nu_2\big(\Phi_1(a)\big) + \nu_2\big(\Phi_2(a)\big) - 1$ for any positive integer $r$.

Now, we deal with bounding $\nu_2(a^r+1)$. Observe

$$a^r + 1 = \frac{a^{2r}-1}{a^r-1} = \prod_{d \mid 2r,\ d \nmid r} \Phi_d(a).$$

First, assume $r$ is even. Then, the latter product does not include $d = 1$ or 2. Moreover, the only divisors $d$ in the above product divisible by 2 satisfy $2^{\nu_2(r)+1} \mid d$. Let $x = \nu_2(r) + 1$. Since $r$ is even, we have $x \geq 2$. By Lemma 1, we deduce $\nu_2(a^r+1) = \nu_2\big(\Phi_{2^x}(a)\big) = 1$. On the other hand, if $r$ is odd, then it follows from Lemma 1 that $\nu_2(a^r+1) = \nu_2\big(\Phi_2(a)\big) = \nu_2(a+1)$. All together, we deduce that $\nu_2(a^r+1) \leq \nu_2(a+1)$. □

The following lemma is classical and we state it without proof. It can be proven readily with the prime number theorem.

**Lemma 2.** *Let $\omega(N)$ denote the number of distinct prime divisors for a positive integer $N$. If $N > 3$, then $\omega(N) \ll \frac{\log N}{\log\log N}$.*

## 3. Beginnings of Proofs

Within this section we propose sufficient assumptions to prove Theorems 1, 2, and 3 within Proposition 1. The proposition encompasses the main approach of this paper and outlines our procedure to deduce such theorems.

For the following we fix an $s$-gonal sequence $S_k = S_k(s)$ with $s \geq 3$. We write $S_k = (1/2) \cdot k \cdot k^*$ and note that $k^* = (s-2)k - (s-4)$ is an integer.

**Proposition 1.** *Let $\mathcal{K}$ denote a congruence class with relatively prime residue and modulus, and let $\mathcal{P}$ denote a nonempty finite set of primes. Suppose the following:*

*(i) The finite set of primes $\mathcal{P}$ satisfies for every $k \in \mathcal{K}$ that*

  *(a) Sierpiński case: $S_k$ is odd and $S_k 2^n + 1$ is divisible by a prime in $\mathcal{P}$ for every positive integer $n$,*

  *(b) Riesel case: $S_k$ is odd and $S_k 2^n - 1$ is divisible by a prime in $\mathcal{P}$ for every positive integer $n$,*

  *(c) Sierpiński and Riesel case: $S_k$ is odd and $S_k 2^n + 1$ and $S_k 2^n - 1$ are each divisible by a prime in $\mathcal{P}$ for every positive integer $n$.*

*(ii) There exists a nonempty finite set of primes $\mathcal{Q}$ satisfying the following: for every prime $k \in \mathcal{K}$, one has that $P(S_k) \geq k$ and $\prod_{q \in \mathcal{Q}} q$ divides $k^*$. This produces a finite set of primes*

$$\mathcal{R} = \mathcal{R}(\mathcal{P}, \mathcal{Q}) = \bigcap_{p \in \mathcal{P}} \left\{ \ell \text{ prime} : \ell \mid \prod_{q \in \mathcal{Q}} \operatorname{ord}_q(p) \right\}$$

*with the property*

(a) *Sierpiński case:* $\prod_{\ell \in \mathcal{R}} (1 - \frac{1}{\ell}) < 1/2$,

(b) *Riesel case:* $\prod_{\ell \in \mathcal{R},\, \ell \neq 2} (1 - \frac{1}{\ell}) < 1/2$,

(c) *Sierpiński and Riesel case:* $\prod_{\ell \in \mathcal{R},\, \ell \neq 2} (1 - \frac{1}{\ell}) < 1/2$.

*Then, for every sufficiently large prime $k \in \mathcal{K}$*

(a) *Sierpiński case: $S_k 2^n + 1$ has at least two distinct prime divisors for all $n \geq 1$.*

(b) *Riesel case: $S_k 2^n - 1$ has at least two distinct prime divisors for all $n \geq 1$.*

(c) *Sierpiński and Riesel case: both $S_k 2^n + 1$ and $S_k 2^n - 1$ have at least two distinct prime divisors for all $n \geq 1$.*

For example to prove Theorem 1, we are claiming that assumptions (i)a and (ii)a in the case $S_k$ is the triangular number $T_k$ imply the result. Likewise to prove Theorem 2, we are claiming that assumptions (i)b and (ii)b imply the result. Lastly, we claim that assumptions (i)c and (ii)c imply Theorem 3. In the following subsections, we prove each case separately. Notice that $\mathcal{K}$, $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$ may differ in each case. In the last subsection, we provide an overview of how we obtain the sufficient assumptions for $T_k = S_k(3)$ and $P_k = S_k(5)$.

### 3.1. Sierpiński Case

Assume the assumptions above for this case have been established. With the intent of reaching a contradiction, start by assuming there is a $k \in \mathcal{K}$ and a positive integer $n$ such that $S_k 2^n + 1$ has only one distinct prime divisor. By (i), it follows that $S_k 2^n + 1 = p^r$ for some prime $p \in \mathcal{P}$ and some $r \geq 1$. Thus, we have that $S_k 2^n = p^r - 1$. By (ii), $S_k 2^n = kk^* 2^{n-1}$ with $n$ a positive integer and $S_k$ is divisible by every prime $q \in \mathcal{Q}$. All such primes $q \in \mathcal{Q}$ henceforth divide $p^r - 1$. Therefore, $\mathrm{ord}_q(p) \mid r$ for each $q \in \mathcal{Q}$ which implies every prime in $\mathcal{R}$ divides $r$ for each possible prime $p \in \mathcal{P}$. Thus, independent of the prime $p \in \mathcal{P}$ with $S_k 2^n + 1 = p^r$, we have that $r$ is divisible by every prime in $\mathcal{R}$.

Next, let us recall the fact (cf. [4] or [5])

$$p^{\phi(d) - 2^{\omega(d)}} \leq |\Phi_d(p)| \leq p^{\phi(d) + 2^{\omega(d)}}.$$

In addition, if $k$ is prime, then $k \leq P(S_k) = P(p^r - 1) \leq \max_{d|r} |\Phi_d(p)| \leq \max_{d|r} p^{\phi(d)+2^{\omega(d)}} = p^{\phi(r)+2^{\omega(r)}}$. By assumption $S_k$ is odd for such values of $k$; one deduces that $n = \nu_2(S_k 2^n) = \nu_2(p^r - 1)$. By Corollary 1, we deduce that there exists a constant $c_1 > 0$ such that $\nu_2(r) \geq n + 1 - \nu_2\big(\Phi_1(p)\big) - \nu_2\big(\Phi_2(p)\big) \geq n - c_1$ for any $p \in \mathcal{P}$. Thus, $2^n \ll 2^{\nu_2(r)} \leq r$. Recall that $S_k = S_k(s) = \frac{1}{2} k\big((s-2)k - (s-4)\big)$ where $s \geq 3$ is fixed. Solving for $k$ in $S_k 2^n = p^r - 1$, one finds as $2^n \ll r$ that

$k \approx \sqrt{\frac{p^r}{2^n}}$. It follows that $k$ and $r$ become large uniformly. Thus, there exists a constant $c > 0$ such that

$$p^{r/2 - n\log_p \sqrt{2} + \log_p c} < k \leq p^{\phi(r) + 2^{\omega(r)}}. \tag{3}$$

By (ii), we infer that there exists $\epsilon > 0$ such that

$$\phi(r) = r \prod_{p|r} \left(1 - \frac{1}{p}\right) \leq r \prod_{\ell \in \mathcal{R}} \left(1 - \frac{1}{\ell}\right) \leq \left(\frac{1}{2} - \epsilon\right) r.$$

Since $2^n \ll r$, we deduce $n \ll \log r$. Lastly by Lemma 2, there exists a constant $C > 0$ such that $2^{\omega(r)} \leq r^{C/\log\log r}$. All together, for sufficiently large $k$ or $r$, we have that

$$\phi(r) + 2^{\omega(r)} + n \log_p \sqrt{2} - \log_p c < r/2$$

which contradicts (3). Hence for sufficiently large $k$ satisfying (i) and (ii) that are prime, one has $S_k 2^n + 1 \neq p^r$. In other words, for every sufficiently large prime $k$ satisfying (i) and (ii), we have $S_k 2^n + 1$ is composite and has at least two distinct prime divisors for all $n \geq 1$. We have now established the Sierpiński case.

### 3.2. Riesel Case

We proceed analogously for the Riesel case. Assume assumptions (i) and (ii) have been established for the Riesel case. With the intent of reaching a contradiction, start by assuming there is a $k \in \mathcal{K}$ and a positive integer $n$ such that $S_k 2^n - 1$ has only one distinct prime divisor. By (i), it follows that $S_k 2^n - 1 = p^r$ for some prime $p \in \mathcal{P}$ and some $r \geq 1$. Thus, we have that $S_k 2^n = p^r + 1$. By (ii), $S_k 2^n = kk^* 2^{n-1}$ with $n$ a positive integer and $S_k$ is divisible by every prime $q \in \mathcal{Q}$. All such primes $q \in \mathcal{Q}$ henceforth divide $p^r + 1$. Therefore, $\text{ord}_q(p) \mid 2r$ for each $q \in \mathcal{Q}$ which implies every prime in $\mathcal{R}$ divides $r$ for each possible prime $p \in \mathcal{P}$. Thus, independent of the prime $p \in \mathcal{P}$ with $S_k 2^n - 1 = p^r$, we have that $r$ is divisible by every prime in $\mathcal{R}$.

Since $S_k$ is odd for such values of $k$, one deduces that $n = \nu_2(S_k 2^n) = \nu_2(p^r + 1)$. By Corollary 1, we have that $n \leq \nu_2(p+1)$. Thus, $n$ is bounded since we are working with a finite set of $p \in \mathcal{P}$. Solving for $k$ in $S_k 2^n = p^r + 1$, one finds $k \approx p^{r/2}$. It follows that $k$ and $r$ become large uniformly.

In addition, assume $k$ is prime. Note $S_k 2^n = p^r + 1 = \frac{p^{2r}-1}{p^r-1}$. From this, one finds that $k \leq P(S_k) = P(p^r + 1) \leq \max_{d|2r} |\Phi_d(p)| \leq \max_{d|2r} p^{\phi(d) + 2^{\omega(d)}} = p^{\phi(2r) + 2^{\omega(2r)}}$. All together, there exists a constant $c > 0$ such that

$$p^{r/2 + \log_p c} < k \leq p^{\phi(2r) + 2^{\omega(2r)}}. \tag{4}$$

By (ii), we infer that there exists $\epsilon > 0$ such that

$$\phi(2r) = 2r \prod_{p|2r} \left(1 - \frac{1}{p}\right) \leq r \prod_{\ell \in \mathcal{R}, \ell \neq 2} \left(1 - \frac{1}{p}\right) \leq \left(\frac{1}{2} - \epsilon\right) r.$$

By Lemma 2, we deduce for sufficiently large $r$ that

$$\phi(2r) + 2^{\omega(2r)} - \log_p c < r/2$$

which contradicts (4). Hence for sufficiently large $k$ satisfying (i) and (ii) that are prime, one has that $S_k 2^n - 1 \neq p^r$. In other words, for every sufficiently large prime $k$ satisfying (i) and (ii), one has that $S_k 2^n - 1$ is composite and has at least two distinct prime divisors for all $n \geq 1$.

### 3.3. Sierpiński and Riesel Case

To prove this case, it is now enough to proceed so that both the Sierpiński case and Riesel case hold for the same congruence class $\mathcal{K}$. We claim that it suffices to have both (i)c and (ii)c hold. Indeed, first take note that (i)c yields a congruence class producing $S_k$ that are both Sierpiński and Riesel. In both the Sierpiński case and the Riesel case, we made use of a set of primes $\mathcal{R}$ that contained prime divisors of $r$ (independent of the possible prime $p \in \mathcal{P}$). Notice the inequality in (ii)b implies the inequality in (ii)a. Thus, (ii)c implies both (ii)a and (ii)b hold; henceforth, we establish the Sierpiński and Riesel case.

### 3.4. Overview of How to Obtain Assumptions for $T_k$ (and $P_k$)

Fix $s = 3$ or $5$ so that $S_k$ is the triangular numbers $T_k$ or pentagonal numbers $P_k$. From the work above, we demonstrated how the assumptions in the proposition imply results like Theorems 1, 2, and 3. We now provide an overview of how to obtain the assumptions in the proposition. We begin by utilizing the work of the authors et al. [1] to deduce the existence of a congruence class for $k$ producing $T_k$ (or $P_k$) which are also Sierpiński, Riesel, or Sierpiński-Riesel. This alone would provide us with just (i). By introducing some additional constraints on $k$, more precisely additional congruences that intersect the congruences for $k$ from the authors et al., we develop a set of primes $\mathcal{Q}$ such that $q \mid T_k$ (or $q \mid P_k$) for every $q \in \mathcal{Q}$. In particular, $k^* = k+1$ (or $k^* = 3k-1$) is divisible by every prime in $\mathcal{Q}$. When using a congruence for $k$ with relatively prime residue and modulus, Dirichlet's classical theorem on primes in arithmetic progressions guarantees that there are infinitely many primes in the congruence for $k$ above. Then, we only work with primes $k$ satisfying the above criterion. Now, let us separately discuss how to also obtain (ii) in each of the cases Sierpiński, Riesel, and Sierpiński and Riesel. We are now ready to prove the theorems.

| | | | | |
|---|---|---|---|---|
| $n \equiv 1 \pmod{2}$ | & | $k \equiv 1 \pmod{3}$ | $\Longrightarrow$ | $3 \mid (T_k \cdot 2^n + 1)$ |
| $n \equiv 0 \pmod{3}$ | & | $k \equiv 3 \pmod{7}$ | $\Longrightarrow$ | $7 \mid (T_k \cdot 2^n + 1)$ |
| $n \equiv 2 \pmod{4}$ | & | $k \equiv 1$ or $3 \pmod{5}$ | $\Longrightarrow$ | $5 \mid (T_k \cdot 2^n + 1)$ |
| $n \equiv 4 \pmod{8}$ | & | $k \equiv 1$ or $15 \pmod{17}$ | $\Longrightarrow$ | $17 \mid (T_k \cdot 2^n + 1)$ |
| $n \equiv 8 \pmod{12}$ | & | $k \equiv 4$ or $8 \pmod{13}$ | $\Longrightarrow$ | $13 \mid (T_k \cdot 2^n + 1)$ |
| $n \equiv 16 \pmod{24}$ | & | $k \equiv 53$ or $187 \pmod{241}$ | $\Longrightarrow$ | $241 \mid (T_k \cdot 2^n + 1)$ |

TABLE 1

## 4. Sierpiński Triangular

First, let us display the idea behind the proof of the existence of infinitely many triangular numbers that are also Sierpiński numbers [1]. Consider the implications in Table 1 below.

The congruences for $n$ in Table 1 form a covering. In each row, the congruences for $k$ result in $T_k \cdot 2^n + 1$ being divisible by one of the primes in $\mathcal{P} = \{3, 5, 7, 13, 17, 241\}$ for every positive integer in the congruence of $n$. By choosing the values of $k$ in each row large enough from the congruence for $k$ so that $T_k$ is greater than the prime from $\mathcal{P}$ in that row, one has that each of these $T_k \cdot 2^n + 1$ must be composite for every positive integer in the congruence of $n$. For our purpose, we then use the Chinese remainder theorem to find the intersection of all congruences for $k$ in Table 1, $k \equiv -1 \pmod{19 \cdot 23}$, and $k \equiv 1 \pmod{4}$. (The latter congruence ensures that both $T_k$ and $k$ are odd.) There are $2^4$ possibilities for congruence classes of $k$, all of which result in $T_k$ being a Sierpiński number. Any of these congruences for $k$ can represent $\mathcal{K}$ in the statement of Proposition 1. One such example congruence for $k$ satisfying all of the congruences in Table 1 and the two other mentioned congruences would be

$$k \equiv 497885461 \pmod{9775523940}.$$

Now, we work toward justifying that (ii) holds. Dirichlet's classical theorem on primes in arithmetic progressions guarantees that there are infinitely many primes in the congruence $\mathcal{K}$ (as each congruence possesses relatively prime residue and modulus). Now, one works with primes $k$ from $\mathcal{K}$.

Assume that there is an $n$ and $k$ such that $T_k 2^n + 1$ is divisible by only one distinct prime. Then, $T_k 2^n = p^r - 1$ for some integer $r \geq 1$. Since 19 and 23 divide $k + 1$, they both divide $p^r - 1$. Thus, $\mathrm{ord}_{19}(p)$ and $\mathrm{ord}_{23}(p)$ divide $r$. Easy calculations provide us with $2 \cdot 3 \cdot 11 \mid r$ for each prime $p \in \mathcal{P}$. Here we have obtained $\mathcal{Q} = \{19, 23\}$ and $\mathcal{R} = \{2, 3, 11\}$. Lastly, observe that $\prod_{\ell \in \mathcal{R}} (1 - \frac{1}{p}) = \frac{10}{33} < \frac{1}{2}$. Now, the assumptions in Proposition 1 hold, and we deduce Theorem 1.

$$
\begin{array}{llcll}
n \equiv 0 \pmod 2 & \& & k \equiv 1 \pmod 3 & \Longrightarrow & 3 \mid (T_k \cdot 2^n - 1) \\
n \equiv 0 \pmod 3 & \& & k \equiv 1 \text{ or } 5 \pmod 7 & \Longrightarrow & 7 \mid (T_k \cdot 2^n - 1) \\
n \equiv 1 \pmod 4 & \& & k \equiv 2 \pmod 5 & \Longrightarrow & 5 \mid (T_k \cdot 2^n - 1) \\
n \equiv 7 \pmod 8 & \& & k \equiv 8 \pmod{17} & \Longrightarrow & 17 \mid (T_k \cdot 2^n - 1) \\
n \equiv 11 \pmod{12} & \& & k \equiv 5 \text{ or } 7 \pmod{13} & \Longrightarrow & 13 \mid (T_k \cdot 2^n - 1) \\
n \equiv 16 \pmod{24} & \& & k \equiv 5 \text{ or } 235 \pmod{241} & \Longrightarrow & 241 \mid (T_k \cdot 2^n - 1)
\end{array}
$$

Table 2

## 5. Riesel Triangular

In this section, we first prove (i) holds. Consider the implications in Table 2 below.

The congruences for $n$ in Table 2 form a covering. In each row, the congruences for $k$ result in $T_k \cdot 2^n - 1$ being divisible by one of the primes in $\mathcal{P} = \{3, 5, 7, 13, 17, 241\}$ for every positive integer in the congruence of $n$. By choosing the values of $k$ in each row large enough from the congruence for $k$ so that $T_k$ is greater than the prime from $\mathcal{P}$ in the row one has that each of these $T_k \cdot 2^n - 1$ must be composite for every positive integer in the congruence of $n$. For our purpose we then use the Chinese remainder theorem to find the intersection of all congruences for $k$ in Table 1, $k \equiv -1 \pmod{11 \cdot 23 \cdot 31}$, and $k \equiv 1 \pmod 4$. (The latter congruence ensures that both $T_k$ and $k$ are odd.) There are $2^3$ possibilities for congruence classes of $k$, all of which result in $T_k$ being a Riesel number. Any of these congruences for $k$ can represent $\mathcal{K}$ in the statement of Proposition 1. One such example congruence for $k$ satisfying all of the congruences in Table 1 and the two other mentioned congruences would be

$$k \equiv 55032182017 \pmod{175444929660}.$$

Now, we work toward justifying that (ii) holds. Dirichlet's classical theorem on primes in arithmetic progressions guarantees that there are infinitely many primes in the congruence $\mathcal{K}$ (as each congruence possesses relatively prime residue and modulus). Now, one works with primes $k$ from $\mathcal{K}$.

Assume that there is an $n$ and $k$ such that $T_k 2^n - 1$ is divisible by only one distinct prime. Then, $T_k 2^n = p^r + 1$ for some integer $r \geq 1$. Here we have $\mathcal{Q} = \{11, 23, 31\}$. Since each $q \in \mathcal{Q}$ satisfies $q \mid (k+1)$, we also obtain that $q \mid p^r + 1$ for every $q \in \mathcal{Q}$. Thus, $\text{ord}_q(p)$ divides $2r$ for each $q \in \mathcal{Q}$. Straightforward calculations provide us with $2 \cdot 3 \cdot 5 \cdot 11 \mid 2r$ for each prime $p \in \mathcal{P}$. Here we have obtained $\mathcal{R} = \{2, 3, 5, 11\}$. Lastly, observe that $\prod_{\ell \in \mathcal{R}, \ell \neq 2} (1 - \frac{1}{p}) = \frac{16}{33} < \frac{1}{2}$. Thus, the assumptions in Proposition 1 hold, and we deduce Theorem 2.

## 6. Sierpiński-Riesel Numbers

Again, we start by justifying (i). Table 3 below gives congruences for $k$ to construct triangular numbers $T_k$ that are simultaneously Sierpiński and Riesel numbers. The congruences for $n$ above the horizontal line form a covering. This part of the table ensures that the congruences for $k$ (along with another congruence $k \equiv 1 \pmod 4$) ensuring $T_k$ is odd) yield a Sierpiński number $T_k$. In addition, the congruences for $n$ below the horizontal line also form a covering. The bottom part of the table ensures that the congruences for $k$ (along with another congruence $k \equiv 1 \pmod 4$ ensuring $T_k$ is odd) yield a Riesel number. Notice that the congruences for $k$ above the line and those below the line have an intersection by the Chinese remainder theorem (the only modulus that is repeated in the two parts of the table is 3, and in both instances $k \equiv 1 \pmod 3$).

In each row of Table 3, the congruences for $k$ result in $T_k \cdot 2^n + 1$ and $T_k \cdot 2^n - 1$ being divisible by one of the primes in

$$\mathcal{P} = \{3, 5, 7, 11, 13, 17, 19, 31, 37, 41, 61, 73, 97, 109, 151, 241, 257, 673\}$$

for every positive integer in the congruence of $n$. By choosing the values of $k$ in each row large enough from the congruence for $k$ so that $T_k$ is greater than the prime from $\mathcal{P}$ in the row, one has that each of these $T_k \cdot 2^n + 1$ and $T_k \cdot 2^n - 1$ must be composite for every positive integer in the corresponding congruence of $n$. For our purpose, we then use the Chinese remainder theorem to find the intersection of all congruences for $k$ in Table 3, $k \equiv -1 \pmod{23 \cdot 29 \cdot 43 \cdot 47 \cdot 67}$, and $k \equiv 1 \pmod 4$. (The latter congruence ensures that both $T_k$ and $k$ are odd.) There are $2^{14}$ possibilities for congruence classes of $k$, all of which result in $T_k$ being a Sierpiński number and a Riesel number. Any of these congruences for $k$ can represent $\mathcal{K}$ in the statement of Proposition 1. One such example congruence for $k$ satisfying all of the congruences in Table 3 and the two other mentioned congruences would be

$$k \equiv 969305265377969309051168953526377 \pmod{4 \cdot \prod_{p \in \mathcal{P} \cup \mathcal{Q}} p}$$

where $\mathcal{Q} = \{23, 29, 43, 47, 67\}$.

Now we work toward justifying that (ii) holds. Dirichlet's classical theorem on primes in arithmetic progressions guarantees that there are infinitely many primes in the congruence $\mathcal{K}$ (as each congruence possesses relatively prime residue and modulus). One works with primes $k$ from $\mathcal{K}$. Every $q \in \mathcal{Q}$ satisfies $q \mid p^r - 1$ or $q \mid p^r + 1$. Thus, $\mathrm{ord}_q(p)$ divides $2r$ for each $q \in \mathcal{Q}$. Straightforward calculations provide us with $2 \cdot 3 \cdot 7 \cdot 11 \cdot 23$ divides $2r$ for each prime $p \in \mathcal{P}$ so that $\mathcal{R} = \{2, 3, 7, 11, 23\}$. Lastly, $\prod_{\ell \in \mathcal{R}, \ell \neq 2}(1 - \frac{1}{p}) = \frac{80}{161} < \frac{1}{2}$. Hence, the assumptions in Proposition 1 hold, and we deduce Theorem 3.

| | | | | |
|---|---|---|---|---|
| $n \equiv 1 \pmod{2}$ | & | $k \equiv 1 \pmod{3}$ | $\Longrightarrow$ | $3 \mid (T_k \cdot 2^n + 1)$ |
| $n \equiv 1 \pmod{3}$ | & | $k \equiv 2 \text{ or } 4 \pmod{7}$ | $\Longrightarrow$ | $7 \mid (T_k \cdot 2^n + 1)$ |
| $n \equiv 5 \pmod{9}$ | & | $k \equiv 23 \text{ or } 49 \pmod{73}$ | $\Longrightarrow$ | $73 \mid (T_k \cdot 2^n + 1)$ |
| $n \equiv 6 \pmod{12}$ | & | $k \equiv 1 \text{ or } 11 \pmod{13}$ | $\Longrightarrow$ | $13 \mid (T_k \cdot 2^n + 1)$ |
| $n \equiv 8 \pmod{18}$ | & | $k \equiv 6 \text{ or } 12 \pmod{19}$ | $\Longrightarrow$ | $19 \mid (T_k \cdot 2^n + 1)$ |
| $n \equiv 2 \pmod{36}$ | & | $k \equiv 15 \text{ or } 21 \pmod{37}$ | $\Longrightarrow$ | $37 \mid (T_k \cdot 2^n + 1)$ |
| $n \equiv 20 \pmod{36}$ | & | $k \equiv 24 \text{ or } 84 \pmod{109}$ | $\Longrightarrow$ | $109 \mid (T_k \cdot 2^n + 1)$ |
| $n \equiv 4 \pmod{5}$ | & | $k \equiv 13 \text{ or } 17 \pmod{31}$ | $\Longrightarrow$ | $31 \mid (T_k \cdot 2^n + 1)$ |
| $n \equiv 6 \pmod{10}$ | & | $k \equiv 3 \text{ or } 7 \pmod{11}$ | $\Longrightarrow$ | $11 \mid (T_k \cdot 2^n + 1)$ |
| $n \equiv 8 \pmod{20}$ | & | $k \equiv 9 \text{ or } 31 \pmod{41}$ | $\Longrightarrow$ | $41 \mid (T_k \cdot 2^n + 1)$ |
| $n \equiv 0 \pmod{15}$ | & | $k \equiv 69 \text{ or } 81 \pmod{151}$ | $\Longrightarrow$ | $151 \mid (T_k \cdot 2^n + 1)$ |
| $n \equiv 12 \pmod{60}$ | & | $k \equiv 20 \text{ or } 40 \pmod{61}$ | $\Longrightarrow$ | $61 \mid (T_k \cdot 2^n + 1)$ |
| $n \equiv 0 \pmod{2}$ | & | $k \equiv 1 \pmod{3}$ | $\Longrightarrow$ | $3 \mid (T_k \cdot 2^n - 1)$ |
| $n \equiv 1 \pmod{4}$ | & | $k \equiv 2 \pmod{5}$ | $\Longrightarrow$ | $5 \mid (T_k \cdot 2^n - 1)$ |
| $n \equiv 7 \pmod{8}$ | & | $k \equiv 8 \pmod{17}$ | $\Longrightarrow$ | $17 \mid (T_k \cdot 2^n - 1)$ |
| $n \equiv 11 \pmod{16}$ | & | $k \equiv 128 \pmod{257}$ | $\Longrightarrow$ | $257 \mid (T_k \cdot 2^n - 1)$ |
| $n \equiv 11 \pmod{24}$ | & | $k \equiv 90 \text{ or } 150 \pmod{241}$ | $\Longrightarrow$ | $241 \mid (T_k \cdot 2^n - 1)$ |
| $n \equiv 3 \pmod{48}$ | & | $k \equiv 41 \text{ or } 55 \pmod{97}$ | $\Longrightarrow$ | $97 \mid (T_k \cdot 2^n - 1)$ |
| $n \equiv 19 \pmod{48}$ | & | $k \equiv 315 \text{ or } 357 \pmod{673}$ | $\Longrightarrow$ | $673 \mid (T_k \cdot 2^n - 1)$ |

Table 3

## 7. Hexagonal-Sierpiński-Riesel Numbers

If we include the congruence $k \equiv 1 \pmod{2}$ with the congruences in the previous subsection, we then have triangular numbers that are also hexagonal, in addition to being both Sierpiński and Riesel. This inherently provides us with the following theorem.

**Theorem 4.** *There exist infinitely many integers $k$ such that the hexagonal number $H_k$ is both a Sierpiński number and Riesel number; moreover, $H_k 2^n + 1$ and $H_k 2^n - 1$ have at least two distinct prime divisors for every positive integer $n$.*

## 8. Pentagonal Numbers

Within this section we let $P_k$ denote the $k^{\text{th}}$ pentagonal number; that is, in terms of the $s$-gonal numbers, we let $P_k = S_k(5) = k(3k-1)/2$. We utilize Proposition 1 again and use similar proofs as those mentioned previously. The only subtle change with pentagonal numbers is that $k^* = 3k-1$; in other words, $P_k = (1/2)k(3k-1)$. All proofs are analogous to those encountered in previous sections.

It is known that there are infinitely many pentagonal-Sierpiński numbers, infinitely many pentagonal-Riesel numbers, and infinitely many pentagonal numbers that are simultaneously Sierpiński and Riesel [1].

### 8.1. Pentagonal-Sierpiński Numbers

Consider the implications in Table 4 below.

$$
\begin{array}{llcll}
n \equiv 1 \pmod{2} & \& & k \equiv 1 \pmod{3} & \Longrightarrow & 3 \mid (P_k \cdot 2^n + 1) \\
n \equiv 2 \pmod{3} & \& & k \equiv 2 \text{ or } 3 \pmod{7} & \Longrightarrow & 7 \mid (P_k \cdot 2^n + 1) \\
n \equiv 2 \pmod{4} & \& & k \equiv 1 \pmod{5} & \Longrightarrow & 5 \mid (P_k \cdot 2^n + 1) \\
n \equiv 4 \pmod{8} & \& & k \equiv 1 \text{ or } 5 \pmod{17} & \Longrightarrow & 17 \mid (P_k \cdot 2^n + 1) \\
n \equiv 0 \pmod{12} & \& & k \equiv 3 \text{ or } 6 \pmod{13} & \Longrightarrow & 13 \mid (P_k \cdot 2^n + 1) \\
n \equiv 16 \pmod{24} & \& & k \equiv 189 \text{ or } 213 \pmod{241} & \Longrightarrow & 241 \mid (P_k \cdot 2^n + 1)
\end{array}
$$

Table 4

Consider $k$ satisfying all congruences for $k$ in Table 4, $k \equiv 3^{-1} \pmod{11 \cdot 19 \cdot 23}$, and $k \equiv 1 \pmod{4}$. (The latter congruence ensures that both $P_k$ and $k$ are odd.) There are $2^4$ such congruences for $k$, all of which result in $P_k$ being a Sierpiński number when proceeding as mentioned in previous sections. One such example is

$$k \equiv 7225443361 \pmod{107530763340}.$$

Let $\mathcal{K}$ denote such a congruence for $k$, and consider primes in $\mathcal{K}$.

Assume that there is an $n$ and $k$ such that $P_k 2^n + 1$ is divisible by only one distinct prime. Then, $P_k 2^n = p^r - 1$ for some integer $r \geq 1$. Since $q$ divides $3k-1$ for every $q \in \mathcal{Q} := \{11, 19, 23\}$, we deduce $q \mid p^r - 1$ for every $q \in \mathcal{Q}$. We obtain $\mathcal{R} = \{2, 3, 11\}$ which implies $\prod_{\ell \in \mathcal{R}}(1 - \frac{1}{p}) = \frac{10}{33}$. Hence, we have the following result by Proposition 1.

**Theorem 5.** *There exist infinitely many $k$ such that the pentagonal number $P_k$ is a Sierpiński number and $P_k 2^n + 1$ has at least two distinct prime divisors for every positive integer $n$.*

### 8.2. Pentagonal-Riesel Numbers

We prove a similar result for the Riesel case.

**Theorem 6.** *There exist infinitely many $k$ such that the pentagonal number $P_k$ is a Riesel number and $P_k 2^n - 1$ has at least two distinct prime divisors for every positive integer $n$.*

Consider the following table:

| | | | | |
|---|---|---|---|---|
| $n \equiv 0 \pmod{2}$ | & | $k \equiv 1 \pmod{3}$ | $\Longrightarrow$ | $3 \mid (P_k \cdot 2^n - 1)$ |
| $n \equiv 2 \pmod{3}$ | & | $k \equiv 6 \pmod{7}$ | $\Longrightarrow$ | $7 \mid (P_k \cdot 2^n - 1)$ |
| $n \equiv 3 \pmod{4}$ | & | $k \equiv 3$ or $4 \pmod{5}$ | $\Longrightarrow$ | $5 \mid (P_k \cdot 2^n - 1)$ |
| $n \equiv 1 \pmod{8}$ | & | $k \equiv 10$ or $13 \pmod{17}$ | $\Longrightarrow$ | $17 \mid (P_k \cdot 2^n - 1)$ |
| $n \equiv 1 \pmod{12}$ | & | $k \equiv 11 \pmod{13}$ | $\Longrightarrow$ | $13 \mid (P_k \cdot 2^n - 1)$ |
| $n \equiv 21 \pmod{24}$ | & | $k \equiv 61$ or $100 \pmod{241}$ | $\Longrightarrow$ | $241 \mid (P_k \cdot 2^n - 1)$ |

Table 5

Consider $k$ satisfying all congruences for $k$ in Table 5, $k \equiv 3^{-1} \pmod{11 \cdot 23 \cdot 31}$, and $k \equiv 1 \pmod{4}$. (The latter congruence ensures that both $P_k$ and $k$ are odd.) There are $2^3$ such congruences for $k$, all of which result in $P_k$ being a Riesel number when proceeding as mentioned in previous sections. One such example is

$$k \equiv 54303588233 \pmod{175444929660}.$$

Let $\mathcal{K}$ denote such a congruence for $k$, and consider primes in $\mathcal{K}$.

Assume that there is an $n$ and $k$ such that $T_k 2^n - 1$ is divisible by only one distinct prime. Then, $T_k 2^n = p^r + 1$ for some integer $r \geq 1$. Since $q$ divides $3k - 1$ for every $q \in \mathcal{Q} := \{11, 23, 31\}$, we obtain $\mathcal{R} = \{2, 3, 5, 11\}$ which implies $\prod_{\ell \in \mathcal{R}\, \ell \neq 2}(1 - \frac{1}{p}) = \frac{16}{33}$. Hence, the theorem mentioned within this subsection holds by Proposition 1.

### 8.3. Pentagonal-Sierpiński-Riesel

We now prove the Sierpiński-Riesel case.

**Theorem 7.** *There exist infinitely many $k$ such that the pentagonal number $P_k$ is a Sierpiński number and Riesel number; moreover, both $P_k 2^n + 1$ and $P_k 2^n - 1$ have at least two distinct prime divisors for every positive integer $n$.*

Consider the table below.

In Table 6 the congruences for $n$ above the horizontal line form a covering of the integers; therefore, the congruences for $k$ above this line (along with another congruence $k \equiv 1 \pmod{4}$ ensuring $P_k$ is odd) yield pentagonal numbers $P_k$ that are Sierpiński. Similarly, the congruences for $n$ below the horizontal line also form

| | | | | |
|---|---|---|---|---|
| $n \equiv 0 \pmod{2}$ | & | $k \equiv 2 \pmod{3}$ | $\Longrightarrow$ | $3 \mid (P_k \cdot 2^n + 1)$ |
| $n \equiv 1 \pmod{4}$ | & | $k \equiv 3$ or $4 \pmod{5}$ | $\Longrightarrow$ | $5 \mid (P_k \cdot 2^n + 1)$ |
| $n \equiv 1 \pmod{10}$ | & | $k \equiv 2 \pmod{11}$ | $\Longrightarrow$ | $11 \mid (P_k \cdot 2^n + 1)$ |
| $n \equiv 7 \pmod{8}$ | & | $k \equiv 9$ or $14 \pmod{17}$ | $\Longrightarrow$ | $17 \mid (P_k \cdot 2^n + 1)$ |
| $n \equiv 3 \pmod{18}$ | & | $k \equiv 15$ or $17 \pmod{19}$ | $\Longrightarrow$ | $19 \mid (P_k \cdot 2^n + 1)$ |
| $n \equiv 11 \pmod{24}$ | & | $k \equiv 162$ or $240 \pmod{241}$ | $\Longrightarrow$ | $241 \mid (P_k \cdot 2^n + 1)$ |
| $n \equiv 3 \pmod{16}$ | & | $k \equiv 140$ or $203 \pmod{257}$ | $\Longrightarrow$ | $257 \mid (P_k \cdot 2^n + 1)$ |
| $n \equiv 43 \pmod{48}$ | & | $k \equiv 32$ or $33 \pmod{97}$ | $\Longrightarrow$ | $97 \mid (P_k \cdot 2^n + 1)$ |
| $n \equiv 27 \pmod{48}$ | & | $k \equiv 112$ or $337 \pmod{673}$ | $\Longrightarrow$ | $673 \mid (P_k \cdot 2^n + 1)$ |
| $n \equiv 1 \pmod{2}$ | & | $k \equiv 2 \pmod{3}$ | $\Longrightarrow$ | $3 \mid (P_k \cdot 2^n - 1)$ |
| $n \equiv 6 \pmod{10}$ | & | $k \equiv 2 \pmod{11}$ | $\Longrightarrow$ | $11 \mid (P_k \cdot 2^n - 1)$ |
| $n \equiv 4 \pmod{12}$ | & | $k \equiv 4$ or $5 \pmod{13}$ | $\Longrightarrow$ | $13 \mid (P_k \cdot 2^n - 1)$ |
| $n \equiv 12 \pmod{18}$ | & | $k \equiv 15$ or $17 \pmod{19}$ | $\Longrightarrow$ | $19 \mid (P_k \cdot 2^n - 1)$ |
| $n \equiv 24 \pmod{36}$ | & | $k \equiv 29$ or $33 \pmod{37}$ | $\Longrightarrow$ | $37 \mid (P_k \cdot 2^n - 1)$ |
| $n \equiv 10 \pmod{20}$ | & | $k \equiv 19$ or $36 \pmod{41}$ | $\Longrightarrow$ | $41 \mid (P_k \cdot 2^n - 1)$ |
| $n \equiv 58 \pmod{60}$ | & | $k \equiv 50$ or $52 \pmod{61}$ | $\Longrightarrow$ | $61 \mid (P_k \cdot 2^n - 1)$ |
| $n \equiv 6 \pmod{36}$ | & | $k \equiv 83$ or $99 \pmod{109}$ | $\Longrightarrow$ | $109 \mid (P_k \cdot 2^n - 1)$ |
| $n \equiv 2 \pmod{3}$ | & | $k \equiv 6 \pmod{7}$ | $\Longrightarrow$ | $7 \mid (P_k \cdot 2^n - 1)$ |
| $n \equiv 0 \pmod{9}$ | & | $k \equiv 1$ or $48 \pmod{73}$ | $\Longrightarrow$ | $73 \mid (P_k \cdot 2^n - 1)$ |
| $n \equiv 4 \pmod{5}$ | & | $k \equiv 22$ or $30 \pmod{31}$ | $\Longrightarrow$ | $31 \mid (P_k \cdot 2^n - 1)$ |
| $n \equiv 7 \pmod{15}$ | & | $k \equiv 28$ or $73 \pmod{151}$ | $\Longrightarrow$ | $151 \mid (P_k \cdot 2^n - 1)$ |

Table 6

a covering of the integers; therefore, the corresponding congruences for $k$ in the bottom part of the table (along with another congruence $k \equiv 1 \pmod{4}$ ensuring $P_k$ is odd) yield pentagonal numbers $P_k$ that are also Riesel.

Let $\mathcal{K}$ denote a congruence for $k$ satisfying all congruences for $k$ in Table 6, $k \equiv -1 \pmod{23 \cdot 29 \cdot 43 \cdot 47 \cdot 67}$, and $k \equiv 1 \pmod{4}$. (The latter congruence ensures that both $P_k$ and $k$ are odd.) There are $2^{16}$ such congruences for $k$, all of which result in $P_k$ being a Sierpiński and Riesel number when proceeding as mentioned in previous sections. One such example is

$$k \equiv 1257997264347309332138237749784609 \pmod{4 \cdot \prod_{p \in \mathcal{P} \cup \mathcal{Q}} p}$$

where $\mathcal{P}$ is the set of prime moduli in Table 6 and $\mathcal{Q} := \{23, 29, 43, 47, 67\}$. Let $\mathcal{K}$ denote such a congruence for $k$, and consider primes in $\mathcal{K}$.

Since $q$ divides $3k-1$ for every $q \in \mathcal{Q} := \{23, 29, 43, 47, 67\}$, we obtain $\mathcal{R} = \{2, 3, 7, 11, 23\}$ which implies $\prod_{\ell \in \mathcal{R}, \ell \neq 2}(1-\frac{1}{p}) = \frac{80}{161}$. Hence, the theorem mentioned within this subsection holds.

## 9. Polygonal Numbers

**Theorem 8.** *For infinitely many values of $s$, there exist infinitely many positive integers $k$ such that the $s$-gonal number $S_k(s)$ is a Sierpiński number and $S_k(s)\cdot 2^n+1$ has at least two distinct prime divisors for every positive integer $n$.*

*Proof.* We will use the same congruences found in the Sierpiński triangular case and claim that basically the same proof can be used to deduce the theorem. Recall the $k^{\text{th}}$ $s$-gonal number is given by

$$S_k(s) = \frac{1}{2}k\big((s-2)k-(s-4)\big).$$

Consider $s \equiv 3 \pmod{\prod_{p \in \mathcal{P}} p}$ where $\mathcal{P} := \{3, 5, 7, 13, 17, 241\}$ with $\gcd(s-2, q) = 1$ for every $q \in \mathcal{Q} := \{19, 23\}$, which are the same $\mathcal{P}$ and $\mathcal{Q}$ as in the Sierpiński triangular case. (As an example, one could just take $s \equiv 3 \pmod{\prod_{p \in \mathcal{P} \cup \mathcal{Q}} p}$.) Using the congruences in Table 1, since $s \equiv 3 \pmod{p}$ for each $p$ in the set $\mathcal{P}$,

$$T_k \equiv S_k(s) \pmod{\prod_{p \in \mathcal{P}} p}$$

and each of the congruences in Table 1 holds with $T_k$ replaced with $S_k(s)$. This implies that the expression $S_k(s) \cdot 2^n + 1$ is composite for all positive integers $n$ if $k$ lies in the intersection of the congruence classes listed in Table 1 (since the congruences for $n$ form a covering of the integers). If we also include $k \equiv 1 \pmod{4}$ as done in the Sierpiński triangular case, then the resulting polygonal number $S_k(s)$ is odd, since $k = 4\ell + 1$ implies $S_k(s) = (4\ell+1)(2\ell s - 4\ell + 1)$ which is clearly odd. In addition, assume that $k \equiv (s-4) \cdot (s-2)^{-1} \pmod{19 \cdot 23}$ where the modular inverse of $s-2$ exists as $\gcd(s-2, 19 \cdot 23) = 1$. Thus, every $q \in \mathcal{Q} = \{19, 23\}$ divides $k^* = (s-2)k - (s-4)$. Using all these congruences mentioned for $k$, we deduce (i) from the proposition. Part (ii) of the proposition follows with the same $\mathcal{Q}$ and $\mathcal{R}$ as in the proof of the Sierpiński triangular case mentioned in Theorem 1. $\square$

Using the same technique, we also have the following results.

**Theorem 9.** *For infinitely many values of $s$, there exist infinitely many positive integers $k$ such that the $s$-gonal number $S_k(s)$ is a Riesel number and $S_k(s) \cdot 2^n - 1$ has at least two distinct prime divisors for every positive integer $n$.*

**Theorem 10.** *For infinitely many values of $s$, there exist infinitely many positive integers $k$ such that the $s$-gonal number $S_k(s)$ is both a Sierpiński number and a Riesel number; moreover, both $S_k(s) \cdot 2^n + 1$ and $S_k(s) \cdot 2^n - 1$ have at least two distinct prime divisors for every positive integer $n$.*

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