THE LIND-LEHMER CONSTANT FOR $\mathbb{Z}_m \times \mathbb{Z}_p^n$

Vincent Pigno
Department of Mathematics & Statistics, University of California, Sacramento, California
vincent.pigno@csus.edu

Chris Pinner
Department of Mathematics, Kansas State University, Manhattan, Kansas
pinner@math.ksu.edu

Wasin Vipismakul
Department of Mathematics, Burapha University, Chonburi, Thailand
wvipismakul@gmail.com

Received: 12/21/15, Accepted: 6/12/16, Published: 7/7/16

Abstract

We give bounds on the Lind-Lehmer constant for groups of the form

$$\mathbb{Z}_m \times \mathbb{Z}_p^n, \ \ p \nmid m$$

that are in many cases sharp. In particular we obtain the Lind-Lehmer constant for groups of the form $\mathbb{Z}_2 \times \mathbb{Z}_p^n, \ \ p \geq 3$.

1. Introduction

For a polynomial F in $\mathbb{Z}[x_1, \ldots, x_k]$ and a finite abelian group

$$G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}, \tag{1}$$

one defines the Lind-Mahler measure [4] of F with respect to G by

$$M_G(F) = |P_G(F)|^{1/|G|},$$

where $P_G(F)$ is the integer

$$P_G(F) := \prod_{j_1=1}^{n_1} \cdots \prod_{j_k=1}^{n_k} F\left(e^{2\pi i j_1/n_1}, \ldots, e^{2\pi i j_k/n_k}\right).$$

That is, instead of the classical logarithmic Mahler measure

$$\log M(F)=\int_0^1\cdots\int_0^1 \log|F(e^{2\pi i x_1},\ldots,e^{2\pi i x_k})|dx_1\cdots dx_k,$$

one defines

$$\log M_G(F)=\frac{1}{|G|}\sum_{x_1=1}^{n_1}\cdots\sum_{x_k=1}^{n_k}\log|F(e^{2\pi i x_1/n_1},\ldots,e^{2\pi i x_k/n_k})|.$$

Mirroring the Lehmer problem for the classical measure, one can ask for the minimal positive logarithmic Lind-Mahler measure, and define a Lind-Lehmer constant for G

$$\lambda(G)=\frac{1}{|G|}\log\mathcal{P}_G,$$

where

$$\mathcal{P}_G=\min\Big\{|P_G(F)|\ :\ |P_G(F)|\geq 2,\ F\in\mathbb{Z}[x_1,\ldots,x_k]\Big\}.$$

For cyclic groups $G=\mathbb{Z}_m$ Kaiblinger [2] gave the bounds

$$\min\left\{\min_{q\nmid m} q,\ \min_{q^\alpha||m} q^{\alpha+1}\right\}\leq \mathcal{P}_{\mathbb{Z}_m}\leq \min\left\{\min_{q\nmid m} q,\ \min_{q^\alpha||m} q^{q^\alpha}\right\}, \tag{2}$$

with equality in these upper and lower bounds when $420\nmid m$ (see [5] for $\lambda(\mathbb{Z}_m)$ when $892371480\nmid m$). Here p and q will always denote primes. Writing

$$\mathcal{M}_j:=\{a^{p^{j-1}}-tp^j\ :\ 1\leq a<p,\ t\in\mathbb{Z}\}, \tag{3}$$

and

$$\mathcal{M}_j^*:=\min\{|b|\geq 2\ :\ b\in\mathcal{M}_j\}, \tag{4}$$

the second author showed in [1] that for $G=\mathbb{Z}_p^n$, we have

$$\mathcal{P}_G=\mathcal{M}_n^*.$$

In his thesis [7, Theorem 2.1.5] the third author extended this to general p-groups

$$G_p:=\mathbb{Z}_{p^{l_1}}\times\cdots\times\mathbb{Z}_{p^{l_n}},\quad l_1\leq\cdots\leq l_n,\quad N=\sum_{i=1}^n l_i, \tag{5}$$

showing the bounds

$$\mathcal{M}_n^*\leq\mathcal{P}_{G_p}\leq\mathcal{M}_N^*. \tag{6}$$

In this note we obtain the counterpart of (2) and (6) for $G=\mathbb{Z}_m\times G_p$, $p\nmid m$. When $G_p=\mathbb{Z}_p^n$ (i.e., $N=n$) we seem to have equality in many cases, including $m=2$.

2. Results

We define

$$\mathcal{M}_j^-(r) := \min\{b > 1 \ : \ b \in \mathcal{M}_j,\ (b, r) \neq 1\},$$
$$\mathcal{M}_j^+(r) := \min\{b > 1 \ : \ b \in \mathcal{M}_j,\ (b, r) = 1\}.$$

Note that $-1 + mp^j$ is in $\mathcal{M}_j$ so that

$$\mathcal{M}_j^+(r) \leq mp^j - 1. \tag{7}$$

Theorem 1. *If $G = \mathbb{Z}_m \times G_p$ with G_p as in (5) and $p \nmid m\phi(m)$, then*

$$\min\left\{\mathcal{M}_n^+(m),\ \min_{q^\alpha || m} \mathcal{M}_n^-(q)^{\alpha+1},\ p^{B(G_p)}\right\} \leq \mathcal{P}_G \leq \min\left\{\mathcal{M}_N^+(m),\ {\mathcal{M}_N^*}^{m_1}\right\}, \tag{8}$$

where $m_1 = \prod_{q^\alpha || m, q | \mathcal{M}_N^*} q^\alpha$ and

$$B(G_p) = (l_1 + 1) + \sum_{i=1}^{n-1} (l_{i+1} - l_i + 1) p^{l_1 + \cdots + l_i}. \tag{9}$$

In view of (7) we can drop the $p^{B(G_p)}$ from the lower bound if $m \leq p^{B(G)-n}$, and we also recover the trivial bound

$$\mathcal{M}_N^+(m) \leq |G| - 1 = \left| P_G \left(-1 + \left(\frac{z^m - 1}{z - 1} \right) \prod_{i=1}^n \left(\frac{x_i^{p^{l_i}} - 1}{x_i - 1} \right) \right) \right|.$$

If $G_p = \mathbb{Z}_p^n$ and $\mathcal{M}_n^+(m) \leq p^{2+p+\cdots+p^{n-1}}$ and $p \nmid m\phi(m)$ we have

$$\mathcal{M}_n^+(m) < {\mathcal{M}_n^*}^2 \quad \text{implies} \quad \mathcal{P}_G = \mathcal{M}_n^+(m). \tag{10}$$

If $G_p = \mathbb{Z}_p^n$ and $m = 2$ we clearly have equality in our upper and lower bounds (8):

Corollary 1. *If $G = \mathbb{Z}_2 \times \mathbb{Z}_p^n$ and $p \geq 3$, then*

$$\mathcal{P}_G = \min\left\{\mathcal{M}_n^+(2)\ ,\ \mathcal{M}_n^-(2)^2\right\}.$$

The lower bound in Theorem 1 will come from observing that if $p \mid P_G(F)$ then $p^{B(G_p)} \mid P_G(F)$, and that if $p \nmid P_G(F)$ then $P_G(F)$ must be a product of $d(m)$ elements of $\mathcal{M}_n$ (which includes 1); moreover that if $q \mid P_G(F)$ and $q^\alpha \mid\mid m$, then at least $(\alpha + 1)$ of them are divisible by q. The upper bounds are constructive. We can drop the assumption $p \nmid \phi(m)$ in Theorem 1 if we replace the $\mathcal{M}_N^+(m)$ in our upper bound by the smallest element of $\mathcal{M}_N$ which is coprime to m and a p^{n-1}st power mod m and add to m_1 any $q^\alpha \mid\mid m$, $q \equiv 1 \bmod p$, such that $\mathcal{M}_N^*$ is not a p^{n-1} power mod q.

3. Proofs

The bound $\mathcal{P}_G \leq \mathcal{M}_N^+(m)$ follows at once from the following lemma and the observation that if $p \nmid \phi(m)$ and $(s,m)=1$ then s is a p^{N-1}st power mod m.

Lemma 1. *Let $G = \mathbb{Z}_m \times G_p$, $p \nmid m$. If $s = a^{p^{N-1}} - tp^N$ has $(s,pm)=1$ and is a p^{N-1}st power mod m then there is a polynomial F in $\mathbb{Z}[z,x_1,\ldots,x_n]$ with $P_G(F)=s$.*

Proof. The proof is entirely constructive and similar to Lemma 2.2 of [1].

Suppose that $s \equiv a_0^{p^{N-1}} \bmod m$. Since $p \nmid m$ we can find an integer λ such that $a+p\lambda$ is a positive integer satisfying $a+p\lambda \equiv a_0 \bmod m$. Hence we can write

$$s = a^{p^{N-1}} - tp^N = (a+\lambda p)^{p^{N-1}} - t_1 p^N$$

for some t_1 which must satisfy $m \mid t_1$. Thus we can assume that a is a positive integer and that $m \mid t$. Notice also that $(s,pm)=1$ ensures that $(a,pm)=1$.

We define $H_1(z,y),\ldots,H_{N-1}(z,y)$ in $\mathbb{Z}[z,y]$ by

$$(1+(zy)+\cdots+(zy)^{a-1})^{p^i} = \left(\sum_{j=0}^{a-1} z^{pj}\right)^{p^{i-1}} + p^i H_i(z,y) \bmod y^p-1. \qquad (11)$$

To see (11) for $i=1$ we have

$$\begin{aligned}(1+(zy)+\cdots+(zy)^{a-1})^p &= 1+(zy)^p+\cdots+(zy)^{p(a-1)} + pH_1(z,y)\\ &\equiv (1+z^p+\cdots+z^{p(a-1)}) + pH_1(z,y) \bmod y^p-1,\end{aligned}$$

and for $i \geq 1$ successively

$$\begin{aligned}(1+(zy)+\cdots+(zy)^{a-1})^{p^{i+1}} &\equiv \left(\left(\sum_{j=0}^{a-1} z^{pj}\right)^{p^{i-1}} + p^i H_i(z,y)\right)^p \bmod y^p-1\\ &= \left(\sum_{j=0}^{a-1} z^{pj}\right)^{p^i} + p^{i+1}H_{i+1}(z,y).\end{aligned}$$

We define $\alpha(1),\ldots,\alpha(N)$ by $\underbrace{1,\ldots,1}_{l_1}, \underbrace{2,\ldots,2}_{l_2},\ldots,\underbrace{n,\ldots,n}_{l_n}$ and $\beta(1),\ldots,\beta(N)$ by $\underbrace{p^{l_1-1},p^{l_1-2},\ldots,1}_{l_1}, \underbrace{p^{l_2-1},\ldots,1}_{l_2},\ldots,\underbrace{p^{l_n-1},\ldots,1}_{l_n}$. Recalling the pth cyclotomic polynomial

$$\Phi_p(x) = 1+x+\cdots+x^{p-1} = \frac{x^p-1}{x-1},$$

we take a positive integer r such that $rp \equiv 1 \bmod m$ and set

$$F(z, x_1, \ldots, x_n) = \left(1 + \left(zx_1^{p^{l_1-1}}\right) + \cdots + \left(zx_1^{p^{l_1-1}}\right)^{a-1}\right) + \sum_{j=1}^{N-1} H_j\left(z^{r^j}, x_{\alpha(j+1)}^{\beta(j+1)}\right) \prod_{i=1}^{j} \Phi_p\left(x_{\alpha(i)}^{\beta(i)}\right) - \frac{t}{m}\left(\frac{z^m-1}{z-1}\right) \prod_{i=1}^{n}\left(\frac{x_i^{p^{l_i}}-1}{x_i-1}\right).$$

Suppose that w is a primitive p^{l_1}th root of unity and z is an mth root of unity which is a primitive lth root of unity. Then $w' = w^{p^{l_1-1}}$ is a primitive pth root of unity and, since $(a, pm) = 1$ and $(m, p) = 1$, both zw' and $(zw')^a$ are primitive plth roots of unity. Thus $1 - (zw')^a$ and $1 - (zw')$ have the same norm and

$$F(z, w, \ldots) = 1 + (zw') + \cdots + (zw')^{a-1} = \frac{1 - (zw')^a}{1 - (zw')}$$

is a unit of norm 1.

Similarly, suppose $x_k = w$ is a primitive $p^{l_k - j}$th root of unity with $0 \le j \le l_k - 1$ (with $j \ge 1$ if $k = 1$) and $x_i = 1$ for any $1 \le i < k$. We set $J = l_1 + \cdots + l_{k-1} + 1 + j$. Then $x_{\alpha(i)}^{\beta(i)} = 1$ and $\Phi_p\left(x_{\alpha(i)}^{\beta(i)}\right) = p$ for all the $i < J$, and $w' = x_{\alpha(J)}^{\beta(J)} = w^{p^{l_k-1-j}}$ is a primitive pth root of unity and $\Phi_p\left(x_{\alpha(J)}^{\beta(J)}\right) = 0$. Hence

$$\begin{aligned} F(z, 1, \ldots, 1, w, \ldots) &= \sum_{j=0}^{a-1} z^j + \sum_{i=1}^{J-2} p^i H_i(z^{r^i}, 1) + p^{J-1} H_{J-1}(z^{r^{J-1}}, w') \\ &= \left(\sum_{j=0}^{a-1} (z^r)^{pj} + pH_1(z^r, 1)\right) + \sum_{i=2}^{J-2} p^i H_i(z^{r^i}, 1) + p^{J-1} H_{J-1}\left(z^{r^{J-1}}, w'\right) \\ &= \left(\sum_{j=0}^{a-1}\left(z^{r^2}\right)^{pj}\right)^p + \sum_{i=2}^{J-2} p^i H_i(z^{r^i}, 1) + p^{J-1} H_{J-1}\left(z^{r^{J-1}}, w'\right) \\ &= \left(\sum_{j=0}^{a-1}\left(z^{r^{J-1}}\right)^{pj}\right)^{p^{J-2}} + p^{J-1} H_{J-1}(z^{r^{J-1}}, w') \\ &= \left(\sum_{j=0}^{a-1}\left(z^{r^{J-1}} w'\right)^j\right)^{p^{J-1}} = \left(\frac{1 - \left(z^{r^{J-1}} w'\right)^a}{1 - \left(z^{r^{J-1}} w'\right)}\right)^{p^{J-1}} \end{aligned}$$

is again a unit of norm 1. Finally, if $z \neq 1$ is an mth root of unity, then

$$F(z, 1, \ldots, 1) = \sum_{j=0}^{a-1} z^j + \sum_{i=1}^{N-1} p^i H_i(z^{r^i}, 1) = \left(\sum_{j=0}^{a-1} z^{jr^{N-1}}\right)^{p^{N-1}} = \left(\frac{1 - z^{ar^{N-1}}}{1 - z^{r^{N-1}}}\right)^{p^{N-1}}$$

is a unit of norm 1,

$$F(1,1,\ldots,1)=a+\sum_{i=1}^{N-1}p^iH_i(1,1)-tp^N=(1+1+\cdots+1)^{p^{N-1}}-tp^N=a^{p^{N-1}}-tp^N,$$

and $P_G(F(z,x_1,\ldots,x_n))=a^{p^{N-1}}-tp^N$. $\square$

We observe that if p divides a G_p measure then a high power of p divides the measure.

Lemma 2. *Suppose that $p^m \mid\mid P_{G_p}(F)$. Then $m=0$ or*

$$m\geq (l_1+1)+\sum_{i=1}^{n-1}(l_{i+1}-l_i+1)p^{l_1+\cdots+l_i}. \tag{12}$$

We can also replace this by a more precise but less digestible bound; if $l_1,\ldots,l_n$ take the values $k_1<\cdots<k_L$ with multiplicities $m_1,\ldots,m_L$ and $k_0=0$, then the right-hand side of (12) can be replaced by

$$1+\sum_{l=0}^{L-1}\sum_{j=0}^{k_{L-l}-k_{L-l-1}-1}\sum_{i=0}^{m_L+\cdots+m_{L-l}-1}p^{m_1k_1+\cdots+m_{L-l-1}k_{L-l-1}+(m_L+\cdots+m_{L-l}-1)(k_{L-l}-j)-i}. \tag{13}$$

Either bound (12) or (13) can be used for $B(G_p)$ in Theorem 1. When $G_p=\mathbb{Z}_p^n$ both give the bound $2+p+\cdots+p^{n-1}=1+\frac{p^n-1}{p-1}$ used in [1]. A simpler bound on $B(G_p)$ is given in [7, Theorem 2.1.3].

Proof. Observe that if w_{p^j} denotes a primitive p^jth root of unity, then

$$\text{Norm}_{\mathbb{Q}(w_{p^s})/\mathbb{Q}}F\left(w_{p^{s_1}},\ldots,w_{p^{s_n}}\right)=\prod_{\substack{j=1\\(j,p)=1}}^{p^s}F\left(w_{p^{s_1}}^j,\ldots,w_{p^{s_n}}^j\right)\in\mathbb{Z},$$

where

$$s=\max\{s_1,\ldots,s_n\},$$

and $M_{G_p}(F)$ can be written as a product of such integer norms. Moreover, extending the p-adic absolute value to $\mathbb{Q}(w_{p^s})$, we have $|w_{p^{s_i}}-1|_p<1$ and so plainly

$$\text{Norm}_{\mathbb{Q}(w_{p^s})/\mathbb{Q}}F\left(w_{p^{s_1}},\ldots,w_{p^{s_n}}\right)\equiv F(1,\ldots,1)^{\phi(p^s)} \bmod p.$$

Hence if $p\mid P_{G_p}(F)$, then $p\mid F(1,\ldots,1)$ and p divides all the norms. Thus the bounds (12) and (13) represent a bound on the number of integer norms that make up M_{G_p}. For (12) we proceed by induction on n; for $n=1$ we have l_1+1 norms, namely the value $F(1)$ and the norms of $F(w_{p^j})$, $j=1,\ldots,l_1$. For $n>1$ and a primitive p^jth root of unity w_{p^j} with $l_{n-1} \leq j \leq l_n$ the $F(x_1, \ldots, x_{n-1}, w_{p^j})$ produce a different norm for each choice of $x_1, \ldots, x_{n-1}$, giving $(l_n - l_{n-1} + 1)p^{l_1 + \cdots + l_{n-1}}$ norms. Discarding any terms $F(x_1, \ldots, x_{n-1}, w_{p^j})$ with $1 \leq j < l_{n-1}$, the remaining terms in (12) come from the $n-1$ variable $\mathbb{Z}_{p^{l_1}} \times \cdots \times \mathbb{Z}_{p^{l_{n-1}}}$ measure of $F(x_1, \cdots, x_{n-1}, 1)$.

Retaining the terms $F(x_1, \ldots, x_{n-1}, w_{p^j})$ with $1 \leq j < l_{n-1}$ gives (13); taking $x_n = w_{p^{k_L}}$ we have the $p^{m_1 k_1 + \cdots + m_{L-1} k_{L-1} + (m_L - 1)k_L}$ choices of the other x_i. The remaining norms then have a k_L replaced by $k_L - 1$. When $m_L > 1$ one successively reduces the remaining k_L to $k_L - 1$ contributing $p^{\sum_{t=1}^{L-1} m_t k_t + (m_L - 1)k_L - i}$ for $i = 0$ to $m_L - 1$. When $k_L - 1 > k_{L-1}$ one continues to reduce all the m_L exponents $k_L - 1$ until one has $m_L + m_{L-1}$ values k_{L-1} (the j sum). One repeats (the l sum) until left with $m_1 + \cdots + m_L$ exponents k_1 and finally the single term $F(1, \ldots, 1)$. $\square$

Proof of Theorem 1. Let w_r denote a primitive rth root of unity. For the lower bound observe that we can write

$$P_G(F) = P_{G_p}(F_1) = \prod_{d|m} P_{G_p}(f_d),$$

where

$$F_1 := \prod_{j=0}^{m} F(w_m^j, x_1, \ldots, x_n) = \prod_{d|m} f_d(x_1, \ldots, x_n)$$

with

$$f_d(x_1, \ldots, x_n) := \prod_{\substack{j=1 \\ (j,d)=1}}^{d} F(w_d^j, x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n].$$

From Lemma 2 if $p \mid P_{G_p}(f_d)$ then $p^{B(G_p)} \mid P_{G_p}(f_d)$. It was shown in [1, Lemma 2.1] for $G_p = \mathbb{Z}_p^n$ and in [7, Theorem 2.1.2] for general G_p that if $p \nmid P_{G_p}(f_d)$ then $P_{G_p}(f_d)$ lies in $\mathcal{M}_n$. Since for a prime q and $(l, q) = 1$ we can write $w_{lq^j} = w_l w_{q^j}$ with $|w_{q^j} - 1|_q < 1$, we have

$$f_{lq^j} \equiv f_l^{\phi(q^j)} \bmod q$$

and

$$P_{G_p}(f_{lq^j}) \equiv P_{G_p}(f_l)^{\phi(q^j)} \bmod q.$$

Hence if $q^\alpha \parallel m$ has $q \mid P_G(F)$, then $q \mid P_{G_p}(f_{lq^j})$ for some l with $q \nmid l$ and $0 \leq j \leq \alpha$, and $q \mid P_{G_p}(f_{lq^i})$ for all $0 \leq i \leq \alpha$, and $|P_{G_p}(f_{lq^i})| \geq \mathcal{M}_n^-(q)$ for all i. So $|P_G(F)| \geq \mathcal{M}_n^-(q)^{\alpha+1}$ and the lower bound is plain.

From Lemma 1 we have $\mathcal{P}_G \leq \mathcal{M}_N^+(m)$. For the remaining upper bound observe that if a is in $\mathcal{M}_N$ and we write $m = m_1 m_2$, where $m_1 = \prod_{q^\alpha || m, q|a} q^\alpha$ and $(m_2, a) = 1$, then we know that for $G_2 := \mathbb{Z}_{m_2} \times \mathbb{Z}_p^n$ there is an $f(z, x_1, \ldots, x_k)$ with $P_{G_2}(f) =$

a. Hence $F(z, x_1, \ldots, x_k) = f(z^{m_1}, x_1, \ldots, x_k)$ will have $P_G(F) = P_{G_2}(f)^{m_1} = a^{m_1}$. Taking $a = \mathcal{M}_N^*$ gives the bound stated. Note, taking the polynomial $F(x_1, \ldots, x_n)$ achieving $\mathcal{P}_{G_p}$, we similarly have the trivial bound $\mathcal{P}_G \leq \mathcal{P}_{G_p}^m$. □

4. Examples

Notice that the smallest possible value of $\mathcal{P}_{\mathbb{Z}_2 \times \mathbb{Z}_p^2}$ is 3, achievable exactly when $3^{p-1} \equiv 1 \bmod p^2$. The only known such Mirimanoff primes (Wieferich primes base 3) are $p = 11$ and $p = 1006003$; see for example [3, p.150] or [6, p.347]. The two known Wieferich primes, $p = 1093$ and 3511, have $\mathcal{P}_{\mathbb{Z}_2 \times \mathbb{Z}_p^2} = \mathcal{M}_2^-(2)^2 = 4$.

The following tables give the $\mathcal{M}_n^*$ and $\mathcal{M}_n^+(m)$ for $G = \mathbb{Z}_m \times \mathbb{Z}_p^n$, with $3 \leq p \leq 103$, $n = 2, 3, 4$, and m of the form $2^\alpha, 3^\alpha, 5^\alpha, 2^\alpha 3^\beta, 7^\alpha, 2^\alpha 5^\beta$ or 11^α. For $p \nmid m\phi(m)$ we have $\mathcal{M}_n^+(m) < {\mathcal{M}_n^*}^2$ and $\mathcal{P}_G = \mathcal{M}_n^+(m)$ except for the following few unresolved cases:

G	$\mathcal{P}_G$	G	$\mathcal{P}_G$
$\mathbb{Z}_{2^\alpha \cdot 3} \times \mathbb{Z}_{11}^2, \ \alpha \geq 0,$	9 or 27	$\mathbb{Z}_{3^2} \times \mathbb{Z}_{11}^2$	27 or 40
$\mathbb{Z}_{2^\alpha \cdot 3^2} \times \mathbb{Z}_{11}^2, \ \alpha \geq 1,$	27, 81 or 161	$\mathbb{Z}_6 \times \mathbb{Z}_{37}^2$	324 or 437
$\mathbb{Z}_{2^\alpha \cdot 3^3} \times \mathbb{Z}_{11}^2, \ \alpha \geq 1,$	81 or 161		

Since 8 is a cube, the restriction $p \nmid \phi(m)$ only affects $\mathbb{Z}_7 \times \mathbb{Z}_3^n$, $n = 3, 4$ and $\mathbb{Z}_{11} \times \mathbb{Z}_5^n$, $n = 2, 3, 4$.

TABLE OF M_n^*:

	$n=2$	$n=3$	$n=4$		$n=2$	$n=3$	$n=4$
$p=3$	8	26	80	$p=47$	53	295	224444
$p=5$	7	57	182	$p=53$	338	1468	189323
$p=7$	18	18	1047	$p=59$	53	2511	11550
$p=11$	3	124	1963	$p=61$	264	15458	397575
$p=13$	19	239	239	$p=67$	143	3859	201305
$p=17$	38	158	4260	$p=71$	11	6372	15384
$p=19$	28	333	2819	$p=73$	306	923	840838
$p=23$	28	42	19214	$p=79$	31	1523	1372873
$p=29$	14	1215	2463	$p=83$	99	5436	1576656
$p=31$	115	513	15714	$p=89$	184	1148	278454
$p=37$	18	691	51344	$p=97$	53	412	1721322
$p=41$	51	1172	20677	$p=101$	181	4943	48072
$p=43$	19	3038	3038	$p=103$	43	4432	281007

TABLE OF $\mathcal{P}_G$ FOR $G = \mathbb{Z}_{2^\alpha} \times \mathbb{Z}_p^n$, $n = 2, 3, 4$, $p \leq 103$. In all these cases $\mathcal{P}_G = \mathcal{M}_n^+(2)$:

	$n=2$	$n=3$	$n=4$		$n=2$	$n=3$	$n=4$
$p=3$	17	53	161	$p=47$	53	295	225947
$p=5$	7	57	443	$p=53$	413	9283	189323
$p=7$	19	19	1047	$p=59$	53	2511	111529
$p=11$	3	161	1963	$p=61$	601	28743	397575
$p=13$	19	239	239	$p=67$	143	3859	201305
$p=17$	65	399	15541	$p=71$	11	8327	557381
$p=19$	69	333	2819	$p=73$	527	923	1551509
$p=23$	63	803	60793	$p=79$	31	1523	1372873
$p=29$	41	1215	2463	$p=83$	99	6509	2864371
$p=31$	115	513	126279	$p=89$	605	1485	6251225
$p=37$	117	691	216739	$p=97$	53	34557	6313037
$p=41$	51	9325	20677	$p=101$	181	4943	571075
$p=43$	19	3623	162637	$p=103$	43	26319	281007

Table of $\mathcal{M}_n^+(3) \neq \mathcal{M}_n^*$ for $G = \mathbb{Z}_{3^\alpha} \times \mathbb{Z}_p^n$, $n = 2, 3, 4$, $p \leq 103$:

	$n=2$		$n=3$		$n=4$
$p=7$	19	$p=5$	68	$p=7$	1048
$p=11$	40	$p=7$	19	$p=17$	15541
$p=37$	76	$p=19$	623	$p=29$	23174
$p=41$	148	$p=23$	803	$p=31$	51266
$p=61$	572	$p=29$	4850	$p=59$	111529
$p=73$	368	$p=31$	5995	$p=61$	846695
$p=83$	161	$p=59$	18511	$p=71$	557381
		$p=71$	8327	$p=83$	2864371
		$p=83$	6509	$p=89$	381718
				$p=97$	3464764
				$p=101$	571075
				$p=103$	4717448

Table of $\mathcal{M}_n^+(5) \neq \mathcal{M}_n^*$ for $G = \mathbb{Z}_{5^\alpha} \times \mathbb{Z}_p^n$, $n = 2, 3, 4$, $p \leq 103$:

	$n=2$		$n=3$		$n=4$
$p=31$	117	$p=29$	1872	$p=3$	161
		$p=47$	4757	$p=17$	15541
				$p=59$	111529
				$p=61$	648103
				$p=67$	201306

TABLE OF $\mathcal{M}_n^+(6) \neq \mathcal{M}_n^*$, for primes such that $3 \mid \mathcal{M}_n^+(2) \neq \mathcal{M}_n^*$ or $2 \mid \mathcal{M}_n^+(3) \neq \mathcal{M}_n^*$ for $G = \mathbb{Z}_{2^\alpha . 3^\beta} \times \mathbb{Z}_p^n$, $n = 2, 3, 4$, $p \leq 103$:

	$n=2$		$n=3$		$n=4$
$p=11$	161	$p=5$	193	$p=7$	2549
$p=19$	127	$p=17$	653	$p=29$	78017
$p=23$	263	$p=19$	623	$p=31$	298423
$p=37$	437	$p=29$	10133	$p=61$	846695
$p=41$	313	$p=31$	5995	$p=89$	6251225
$p=61$	601	$p=59$	18511	$p=97$	6313037
$p=73$	527	$p=61$	38447	$p=103$	6280381
$p=83$	161	$p=89$	24833		
		$p=97$	34675		
		$p=103$	50645		

TABLE OF $\mathcal{M}_n^+(7) \neq \mathcal{M}_n^*$ FOR $G = \mathbb{Z}_{7^\alpha} \times \mathbb{Z}_p^n$, $n = 2, 3, 4$, $p \leq 103$:

	$n=2$		$n=3$		$n=4$
$p=5$	18	$p=23$	803	$p=5$	443
$p=19$	54	$p=43$	3623	$p=43$	45922
$p=23$	118	$p=89$	1485	$p=59$	111529
$p=29$	41				

TABLE OF $\mathcal{M}_n^+(10) \neq \mathcal{M}_n^*$, for primes such that $5 \mid \mathcal{M}_n^+(2) \neq \mathcal{M}_n^*$ or $2 \mid \mathcal{M}_n^+(5) \neq \mathcal{M}_n^*$ for $G = \mathbb{Z}_{2^\alpha . 5^\beta} \times \mathbb{Z}_p^n$, $n = 2, 3, 4$, $p \leq 103$:

	$n=2$		$n=3$		$n=4$
$p=31$	117	$p=29$	2463	$p=61$	548103
$p=89$	707	$p=41$	10399	$p=67$	1057933
		$p=89$	24833	$p=89$	7552311
				$p=101$	1358891

TABLE OF $\mathcal{M}_n^+(11) \neq \mathcal{M}_n^*$ FOR $G = \mathbb{Z}_{11^\alpha} \times \mathbb{Z}_p^n$, $n = 2, 3, 4$, $p \leq 103$:

	$n=2$		$n=4$
$p=61$	432	$p=47$	225947
$p=67$	248	$p=59$	905953
$p=71$	26	$p=89$	381718
$p=83$	161		

Similarly, fixing p we can evaluate $\mathcal{P}_G$ for varying m:

Example 4.1. *Suppose that* $G = \mathbb{Z}_m \times \mathbb{Z}_3^2$ *with* $3 \nmid m$. *Then*

$$\begin{aligned}
\mathcal{P}_G =& 8 \quad \textit{if } 2 \nmid m,\\
\mathcal{P}_G =& 17 \quad \textit{if } m = 2n,\ 17 \nmid n,\ 3 \nmid \phi(n),\\
\mathcal{P}_G =& 19 \quad \textit{if } m = 2 \cdot 17n,\ 3 \nmid \phi(n),\\
\mathcal{P}_G =& 64 \quad \textit{if } m = 2 \cdot 5 \cdot 17 \cdot 19 \cdot 37 \cdot 53n \ \ \textit{or } m = 2 \cdot 7 \cdot 11 \cdot 17 \cdot 19 \cdot 37 \cdot 53n,\ \ 2 \nmid n.
\end{aligned}$$

Example 4.2. *Suppose that* $G = \mathbb{Z}_m \times \mathbb{Z}_5^2$ *with* $5 \nmid m\phi(m)$. *Then*

$$\begin{aligned}
\mathcal{P}_G =& 7 \quad \textit{if } 7 \nmid m,\\
\mathcal{P}_G =& 18 \quad \textit{if } m = 7n,\ (6, n) = 1,\\
\mathcal{P}_G =& 26 \quad \textit{if } m = 3 \cdot 7n,\ (26, n) = 1,\\
\mathcal{P}_G =& 32 \quad \textit{if } m = 3 \cdot 7 \cdot 13n,\ 2 \nmid n,\\
\mathcal{P}_G =& 43 \quad \textit{if } m = 2 \cdot 7n,\ 43 \nmid n.
\end{aligned}$$

For $m = 2 \cdot 7 \cdot 43$ *we have* $\mathcal{P}_G = 49$ *or 51. Since 32 is a fifth power we can drop the restriction* $5 \nmid \phi(m)$ *when* $m = 3 \cdot 7 \cdot 13n$, $2 \nmid n$.

Example 4.3. *Suppose that* $G = \mathbb{Z}_m \times \mathbb{Z}_7^2$ *with* $7 \nmid m\phi(m)$. *Then*

$$\begin{aligned}
\mathcal{P}_G =& 18 \quad \textit{if } (6, m) = 1,\\
\mathcal{P}_G =& 19 \quad \textit{if } m = 2n \textit{ or } m = 3n,\ 19 \nmid n,\\
\mathcal{P}_G =& 31 \quad \textit{if } m = 2 \cdot 19n \textit{ or } m = 3 \cdot 19n,\ 31 \nmid n,\\
\mathcal{P}_G =& 50 \quad \textit{if } m = 3 \cdot 19 \cdot 31n,\ (10, n) = 1,\\
\mathcal{P}_G =& 67 \quad \textit{if } m = 2 \cdot 19 \cdot 31n \textit{ or } m = 3 \cdot 5 \cdot 19 \cdot 31n,\ 67 \nmid n,\\
\mathcal{P}_G =& 68 \quad \textit{if } m = 3 \cdot 5 \cdot 19 \cdot 31 \cdot 67n,\ (2 \cdot 17, n) = 1,\\
\mathcal{P}_G =& 79 \quad \textit{if } m = 2 \cdot 19 \cdot 31 \cdot 67n \textit{ or } m = 3 \cdot 5 \cdot 17 \cdot 19 \cdot 31 \cdot 67n,\ 79 \nmid n,\\
\mathcal{P}_G =& 97 \quad \textit{if } m = 2 \cdot 19 \cdot 31 \cdot 67 \cdot 79n \textit{ or } m = 3 \cdot 5 \cdot 17 \cdot 19 \cdot 31 \cdot 67 \cdot 79n,\ 97 \nmid n,\\
\mathcal{P}_G =& 99 \quad \textit{if } m = 2 \cdot 19 \cdot 31 \cdot 67 \cdot 79 \cdot 97n,\ (3 \cdot 11, n) = 1,\\
\mathcal{P}_G =& 116 \quad \textit{if } m = 3 \cdot 5 \cdot 17 \cdot 19 \cdot 31 \cdot 67 \cdot 79 \cdot 97n,\ 2 \nmid n,\\
\mathcal{P}_G =& 117 \quad \textit{if } m = 2 \cdot 11 \cdot 19 \cdot 31 \cdot 67 \cdot 79 \cdot 97n,\ (3 \cdot 13, n) = 1,\\
\mathcal{P}_G =& 129 \quad \textit{if } m = 2 \cdot 11 \cdot 13 \cdot 19 \cdot 31 \cdot 67 \cdot 79 \cdot 97n,\ 3 \nmid n,\\
\mathcal{P}_G =& 197 \quad \textit{if } m = 2 \cdot 3 \cdot 19 \cdot 31 \cdot 67 \cdot 79 \cdot 97n.
\end{aligned}$$

Since 128 *is a seventh power we can drop the restriction* $7 \nmid \phi(m)$ *and obtain* $\mathcal{P}_G = 128$ *when* $m = 3 \cdot 5 \cdot 17 \cdot 19 \cdot 29 \cdot 31 \cdot 67 \cdot 79 \cdot 97n$, $2 \nmid n$.

Acknowledgment. The second author thanks Felipe Voloch for asking whether $\mathcal{P}_{\mathbb{Z}_2\times\mathbb{Z}_3^2} = 8$ or 17.

References

[1] D. De Silva and C. Pinner, The Lind-Lehmer constant for $\mathbb{Z}_p^n$, *Proc. Amer. Math. Soc.* **142** (2014), 1935-1941.

[2] N. Kaiblinger, On the Lehmer constant of finite cyclic groups, *Acta Arith.* **142** (2010), 79-84.

[3] J. S. Kraft and L. C. Washington, *Elementary Number Theory,* CRC Press, Boca Raton, 2015.

[4] D. Lind, Lehmer's problem for compact abelian groups, *Proc. Amer. Math. Soc.* **133** (2005), 1411-1416.

[5] V. Pigno and C. Pinner, The Lind-Lehmer constant for cyclic groups of order less than 892371480, *Ramanujan J.* **33** (2014), 295-300.

[6] P. Ribenboim, *The New Book of Prime Number Records,* Springer-Verlag, New York, 1996.

[7] W. Vipismakul, *The Stabilizer of the Group Determinant and Bounds for Lehmer's Conjecture on Finite Abelian Groups*, PhD. thesis, University of Texas at Austin, 2013.