VARIATIONS ON AN ERROR-SUM FUNCTION FOR THE CONVERGENTS OF SOME POWERS OF e

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Abstract

Several years ago the second author, playing with different "recognizers of real constants," e.g., the LLL algorithm, the Plouffe inverter, etc., found the following formula empirically. Let p_n/q_n denote the nth convergent of the continued fraction of the constant e. Then

$$\sum_{n \geq 0} |q_n e - p_n| = \frac{e}{4} \left(-1 + 10 \sum_{n \geq 0} \frac{(-1)^n}{(n+1)!(2n^2 + 7n + 3)} \right).$$

The purpose of the present paper is to prove this formula and to give similar formulas for some powers of e.

1. Introduction

Playing with the convergents of e, the second author discovered several years ago the formula

$$\sum_{n \geq 0} |q_n e - p_n| = \frac{e}{4} \left(-1 + 10 \sum_{n \geq 0} \frac{(-1)^n}{(n+1)!(2n^2 + 7n + 3)} \right). \tag{1}$$

While trying to prove the formula rigorously we began being interested in the following quantity. If α is a positive real number, and if p_n/q_n is the nth convergent of its continued fraction, the quantity $|q_n\alpha - p_n|$ tends rapidly to zero. Thus the

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series $\sum_{n\geq 0}|q_n\alpha - p_n|$ converges. This series measures in some sense the "global approximation" of α by its convergents. We then learned from J. Shallit that the quantity $\sum_{n\geq 0}|q_n\alpha - p_n| = \sum_{n\geq 0}(-1)^n(q_n\alpha - p_n)$ was investigated in several papers [5, 8, 9, 6], where the study of the quantity $\sum_{n\geq 0}(q_n\alpha - p_n)$ (first defined in [28] where it is called the error-sum function of α) can also be found. Both quantities $\sum_{n\geq 0}|q_n\alpha - p_n|$ and $\sum_{n\geq 0}(q_n\alpha - p_n)$ are called "error-sum function(s)" in the literature. A natural question is whether the sum of these series can be expressed in terms of α without explicitly using the convergents, in particular in the case where α has a "nice" continued fraction expansion, e.g., when α is quadratic or when $\alpha = e$.

2. Quadratic Numbers

The case of quadratic numbers was addressed in [5] (also see [6]).

Theorem 1 (Elsner). *Let p_n/q_n be the nth convergent of the continued fraction of α. Then the series $\sum_{n\geq 0}(q_n\alpha - p_n)x^n$ converges absolutely at least for $|x| < \frac{1+\sqrt{5}}{2}$ and, if α is a real quadratic number,*

$$\sum_{n\geq 0}(q_n\alpha - p_n)x^n \in \mathbb{Q}[\alpha](x).$$

In particular (taking $x = -1$), $\sum_{n\geq 0}|q_n\alpha - p_n|$ belongs to $\mathbb{Q}[\alpha]$.

Example 1 (Elsner).

- $\sum_{n\geq 0}|q_n\sqrt{7} - p_n| = \frac{7+5\sqrt{7}}{14}$.

- For any integer $n \geq 1$ we have $\sum_{n\geq 0}|q_n(\frac{n+\sqrt{4+n^2}}{2}) - p_n| = \frac{1}{\frac{n+\sqrt{4+n^2}}{2}-1}$.

- In particular, $\sum_{n\geq 0}|q_n(\frac{1+\sqrt{5}}{2}) - p_n| = \frac{1+\sqrt{5}}{2}$.

3. Powers of e

Euler [10] proved that the continued fraction expansion of e is

$$[2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, ...].$$

This expansion is sometimes replaced by the not really regular expression

$$[1, 0, 1, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, ...].$$

After Euler, a large number of papers contained the computation of continued fraction expansions for some expressions containing e (typically certain powers of e possibly multiplied by some rational numbers, or numbers like $\frac{e^{2/k}-1}{e^{2/k}+1}$), see in particular [11, 19, 13, 20, 4, 24, 23, 27, 3, 26, 14, 22, 15, 17, 16, 21, 18, 12].

The fundamental theorem we will use here is due to Komatsu [17, Theorem 6, first part]. Komatsu's theorem contains several previous results.

Theorem 2 (Komatsu). *Let $\ell \geq 2$ and $s \geq 1$ be two integers. Let p_n/q_n be the nth convergent of the continued fraction of*

$$se^{1/(\ell s)} = [s, \ell-1, 1, 2s-1, 3\ell-1, 1, 2s-1, 5\ell-1, 1, 2s-1, \cdots, (2k-1)\ell-1, 1, 2s-1, \cdots].$$

Then for $n \geq 0$,

$$p_{3n} - se^{1/(\ell s)}q_{3n} = -\frac{1}{(\ell s)^{n+1}} \int_0^1 \frac{x^n(x-1)^n}{n!} se^{x/(\ell s)}\mathrm{d}x$$

$$p_{3n+1} - se^{1/(\ell s)}q_{3n+1} = \frac{1}{s(\ell s)^{n+1}} \int_0^1 \frac{(x+s-1)x^n(x-1)^n}{n!} se^{x/(\ell s)}\mathrm{d}x$$

$$p_{3n+2} - se^{1/(\ell s)}q_{3n+2} = \frac{1}{s(\ell s)^{n+1}} \int_0^1 \frac{x^n(x-1)^{n+1}}{n!} se^{x/(\ell s)}\mathrm{d}x.$$

Let $s \geq 1$ be an integer. Let p_n^*/q_n^* be the nth convergent of the continued fraction of

$$se^{1/s} = [s+1, 2s-1, 2, 1, 2s-1, 4, 1, \cdots, 2s-1, 2k, 1, \cdots].$$

Then $p_n^*/q_n^* = p_{n+2}/q_{n+2}$ with p_n/q_n is as above. More precisely, for $n \geq 0$ we have

$$p_{3n}^* - se^{1/s}q_{3n}^* = \frac{1}{s^{n+2}} \int_0^1 \frac{x^n(x-1)^{n+1}}{n!} se^{x/s}\mathrm{d}x$$

$$p_{3n+1}^* - se^{1/s}q_{3n+1}^* = -\frac{1}{s^{n+2}} \int_0^1 \frac{x^{n+1}(x-1)^{n+1}}{(n+1)!} se^{x/s}\mathrm{d}x$$

$$p_{3n+2}^* - se^{1/s}q_{3n+2}^* = \frac{1}{s^{n+3}} \int_0^1 \frac{(x+s-1)x^{n+1}(x-1)^{n+1}}{(n+1)!} se^{x/s}\mathrm{d}x.$$

Using Komatsu's result we can prove the following theorem. First recall that the "error function," erf, is defined by $\mathrm{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\mathrm{d}t$.

Note. The name "error-sum function" (or "error sum function") that goes back to [28] should not be confused with the name "error function." To (try to) avoid any ambiguity, we will always write for the latter "error function erf."

Theorem 3. *The following summations of series hold.*

1. Let $\ell \geq 2$ and $s \geq 1$ be two integers. Let p_n/q_n be the nth convergent of the continued fraction of

$$se^{1/(\ell s)} = [s, \ell-1, 1, 2s-1, 3\ell-1, 1, 2s-1, 5\ell-1, 1, 2s-1, \cdots, (2k-1)\ell-1, 1, 2s-1, \cdots].$$

Then

$$\sum_{n \geq 0} |p_n - se^{1/(\ell s)}q_n| = e^{1/\ell s}\sqrt{\frac{\pi s}{\ell}}\,\mathrm{erf}(1/\sqrt{\ell s}).$$

2. Let $s \geq 1$ be an integer. Let p_n^*/q_n^* be the nth convergent of the continued fraction of

$$se^{1/s} = [s+1, 2s-1, 2, 1, 2s-1, 4, 1, \cdots, 2s-1, 2k, 1, \cdots].$$

Then

$$\sum_{n \geq 0} |p_n^* - se^{1/s}q_n^*| = e^{1/s}\sqrt{\pi s}\,\mathrm{erf}(1/\sqrt{s}) + s(1 - e^{1/s}) - 1.$$

Proof. First we write

$$\sum_{n \geq 0} |p_n - se^{1/\ell s}q_n| = \sum_{0 \leq j \leq 2}\sum_{n \geq 0} |p_{3n+j} - se^{1/\ell s}q_{3n+j}|$$

and

$$\sum_{n \geq 0} |p_n^* - se^{1/s}q_n^*| = |p_0^* - se^{1/s}q_0^*| + \sum_{1 \leq j \leq 3}\sum_{n \geq 0} |p_{3n+j}^* - se^{1/s}q_{3n+j}^*|.$$

Then for each of the sums $\sum_{n \geq 0} |p_{3n+j} - se^{1/\ell s}q_{3n+j}|$, $(j = 0, 1, 2)$, respectively $\sum_{n \geq 0} |p_{3n+j}^* - se^{1/s}q_{3n+j}^*|$ $(j = 1, 2, 3)$, we use the integral expressions given in the two parts of Komatsu's theorem (Theorem 2 above). Then we intervert the signs $\sum$ and $\int$ and recognize the series expansion of some exponential. $\qquad\square$

We deduce the following corollary.

Corollary 1. *The two equalities below hold.*

Let p_n/q_n be the nth convergent of the continued fraction of $e^{1/\ell}$ (with $\ell \geq 2$). Then

$$\sum_{n \geq 0} |p_n - e^{1/\ell}q_n| = e^{1/\ell}\sqrt{\frac{\pi}{\ell}}\,\mathrm{erf}(1/\sqrt{\ell}).$$

Let p_n^*/q_n^* be the nth convergent of the continued fraction of e (recall that the continued fraction of e is given by $e = [2, 1, 2, 1, 1, 4, 1, \cdots, 1, 2n, 1, \cdots]$). Then

$$\sum_{n \geq 0} |p_n^* - eq_n^*| = 2e\int_0^1 e^{-t^2}\,dt - e = e\sqrt{\pi}\,\mathrm{erf}(1) - e.$$

Remark 1. The second result in Corollary 1 above was already obtained by Elsner in [5, p. 2].

Now we prove Formula (1). We begin with a lemma.

Lemma 1. *Let $A(s)$ be defined for a positive real number s by*

$$A(s) := \sum_{n \geq 0} \frac{(-1)^n}{(n+1)!(2n^2 + 7n + 3)s^n}.$$

Then

$$A(s) = \frac{1}{5}s^3 - \frac{1}{2}s + \frac{1}{5}s(2 - s - s^2)e^{-1/s} + \frac{4}{5}\int_0^1 e^{-t^2/s}\mathrm{d}t.$$

Proof. Since

$$\frac{1}{2n^2 + 7n + 3} = \frac{2}{5(2n+1)} - \frac{1}{5(n+3)},$$

it is useful to introduce the series

$$f(x) := \sum_{n \geq 0} \frac{(-1)^{n+1}x^{2n+1}}{(n+1)!(2n+1)s^{n+1}} \quad \text{and} \quad g(x) := \sum_{n \geq 0} \frac{(-1)^{n+1}x^{n+3}}{(n+1)!(n+3)s^{n+1}}.$$

For $f(x)$ we obtain $f'(x) = \frac{1}{x^2}(e^{-x^2/s} - 1)$, so that (note that $f(x) = 0$)

$$f(x) = \frac{1}{x} - \frac{1}{x}e^{-x^2/s} - \sqrt{\frac{\pi}{s}} \cdot \mathrm{erf}(x/\sqrt{s}) = \frac{1}{x} - \frac{1}{x}e^{-x^2/s} - \frac{2}{s}\int_0^x e^{-t^2/s}\mathrm{d}t.$$

For $g(x)$ we have $g'(x) = xe^{-x/s} - x$, so that

$$g(x) = \left(s^2 - \frac{x^2}{2}\right) - s(x+s)e^{-x/s}.$$

Now $A(s) = -s\left(\frac{2}{5}f(1) - \frac{1}{5}g(1)\right)$ which completes the proof of the lemma. $\qquad \square$

Corollary 2. *Formula (1) given in the introduction holds:*

$$\sum_{n \geq 0} |q_n e - p_n| = \frac{e}{4}\left(-1 + 10\sum_{n \geq 0} \frac{(-1)^n}{(n+1)!(2n^2 + 7n + 3)}\right).$$

Proof. From Lemma 1 we have in particular

$$A(1) = -\frac{3}{10} + \frac{4}{5}\int_0^1 e^{-t^2}\mathrm{d}t.$$

Hence

$$\sum_{n\geq 0} |q_n e - p_n| = 2e \int_0^1 e^{-t^2}\, \mathrm{d}t - e = \frac{e}{4}(-1 + 10A(1)).$$

□

Remark 2. Analogously we can define $A(\ell, s) := \sum_{n\geq 0} \frac{(-1)^n}{(n+1)!(2n^2+7n+3)(\ell s)^n}$ for positive reals ℓ and s to obtain similar formulas for $se^{\ell s}$. We also note that we first thought that the quantity $(2n^2 + 7n + 3)$ was somehow crucial in Formula (1): there might even be (though it would be quite surprising) a link with the number of independent parameters of the orthosymplectic group $\mathrm{OSP}(3, 2n)$ which is precisely $(2n^2 + 7n + 3)$ (see, e.g., [2, p. 223]). But this quantity is not crucial; compare with Formula (2) given below which can be proved by using a step of the proof of Corollary 2 above with $s = 1$ and $x = 1$:

$$\sum_{n\geq 0} \frac{(-1)^{n+1}}{(n+1)!(2n+1)} = 1 - e^{-1} - 2\int_0^1 e^{-t^2}\, \mathrm{d}t.$$

This implies

$$\sum_{n\geq 0} |p_n^* - eq_n^*| = e\sum_{n\geq 0} \frac{(-1)^n}{(n+1)!(2n+1)} - 1. \tag{2}$$

Remark 3. The value of $\sum_{n\geq 0} |p_n - \alpha q_n|$ for α equal to one of the above real numbers can also be expressed as another kind of series. Namely a classical series for the error function erf (see, e.g., [1, 7.1.6, p. 297]) reads (recall the notation $(2n+1)!! = 1 \times 3 \times 5 \cdots \times (2n+1)$)

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} e^{-z^2} \sum_{n\geq 0} \frac{2^n}{(2n+1)!!} z^{2n+1}.$$

Using Corollary 1 and the notation therein, this gives in particular the following formulas:

$$\sum_{n\geq 0} |p_n - e^{1/\ell}q_n| = \sum_{n\geq 0} \frac{2^{n+1}}{\ell^{n+1}(2n+1)!!} = \sum_{n\geq 0} \frac{2^{2n+1}\, n!}{\ell^{n+1}(2n+1)!}. \tag{3}$$

for any integer $\ell \geq 2$, and

$$\sum_{n\geq 0} |p_n^* - eq_n^*| = \sum_{n\geq 0} \frac{2^{n+1}}{(2n+1)!!} - e = \sum_{n\geq 0} \frac{2^{2n+1}\, n!}{(2n+1)!} - e. \tag{4}$$

Note that the second author obtained Formula (3) empirically. Also note that the digits of the decimal expansion of the right side of Equation (3) above for $\ell = 2$ and $\ell = 4$ are given in [25] as A060196 and A214869 respectively, and that the expansion of the right side of Equation (4) (up to the $-e$ term) is given in [25] as A125961.

4. More Results for the Error-sum Function

Formulas similar to the formulas in the previous section can be stated by using results on the convergents for continued fractions with "regular" patterns, in particular at least for (some of) the so-called Hurwitz continued fractions, sometimes also called (regular) continued fractions of Hurwitzian type; see, e.g., [22]. We simply list below results that can be used to yield nice formulas for the error-sum function we considered. They give, in terms of integrals for some reals α and their convergents p_n/q_n, the quantity $p_{an+b} - \alpha q_{an+b}$ (for any b in a complete system of residues modulo a), and they are due to Komatsu.

- For $\alpha = \frac{e^{1/\ell s}}{s}$, with s and ℓ any two integers ≥ 2, and $\alpha = \frac{e^{1/s}}{s}$, with $s \geq 2$, integral expressions for $p_{3n+j} - \alpha q_{3n+j}$ with $j \in \{0, 1, 2\}$ are given in [17, Theorem 3, second part];

- for $\alpha = e^{2/s}$, with $s \geq 3$ and odd, integral expressions for $p_{5n+j} - \alpha q_{5n+j}$ with $j \in \{0, 1, 2, 3, 4\}$ are given in [14];

- for $\alpha = \frac{e^{1/(3s+1)}}{3}$ (resp. $\alpha = \frac{e^{1/(3s+2)}}{3}$), integral expressions for $p_{9n+j} - \alpha q_{9n+j}$ with $j \in \{-6, -5, -4, -3, -2, -1, 0, 1, 2\}$ are given in [17];

- for $\alpha = \sqrt{\frac{v}{u}} \tanh \frac{1}{\sqrt{uv}}$, integral expressions for $p_{2n-1} - \alpha q_{2n-1}$ and $p_{2n} - \alpha q_{2n}$ are given in [16];

- for $\alpha = \sqrt{\frac{v}{u}} \tan \frac{1}{\sqrt{uv}}$, integral expressions for $p_{4n-j} - \alpha q_{4n-j}$ with $j = 0, 1, 2, 3$ are given in [16].

The last result we would like to cite here is a nice particular case of a theorem of Hetyei [12, Theorem 2.9] (also see [12, p. 21]) which could be used to compute the error-sum function for $\alpha = \frac{4(11\sin(1/2) - 6\cos(1/2))}{53\cos(1/2) - 97\sin(1/2)}$.

Theorem 4 (Hetyei). *We have the following continued fraction expansion*

$$\frac{4(11\sin(1/2) - 6\cos(1/2))}{53\cos(1/2) - 97\sin(1/2)} = [\,\overline{4, n}\,]_{n=3}^{\infty} = [4, 3, 4, 4, 4, 5, 4, 6, 4, 7, 4, ...].$$

5. Conclusion

The error-sum function of some other continued fractions with "regular" patterns could probably be studied. Another appealing possibility is the definition and study of error-sum functions for similar continued fractions in the function field case (see

in particular [29, 30, 31]). Finally we give a last relation that the second author discovered empirically:

$$\int_0^1 e^{-t^2}\,\mathrm{d}t = 3/8 + \cfrac{5/4}{3 + \cfrac{9}{21 + \cfrac{288}{63 + \cfrac{\ddots}{\ddots + \cfrac{n(n+2)^2(2n-1)^2}{(2n+5)(n^2+n+1) + {}^{\ddots}}}}}}$$

(we did not locate this formula in the literature and did not prove it yet).

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Addendum. After we posted the first version of this paper on ArXiv, C. Elsner kindly sent us the preprint [7], which was written a few months ago, and where the reader can find very nice results about error-sum functions (limit formulas, differential equations, algebraic independence results, relations to Hall's theorem, summation formulas involving the Riemann zeta function), but also a proof of the first result in Corollary 1:

$$\text{for all } \ell \geq 2, \ \sum_{n \geq 0} |p_n - e^{1/\ell} q_n| = e^{1/\ell}\sqrt{\frac{\pi}{\ell}}\,\mathrm{erf}(1/\sqrt{\ell}).$$

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