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THE VALUATIVE CAPACITIES OF THE SETS OF SUMS OF TWO AND OF THREE SQUARES

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Abstract

If A is a subset of $\mathbb{Z}$, then the n-th characteristic ideal of A is the fractional ideal of $\mathbb{Z}$ consisting of 0 and the leading coefficients of polynomials in $\mathbb{Q}[x]$ of degree no more than n which are integer-valued on A. The valuative capacity with respect to a prime p of A is a measure of the rate of growth of the p-adic part of these characteristic ideals of A and is defined, for a given p, to be the value of the limit

$$\lim_{n\to\infty}\frac{\alpha_{A,p}(n)}{n},$$

where $\alpha_{A,p}(n)$ is the p-adic valuation of the inverse of the n-th characteristic ideal of A. In this paper we compute this valuative capacity when A is the set of those integers which are expressible as the sum of two and of three squares.

1. Introduction

For any subset A of $\mathbb{Z}$, the ring of integer-valued polynomials on A is defined to be

$$\mathrm{Int}(A,\mathbb{Z}) = \{f(x)\in\mathbb{Q}[x] : f(A)\subseteq\mathbb{Z}\}.$$

Associated with this ring is its sequence of characteristic ideals $\{I_n : n = 0,1,2,\dots\}$, with I_n the fractional ideal formed by 0 and the leading coefficients of the elements of $\mathrm{Int}(A,\mathbb{Z})$ of degree at most n. For p a prime, the sequence of negatives of the p-adic valuations of the ideals I_n, $\{\alpha_{A,p}(n) : n = 0,1,2,\dots\}$, is called the *characteristic sequence* of A with respect to p. This sequence is super-additive in the sense that

$\alpha_{A,p}(n+m) \geq \alpha_{A,p}(n) + \alpha_{A,p}(m)$ for any nonnegative integers n and m, and so the limit

$$L_{A,p} = \lim_{n\to\infty} \frac{\alpha_{A,p}(n)}{n}$$

always exists (by Fekete's lemma) and is called the *valuative capacity* of A with respect to the prime p.

In some cases this limit can be evaluated by knowing a closed form formula for the terms in the characteristic sequence. For example, if $A = \mathbb{Z}$ then, for any prime, p, we have $\alpha_{\mathbb{Z},p}(n) = \nu_p(n!)$, the p-adic valuation of $n!$, which equals the largest k for which p^k divides $n!$. Since this is given by $\nu_p(n!) = \sum_{k>0} \lfloor n/p^k \rfloor$, it follows that $L_{\mathbb{Z},p} = 1/(p-1)$.

In this paper we evaluate this limit for two subsets of $\mathbb{Z}$ familiar from number theory, for which closed form formulas for $\alpha_{A,p}(n)$ are not available. Let S denote the set of integers which are squares and let $E = S + S$ and $F = S + S + S$ denote the sets of integers which are the sums of two and of three squares, respectively. Of course both of the sets E and F have complete classical number theoretic descriptions:

Theorem 1. *(Fermat) An integer z is in E if and only if every prime congruent to 3 modulo 4 which occurs in its prime factorization, does so with even exponent.*

Theorem 2. *(Legendre) An integer z is in F if and only if it is not of the form $4^a(8b+7)$ for any integers a and b.*

A subset $B \subseteq A \subseteq \mathbb{Z}$ is polynomially dense in A if any rational polynomial that is integer-valued on B, is also integer-valued on A, and so $\mathrm{Int}(B) = \mathrm{Int}(A)$. There is a corresponding idea for p-locally integer-valued polynomials, and an important result is that p-adically dense subsets are also p-locally polynomially dense, for any prime p. Fermat's and Legendre's theorems allow us to describe E and F as p-locally polynomially dense subsets of unions of cosets of powers of primes, and so to use the methods for computing α and L that were developed in [4] and [5]. The results are given in the following theorem.

Theorem 3. *The valuative capacities of the sets E and F are*

$$L_{E,p} = \begin{cases} \frac{1}{p-1} & \text{if } p \equiv 1 \pmod 4, \\ -1 + \sqrt{1 + \frac{2p}{(p-1)^2}} & \text{if } p \equiv 3 \pmod 4, \\ \frac{-1+\sqrt{13}}{2} & \text{if } p = 2. \end{cases}$$

$$L_{F,p} = \begin{cases} \frac{1}{p-1} & \text{if } p > 2, \\ \frac{-25+3\sqrt{705}}{52} & \text{if } p = 2. \end{cases}$$

The answer to the corresponding question for S itself follows from the results in [1] (Example 19). For any prime, p, the characteristic sequence of S is given by

$\alpha_{S,p}(n) = \nu_p((2n)!/2)$ and hence $L_{S,p} = 2/(p-1)$. Since any nonnegative integer is a sum of four squares the answer in that case is, therefore, also known.

2. Valuative Capacity of the Set of Sums of Two Squares

We begin this section with two lemmas, which will be used in the following sections.

Lemma 1. *For any prime p and integer c, the congruence*

$$x^2 + y^2 \equiv c \pmod{p}$$

is solvable.

Proof. If $p = 2$, then this is trivial. For $p > 2$, consider the sequence consisting of those positive integers that are congruent to 1 modulo 4 and also congruent to c modulo p. This is an arithmetic sequence and so, by Dirichlet's theorem, contains a prime q. Since q is congruent to 1 modulo 4, it is a sum of squares, $q = x^2 + y^2$, and so $x^2 + y^2 \equiv c \pmod{p}$, as required. □

Lemma 2. *If p is any odd prime, c is any integer not divisible by p, and k is any positive integer, then the congruence*

$$x^2 + y^2 \equiv c \pmod{p^k}$$

is solvable.

Proof. We proceed by induction on k, with the case $k = 1$ being the previous lemma. Assume that we have found (x_k, y_k) such that

$$x_k^2 + y_k^2 \equiv c \pmod{p^k}$$

and consider the following expansion:

$$(x_k + ap^k)^2 + (y_k + bp^k)^2 = (x_k^2 + y_k^2) + 2p^k(ax_k + by_k) + p^{2k}(a^2 + b^2).$$

We wish to solve the congruence

$$2p^k(ax_k + by_k) \equiv (x_k^2 + y_k^2) - c \pmod{p^{k+1}}$$

for integers a and b. Since $c \not\equiv 0 \pmod{p}$, one of x_k or y_k is not divisible by p. Thus the congruence

$$2(ax_k + by_k) \equiv \frac{c - (x_k^2 + y_k^2)}{p^k} \pmod{p}$$

is solvable for a and b. Taking $x_{k+1} = x_k + ap^k$ and $y_{k+1} = y_k + bp^k$ completes the induction. □

2.1. The Case $p \equiv 1 \pmod 4$

Proposition 1. *If p is a prime congruent to 1 modulo 4, c is any integer, and k is any positive integer, then the congruence*

$$x^2 + y^2 \equiv c \pmod{p^k}$$

is solvable.

Proof. If c is not divisible by p, then this is the previous lemma; so we restrict our attention to the case $c \equiv 0 \pmod p$. Since $p \equiv 1 \pmod 4$, there exists an integer d for which $d^2 \equiv -1 \pmod p$. It follows that $x_1 = d$ and $y_1 = 1$ gives a solution of

$$x^2 + y^2 \equiv c \pmod{p^k}$$

for $k = 1$. Since neither x_1 nor y_1 is divisible by p, the inductive proof in the previous lemma applies to show that this congruence is solvable for all positive k. □

Corollary 1. *If p is a prime congruent to 1 modulo 4, and if k is a positive integer, then*

$$E/(p^k) = \mathbb{Z}/(p^k),$$

and so E is p-adically dense in $\mathbb{Z}$.

Recall, from [2, p. 30] for example, that the characteristic sequence of $\mathbb{Z}$ is given by Legendre's formula

$$\alpha_{\mathbb{Z},p}(n) = \sum_{k\geq 1} \lfloor \frac{n}{p^k} \rfloor = \frac{n - \sum_{i=0}^{m} n_i}{p-1},$$

if $n = \sum_{i=0}^{m} n_i p^i$ is the expansion of n in base p. We thus have the following result:

Corollary 2. *If p is a prime congruent to 1 modulo 4, then the characteristic sequence, $\{\alpha_{E,p}(n) : n = 0, 1, 2, \dots\}$, of E with respect to p is equal to $\alpha_{\mathbb{Z},p}(n)$, and so the valuative capacity of E with respect to p is given by*

$$L_{E,p} = \lim_{n\to\infty} \frac{\alpha_{E,p}(n)}{n} = \lim_{n\to\infty} \frac{\alpha_{\mathbb{Z},p}(n)}{n} = \frac{1}{p-1}.$$

2.2. The case $p \equiv 3 \pmod 4$

We begin with the following notation. If p is a prime congruent to 3 modulo 4, then let $E_0 = E \cap (\mathbb{Z} \setminus p\mathbb{Z})$; for $k > 0$, let $E_k = p^{2k} E_0$.

Lemma 3. *If p is a prime congruent to 3 modulo 4, then for any positive integer k,*

$$E_0/(p^k) = (\mathbb{Z} \setminus p\mathbb{Z})/(p^k)$$

and

$$E = \bigcup_{k \geq 0} E_k.$$

Proof. By Lemma 2, if c is not divisible by p, then $x^2+y^2 \equiv c \pmod{p^k}$ is solvable, and hence $(\mathbb{Z} \setminus p\mathbb{Z})/(p^k) \subseteq E_0/(p^k)$. The reverse inclusion is immediate from the definition of E_0, and so the first equality follows. From Theorem 1, if $c \in E$, then $\nu_p(c)$ is even, and so we have $c = p^{2k}c'$ with $c' \in \mathbb{Z} \setminus p\mathbb{Z}$ and, again by Theorem 1, $c' \in E$. Thus $c \in p^{2k}E_0 = E_k$. Since Theorem 1 also implies $E_k = p^{2k}E_0 \subset E$, the second equality follows. □

To make use of this to evaluate $L_{E,p}$, we recall the following results from [4] and [5]:

Proposition 2. *Let p be a fixed prime.*

1. *If $A \subseteq \mathbb{Z}$ with characteristic sequence $\alpha_{A,p}(n)$, then for any $c \in \mathbb{Z}$ the characteristic sequence of $A + c$ is also $\alpha_{A,p}(n)$, and the characteristic sequence of $p^k A$ is $\alpha_{A,p}(n) + kn$.*

2. *If $B \subseteq \mathbb{Z}$ is another subset of $\mathbb{Z}$ with the property that for any $x \in A$ and $y \in B$ we have $\nu_p(x - y) = 0$, then the characteristic sequence of $A \cup B$ is the disjoint union of the sequences $\alpha_{A,p}(n)$ and $\alpha_{B,p}(n)$ sorted into nondecreasing order. This sequence is called the shuffle product of $\alpha_{A,p}(n)$ and $\alpha_{B,p}(n)$, and is denoted $(\alpha_{A,p} \wedge \alpha_{B,p})(n)$.*

The effect that a union of the sort considered in (ii) above has on valuative capacity is determined by the following algebraic result from [5]:

Proposition 3. *If $\alpha_1(n)$ and $\alpha_2(n)$ are superadditive sequences with $L_1 = \lim \alpha_1(n)/n$ and $L_2 = \lim \alpha_2(n)/n$ then,*

$$\lim \frac{(\alpha_1 \wedge \alpha_2)(n)}{n} = \frac{1}{\dfrac{1}{L_1} + \dfrac{1}{L_2}}.$$

With these results we can prove:

Proposition 4. *If p is a prime congruent to 3 modulo 4, then*

$$L_{E_0,p} = \frac{p}{(p-1)^2}.$$

Proof. The first equality in Lemma 3 shows that E_0 is p-adically, and so polynomially, dense in $\mathbb{Z} \setminus p\mathbb{Z}$. Therefore, these sets have the same characteristic sequence and the same valuative capacity. The set $\mathbb{Z} \setminus p\mathbb{Z}$ is the union of $p-1$ distinct cosets of $p\mathbb{Z}$, each of which has, by Proposition 2(i), characteristic sequence $\alpha_{\mathbb{Z},p}(n)+n$, and so valuative capacity $L_{\mathbb{Z},p}+1 = p/(p-1)$. Proposition 2(ii) applies to this union; hence, by Proposition 3, the valuative capacity of $\mathbb{Z} \setminus p\mathbb{Z}$, and that of E_0 are given by

$$L_{E_0,p} = L_{\mathbb{Z}\setminus p\mathbb{Z},p} = \frac{1}{(p-1)\dfrac{1}{\left(\dfrac{p}{p-1}\right)}} = \frac{p}{(p-1)^2}.$$

□

Proposition 5. *If p is a prime congruent to 3 modulo 4, then*

$$L_{E,p} = -1 + \sqrt{1 + \frac{2p}{(p-1)^2}}.$$

Proof. From Lemma 3 we have

$$E = \bigcup_{k \ge 0} E_k = E_0 \cup (\bigcup_{k \ge 1} E_k) = E_0 \cup (\bigcup_{k \ge 0} p^2 E_k) = E_0 \cup p^2 E.$$

Also, if $x \in E_0$ and $y \in p^2E$, then $\nu_p(x-y) = 0$. It therefore follows from Proposition 2 that, if (kn) denotes the linear sequence whose n-th term is kn, then the characteristic sequence of E with respect to p satisfies the equation

$$\alpha_{E,p} = \alpha_{E_0,p} \wedge (\alpha_{E,p} + (2n)).$$

This implies, by Proposition 3, that

$$L_{E,p} = \frac{1}{\dfrac{1}{L_{E_0,p}} + \dfrac{1}{L_{E,p}+2}}.$$

Solving this for $L_{E,p}$ yields the stated result. □

2.3. The Case $p = 2$

As in the case of odd primes, we will determine the valuative capacity of E for the prime 2 by showing that the 2-adic closure of E is a union of cosets modulo powers of 2; Propositions 2 and 3 can then be applied.

Lemma 4. *If z is an element of E, then $z \not\equiv 2^{k-2}3 \pmod{2^k}$ for any integer $k > 1$.*

Proof. Since $z \in E$, by Theorem 1 its prime expansion is of the form

$$z = 2^a \prod_{p_i \equiv 1 \pmod 4} p_i^{e_i} \prod_{q_i \equiv 3 \pmod 4} q_i^{2f_i}.$$

Since $q_i^2 \equiv 1 \pmod 4$, it follows that $z \equiv 2^a \pmod{2^{a+2}}$. □

Lemma 5. *If c is congruent to 2^ℓ modulo $2^{\ell+2}$ for some $\ell \geq 0$, then the congruence*

$$x^2 + y^2 \equiv c \pmod{2^k}.$$

is solvable, for every $k \geq 0$.

Proof. The proof, as before, is by induction on k. For $k \leq \ell+2$ there is nothing to prove. Since $c \equiv 2^\ell \pmod{2^{\ell+2}}$, we may write $c = 2^\ell + d2^{\ell+2}$. Choose $(x_{\ell+3}, y_{\ell+3})$ as follows:

If ℓ is even, then

$$(x_{\ell+3}, y_{\ell+3}) = \begin{cases} (2^{\ell/2}, 0) & \text{if } d \text{ is even },\\ (2^{\ell/2}, 2^{\ell/2+1}) & \text{if } d \text{ is odd },\end{cases}$$

while if ℓ is odd, then

$$(x_{\ell+3}, y_{\ell+3}) = \begin{cases} (2^{(\ell-1)/2}, 2^{(\ell-1)/2}) & \text{if } d \text{ is even },\\ (2^{(\ell-1)/2}, 2^{(\ell-1)/2} + 2^{(\ell+1)/2}) & \text{if } d \text{ is odd}.\end{cases}$$

Direct calculation shows that these satisfy the required congruence.

We now assume, by induction, that the pair (x_k, y_k) has been found for some $k \geq \ell+3$ and proceed to construct (x_{k+1}, y_{k+1}). We divide the proof into two cases according to the parity of ℓ, and begin by assuming ℓ is even. Also assume, as part of the induction hypothesis, that $\nu_2(x_k) = \ell/2$, and that $\nu_2(y_k) > \ell/2$. Expanding $(x_k + a2^{k-\ell/2-1})^2 + (y_k + b2^{k-\ell/2-1})^2$, we obtain

$$(x_k^2 + y_k^2) + 2^{k-\ell/2}(ax_k + by_k) + 2^{2k-\ell-2}(a^2 + b^2).$$

Since $k \geq \ell+3$, the third term is congruent to 0 modulo 2^{k+1}. Since $\nu_2(x_k) = \ell/2$, the congruence

$$ax_k 2^{k-\ell/2} \equiv c - (x_k^2 + y_k^2) \pmod{2^{k+1}}$$

is solvable for a. Since $\nu_2(y_k) > \ell/2$, taking $b = 0$ gives a solution of

$$(ax_k + by_k)2^{k-\ell/2} \equiv c - (x_k^2 + y_k^2) \pmod{2^{k+1}},$$

and so $(x_{k+1}, y_{k+1}) = (x_k + a2^{k-\ell/2-1}, y_k)$ satisfies the required congruence. The identity $\nu_2(x_{k+1}) = \ell/2$ follows from the inequality $\nu_2(a2^{k-\ell/2-1}) \geq k - \ell/2 - 1 \geq \ell/2 + 2$. The corresponding condition on y_{k+1} is obvious since $y_{k+1} = y_k$.

Next assume that ℓ is odd, and also assume, as part of the induction hypothesis, that $\nu_2(x_k) = \nu_2(y_k) = (\ell-1)/2$. Expanding $(x_k + a2^{k-(\ell+1)/2})^2 + (y_k + b2^{k-(\ell+1)/2})^2$, we obtain

$$(x_k^2 + y_k^2) + 2^{k+1-(\ell+1)/2}(ax_k + by_k) + 2^{2k-\ell-1}(a^2 + b^2).$$

Since $k \geq \ell + 3$, the third term, as before, is congruent to 0 modulo 2^{k+1}. Since $\nu_2(x_k) = \nu_2(y_k) = (\ell-1)/2$, we have

$$\nu_2(x_k 2^{k+1-(\ell+1)/2}) = \nu_2(y_k^{k+1-(\ell+1)/2}) = k,$$

and so the congruence

$$(ax_k + by_k)2^{k+1-(\ell+1)/2} \equiv c - (x_k^2 + y_k^2) \pmod{2^{k+1}}$$

is solvable for a and b. We take $(x_{k+1}, y_{k+1}) = (x_k + a2^{k-(\ell+1)/2}, y_k + b2^{k-(\ell+1)/2})$. Since $\nu_2(a2^{k-(\ell+1)/2}), \nu_2(b2^{k-(\ell+1)/2}) \geq k - (\ell+1)/2 > (\ell-1)/2$, we have $\nu_2(x_{k+1}), \nu_2(y_{k+1}) = (\ell-1)/2$. □

This provides the following description of the 2-adic completion of E:

Corollary 3. *The set E is 2-adically dense in $\bigcup_{\ell \geq 0}(2^\ell + 2^{\ell+2}\mathbb{Z})$.*

The 2-adic valuative capacity of E can now be computed.

Proposition 6. *The valuative capacity of E for the prime 2 is given by*

$$L_{E,2} = \frac{-1+\sqrt{13}}{2}.$$

Proof. Let $\bar{E}$ denote the 2-adic closure of E determined in the previous corollary. Since

$$\bar{E} = \bigcup_{\ell \geq 0}(2^\ell + 2^{\ell+2}\mathbb{Z}) = (1+4\mathbb{Z}) \cup 2(\bigcup_{\ell \geq 0}(2^\ell + 2^{\ell+2}\mathbb{Z})) = (1+4\mathbb{Z}) \cup 2\bar{E},$$

we have

$$L_{E,2} = L_{\bar{E},2} = \cfrac{1}{\cfrac{1}{L_{1+4\mathbb{Z},2}} + \cfrac{1}{L_{\bar{E},2}+1}} = \cfrac{1}{\cfrac{1}{3} + \cfrac{1}{L_{\bar{E},2}+1}} = \cfrac{1}{\cfrac{1}{3} + \cfrac{1}{L_{E,2}+1}},$$

which simplifies to

$$L_{E,2}^2 + L_{E,2} - 3 = 0;$$

this has positive root as stated. □

3. Valuative Capacity of the Set of Sums of Three Squares

Legendre's description of F is equivalent to:

Proposition 7. *The set F can be expressed as*

$$F = \mathbb{Z} \setminus (\bigcup_{a \geq 0} (2^{2a}7 + 2^{2a+3}\mathbb{Z})).$$

From this the valuative capacity of F for odd primes is immediate:

Proposition 8. *For any odd prime, p, the valuative capacity of F with respect to p is*

$$L_{F,p} = L_{\mathbb{Z},p} = \frac{1}{p-1}.$$

Proof. From Proposition 7 it is clear that F contains the coset $2 + 4\mathbb{Z}$. If c is a given integer and $k > 0$, then the congruences

$$\begin{aligned} z &\equiv 2 \pmod{4}, \\ z &\equiv c \pmod{p^k}, \end{aligned}$$

are simultaneously solvable, hence have a solution in F. Thus $F/(p^k) = \mathbb{Z}/(p^k)$, and so F is p-adically dense in $\mathbb{Z}$ and $L_{F,p}$ is as stated. □

It thus only remains to determine $L_{F,p}$ for $p = 2$.

Lemma 6. *The set F satisfies the equation*

$$F = (2 + 4\mathbb{Z}) \cup (\{1, 3, 5\} + 8\mathbb{Z}) \cup 2^2 F.$$

Proof. This is another restatement of Legendre's theorem, this time as a union of cosets:

$$\begin{aligned} F &= (\bigcup_{a \geq 0} (2^{2a+1} + 2^{2a+2}\mathbb{Z})) \cup (\bigcup_{a \geq 0} (\{1, 3, 5\}2^{2a} + 2^{2a+3}\mathbb{Z})) \\ &= (2 + 4\mathbb{Z}) \cup (\bigcup_{a \geq 1} (2^{2a+1} + 2^{2a+2}\mathbb{Z})) \\ &\qquad \cup (\{1, 3, 5\} + 8\mathbb{Z}) \cup (\bigcup_{a \geq 1} (\{1, 3, 5\}2^{2a} + 2^{2a+3}\mathbb{Z})) \\ &= (2 + 4\mathbb{Z}) \cup (\{1, 3, 5\} + 8\mathbb{Z}) \\ &\qquad \cup 2^2 (\bigcup_{a \geq 0} (2^{2a+1} + 2^{2a+2}\mathbb{Z})) \cup (\bigcup_{a \geq 0} (\{1, 3, 5\}2^{2a} + 2^{2a+3}\mathbb{Z})) \\ &= (2 + 4\mathbb{Z}) \cup (\{1, 3, 5\} + 8\mathbb{Z}) \cup 2^2 F, \end{aligned}$$

as claimed. □

By applying Proposition 3 twice to the result of Lemma 6, we obtain:

Lemma 7. *The valuative capacity of F with respect to the prime 2 satisfies the equation*

$$L_{F,2} = \frac{1}{\dfrac{1}{L_{\{1,3,5\}+8\mathbb{Z},2}} + \dfrac{1}{1 + \dfrac{1}{\dfrac{1}{L_{2+4\mathbb{Z},2} - 1} + \dfrac{1}{L_{F,2} + 1}}}}.$$

Two further applications evaluate the first term in the denominator on the right.

Lemma 8. *The valuative capacity of $\{1,3,5\} + 8\mathbb{Z}$ is given by*

$$L_{\{1,3,5\}+8\mathbb{Z},2} = \frac{11}{5}.$$

Proof. Expressing $\{1,3,5\} + 8\mathbb{Z}$ as $(\{1,5\} + 8\mathbb{Z}) \cup (3 + 8\mathbb{Z})$, we obtain

$$L_{\{1,5\}+8\mathbb{Z},2} = 2 + \frac{1}{\dfrac{1}{L_{1+8\mathbb{Z},2} - 2} + \dfrac{1}{L_{5+8\mathbb{Z},2} - 2}} = 2 + \frac{1}{\dfrac{1}{4-2} + \dfrac{1}{4-2}} = 3$$

and

$$\begin{aligned} L_{\{1,3,5\}+8\mathbb{Z},2} &= 1 + \frac{1}{\dfrac{1}{L_{\{1,5\}+8\mathbb{Z},2} - 1} + \dfrac{1}{L_{3+8\mathbb{Z},2} - 1}} \\ &= 1 + \frac{1}{\dfrac{1}{3-1} + \dfrac{1}{4-1}} = \frac{11}{5}. \end{aligned}$$

□

We thus have

$$L_{F,2} = \frac{1}{\dfrac{5}{11} + \dfrac{1}{1 + \dfrac{1}{\dfrac{1}{3-1} + \dfrac{1}{L_{F,2}+1}}}},$$

which simplifies to give the quadratic

$$26L_{F,2}^2 + 25L_{F,2} - 55 = 0,$$

with positive root

$$L_{F,2} = \frac{-25 + 3\sqrt{705}}{52};$$

this concludes the proof of Theorem 3.

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