A CONGRUENCE PROPERTY OF IRREDUCIBLE LAGUERRE POLYNOMIALS IN TWO VARIABLES

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Abstract

In this paper we introduce a version of multivariable Laguerre polynomials irreducible over the rationals. We also prove, for such polynomials in two variables, a congruence property, which is similar to the one obtained by Carlitz for the classical Laguerre polynomials in one variable.

1. Introduction

The generalized Laguerre polynomials in one variable are defined for an arbitrary integer $n \geq 0$ and a parameter $\alpha > -1$ by Rodrigues' relation (see, for example, [4], Section 1.4.2):

$$L_n^\alpha(x) = \frac{1}{n!} e^x x^{-\alpha} \cdot D^n \left(e^{-x} x^{n+\alpha}\right), \quad \text{where} \quad D^n := \frac{d^n}{dx^n}.$$

In this paper we will consider only non-negative integer values for the parameter α, i.e., we assume from now on that $\alpha \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Expanding the definition of $L_n^\alpha(x)$ using the n-fold product rule and the Pochhammer symbol defined for all $x \in \mathbb{R}$ by

$$(x)_0 = 1, \quad (x)_n = \prod_{i=1}^{n} (x + i - 1) \quad \text{for all} \ \ n \in \mathbb{N},$$

one immediately comes to the following explicit formulas ([4], Section 1.4.2):

$$L_n^\alpha(x) = \sum_{j=0}^{n} \frac{(\alpha+1)_n \cdot (-n)_j}{(\alpha+1)_j \cdot n!} \cdot \frac{x^j}{j!} = \sum_{j=0}^{n} \frac{(-1)^j}{j!} \cdot \binom{n+\alpha}{n-j} \cdot x^j. \tag{1}$$

It was established by Schur in 1929 (see [8]) that $L_n(x) = L_n^0(x)$ are irreducible over the rationals for all $n \in \mathbb{N}$. Recently this result was generalized by Filaseta and Lam who proved that for all but finitely many $n \in \mathbb{N}$, the polynomials $L_n^\alpha(x)$, where α is a rational number which is not a negative integer, are irreducible over $\mathbb{Q}$ (see [6]). Note that reducible L_n^α do exist, for example, $L_2^2(x) = 1/2(x-2)(x-6)$. One of the key characteristics of the Laguerre polynomials $L_n^\alpha(x)$ (with a fixed $\alpha > -1$) is that they are orthogonal over the interval $(0, \infty)$ with respect to the weight function $\omega(x) = e^{-x}x^\alpha$ (see Chapter 1 of [4]). They also satisfy other interesting properties, including the one due to Carlitz (see [3]), who proved in 1954 that for all $n, m \in \mathbb{N}$, and a rational number α that is integral $\pmod{m}$,

$$(n+m)!L_{n+m}^\alpha(x) \equiv n!L_n^\alpha(x) \cdot m!L_m^\alpha(x) \pmod{m}. \tag{2}$$

There are various examples of families of orthogonal polynomials in several variables, and certain properties of the following multivariable Laguerre polynomials have been studied in [4] and [2]:

$$L_{n_1,\ldots,n_r}^{\alpha_1,\ldots,\alpha_r}(x_1,\ldots,x_r) = L_{n_1}^{\alpha_1}(x_1) \cdot L_{n_2}^{\alpha_2}(x_2) \cdot \ldots \cdot L_{n_r}^{\alpha_r}(x_r). \tag{3}$$

Such multivariable Laguerre polynomials are orthogonal with respect to the weight function, which is the product of the corresponding weight functions $x_1^{\alpha_1} \cdot \ldots \cdot x_r^{\alpha_r} \cdot e^{-(x_1+\ldots+x_r)}$ over the domain, which is the cartesian product of the corresponding domains $\mathbb{R}_+^d = \{(x_1,\ldots,x_r) \mid 0 < x_j < \infty, \quad j \in \{1,2,\ldots,r\}\}$ (see [2] and [4], Section 2.3.5). It is also clear from (3) that such multiple Laguerre polynomials are reducible as soon as they have more than one variable.

In this paper, we introduce a version of multivariable Laguerre polynomials $L_{n_1,\ldots,n_r}(x_1,\ldots,x_r)$, which are irreducible over the rationals and prove that such Laguerre polynomials in two variables satisfy a congruence relation similar to (2).

The rest of this paper is divided up as follows. In Section 2, we introduce our version of Laguerre polynomials in x and y using Rodrigues' formula with partial derivatives and derive the corresponding explicit formulas similar to (1). In Section 3, we establish several auxiliary lemmas and use them to give another proof of the congruence (2) of Carlitz. Section 4 contains a proof of the corresponding congruence for two-variable Laguerre polynomials (see (12) below). In Section 5, we offer our version of multivariable Laguerre polynomials in an arbitrary number of variables and discuss their irreducibility over $\mathbb{Q}$.

2. Laguerre Polynomials in Two Variables

As we already wrote, the Laguerre polynomials in x are defined for an arbitrary integer $n \geq 0$ and the parameter $\alpha = 0$ by Rodrigues' relation $L_n(x) = \frac{1}{n!}e^x \cdot$

$D^n\left(e^{-x}x^n\right)$. We apply this approach to define the Laguerre polynomials in two variables as follows.

Definition 1. For all $n, m \in \mathbb{N}_0$ let

$$L_{n,m}(x,y) := \frac{1}{n!\cdot m!} e^{(x+y)/2} \cdot D_\partial^{n+m}\left(e^{(-x-y)/2} x^n y^m\right),$$

where $D_\partial(f(x,y)) := f_x(x,y) + f_y(x,y)$.

`Note 1`: Since $e^{(x+y)/2} \cdot D_\partial\left(e^{(-x-y)/2} f(x,y)\right) = f_x(x,y) + f_y(x,y) - f(x,y)$, if it happens that $f(x,y)$ depends only on a single variable x, we obtain

$$e^{(x+y)/2} \cdot D_\partial\left(e^{(-x-y)/2} f(x,y)\right) = \frac{d}{dx} f(x) - f(x) = e^x \cdot D\left(e^{-x} f(x)\right),$$

and hence naturally

$$L_{n,0}(x,y) = \frac{1}{n!\cdot 0!} e^{(x+y)/2} \cdot D_\partial^{n}\left(e^{(-x-y)/2} x^n y^0\right) = L_n(x), \qquad L_{0,m}(x,y) = L_m(y).$$

By the same argument we also have

$$L_{n,m}(x,x) = \frac{1}{n!\cdot m!} e^{(x+y)/2} \cdot D_\partial^{n+m}\left(e^{(-x-y)/2} x^{n+m}\right) = \binom{n+m}{n} \cdot L_{n+m}(x).$$

Before giving the explicit formulas for $L_{n,m}(x,y)$ we prove the following formula.

Lemma 1. *For all $n,\ m,\ t \in \mathbb{N}_0$ we have*

$$e^{\frac{(x+y)}{2}} \cdot D_\partial^t\left(e^{\frac{(-x-y)}{2}} x^n y^m\right) = \sum_{i=0}^{\min(t,m)} \binom{t}{i} \cdot \frac{m!}{(m-i)!} \cdot \left(e^x D^{t-i}\left(e^{-x} x^n\right)\right) \cdot y^{m-i}, \quad (4)$$

where $D = \frac{d}{dx}$ is as in the definition of the classical Laguerre polynomials.

Proof. We use induction on t; so suppose $t = 0$. Hence

$$e^{(x+y)/2} \cdot D_\partial^t(e^{(-x-y)/2} x^n y^m) = x^n y^m,$$

and the right-hand side of (4) also yields a single term $x^n y^m$. Now assume that (4) is true for $t = k-1$. Then, by the induction hypothesis, we obtain

$$e^{\frac{(x+y)}{2}} \cdot D_\partial^k\left(e^{\frac{(-x-y)}{2}} x^n y^m\right) = e^{\frac{(x+y)}{2}} \cdot D_\partial\left(e^{\frac{(-x-y)}{2}} \cdot e^{(x+y)/2} \cdot D_\partial^{k-1}\left(e^{\frac{(-x-y)}{2}} x^n y^m\right)\right)$$

$$= e^{\frac{(x+y)}{2}} \cdot D_\partial\left(e^{\frac{(-x-y)}{2}}\left(\sum_{i=0}^{\min(k-1,m)} \binom{k-1}{i} \cdot \frac{m!}{(m-i)!} \cdot \left(e^x D^{k-1-i}\left(e^{-x} x^n\right)\right) \cdot y^{m-i}\right)\right).$$

Since

$$e^{\frac{(x+y)}{2}} D_\partial \left(e^{\frac{(-x-y)}{2}} g(x)y^l\right) = \big(g'(x) - g(x)\big)y^l + lg(x)y^{l-1} = e^x D(e^{-x}g(x))y^l + lg(x)y^{l-1}$$

for all $l \in \mathbb{N}$, the above summation can be rewritten as

$$\begin{aligned} e^{\frac{(x+y)}{2}} \cdot D_\partial^k \left(e^{\frac{(-x-y)}{2}} x^n y^m\right) &= \sum_{i=0}^{\min(k-1,m)} \binom{k-1}{i} \cdot \frac{m!}{(m-i)!} \cdot \left(e^x D^{k-i}\left(e^{-x}x^n\right)\right) \cdot y^{m-i} \\ &+ \sum_{i=1}^{\min(k-1,m)+1} \binom{k-1}{i-1} \cdot \frac{m!}{(m-i)!} \cdot \left(e^x D^{k-i}\left(e^{-x}x^n\right)\right) \cdot y^{m-i}, \end{aligned}$$

where the expression $m!/(m-i)!$ is to be interpreted as 0 if $i = m+1$ in the last sum. Using the identities

$$\binom{k-1}{0} = \binom{k}{0}, \ \binom{k-1}{k-1} = \binom{k}{k}, \text{ and } \binom{k-1}{i} + \binom{k-1}{i-1} = \binom{k}{i}, \text{ for } 1 \le i \le k-1,$$

and combining the coefficients of terms that have the same degree in y in the last two summations, we see that the identity (4) holds true for $t = k$, which finishes the induction. □

The next theorem gives the explicit formulas for $L_{n,m}(x, y)$ (compare with (1)).

Theorem 1. *For all $n, m \in \mathbb{N}_0$ we have*

$$L_{n,m}(x,y) = \sum_{i=0}^{m} \frac{(-1)^i}{i!} \cdot \binom{m+n}{m-i} \cdot L_n^i(x) \cdot y^i = \sum_{s=0}^{n} \frac{(-1)^s}{s!} \cdot \binom{n+m}{n-s} \cdot L_m^s(y) \cdot x^s, \quad \textit{and}$$

$$L_{n,m}(x,y) = \sum_{i=0}^{m} \sum_{s=0}^{n} \frac{(-1)^{i+s}}{i! \cdot s!} \cdot \binom{m+n}{m-i} \cdot \binom{n+i}{n-s} \cdot x^s \cdot y^i. \tag{5}$$

Proof. Using Lemma 1 and Formula (4) we can write

$$e^{\frac{(x+y)}{2}} \cdot D_\partial^{n+m} \left(e^{\frac{(-x-y)}{2}} x^n y^m\right) = \sum_{j=0}^{m} \binom{m+n}{j} \frac{m!}{(m-j)!} \left(e^x D^{n+m-j}\left(e^{-x}x^n\right)\right) y^{m-j},$$

which implies that for $i := m - j \in \{0, \ldots, m\}$,

$$\begin{aligned} L_{n,m}(x,y) &= \frac{1}{n! \cdot m!} \cdot \sum_{i=0}^{m} \binom{m+n}{m-i} \frac{m!}{i!} \left(e^x D^{n+i}\left(e^{-x}x^n\right)\right) y^i \\ &= \sum_{i=0}^{m} \frac{1}{i!} \binom{m+n}{m-i} \left(\frac{1}{n!} e^x D^{n+i}\left(e^{-x}x^n\right)\right) y^i. \end{aligned} \tag{6}$$

Since for Laguerre polynomials in a single variable x (see [4], Section 1.4.2)

$$\frac{d}{dx}\big(L_k^{\alpha}(x)\big) = -L_{k-1}^{\alpha+1}(x) \quad \text{and} \quad L_k^{\alpha}(x) = L_k^{\alpha+1}(x) - L_{k-1}^{\alpha+1}(x)$$
$$\Rightarrow e^x D^t(e^{-x} L_k^{\alpha}(x)) = (-1)^t \cdot L_k^{\alpha+t}(x)$$

we can continue Formula (6) and write

$$L_{n,m}(x,y) = \sum_{i=0}^{m} \frac{1}{i!}\binom{m+n}{m-i}\Big(e^x D^i\big(e^{-x}L_n(x)\big)\Big)y^i = \sum_{i=0}^{m}\frac{(-1)^i}{i!}\binom{m+n}{m-i}L_n^i(x)y^i,$$

which gives the first formula in (5). The second formula follows either from the symmetry or from an argument similar to the one we just gave. To obtain the third formula recall that by (1),

$$L_n^i(x) = \sum_{s=0}^{n}\frac{(-1)^s}{s!}\cdot\binom{n+i}{n-s}\cdot x^s$$

and hence

$$L_{n,m}(x,y) = \sum_{i=0}^{m}\sum_{s=0}^{n}\frac{(-1)^{i+s}}{i!\cdot s!}\binom{m+n}{m-i}\binom{n+i}{n-s}\cdot x^s\cdot y^i,$$

as was required. □

It is well-known (see [4], Sections 1.4.2 and 2.3.5) that Laguerre polynomials in one variable satisfy the differential equation

$$x\cdot\frac{d^2}{dx^2}L_n^{\alpha}(x) + (\alpha+1-x)\cdot\frac{d}{dx}L_n^{\alpha}(x) + n\cdot L_n^{\alpha}(x) = 0, \tag{7}$$

and the multiple Laguerre polynomials $L_{n_1,\ldots,n_r}^{\alpha_1,\ldots,\alpha_r}(x_1,\ldots,x_r)$ (recall (3) above) satisfy the partial differential equation

$$\sum_{i=1}^{r} x_i\cdot\frac{\partial^2}{\partial x_i^2}L_{n_1,\ldots,n_r}^{\alpha_1,\ldots,\alpha_r} + \sum_{i=1}^{r}\Big((\alpha_i+1-x_i)\cdot\frac{\partial}{\partial x_i}L_{n_1,\ldots,n_r}^{\alpha_1,\ldots,\alpha_r}\Big) + n\cdot L_{n_1,\ldots,n_r}^{\alpha_1,\ldots,\alpha_r} = 0.$$

Here is the corresponding analog for the Laguerre polynomial $L_{n,m}(x,y)$.

Claim 1. *For all $n,m\in\mathbb{N}_0$, $L_{n,m}(x,y)$ satisfies the following system of partial differential equations*

$$\begin{pmatrix} L_{xx} & L_{xy}\\ L_{yx} & L_{yy}\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}L_x & 0\\ 0 & L_y\end{pmatrix}\cdot\begin{pmatrix}1-x\\1-y\end{pmatrix} + \begin{pmatrix}L & 0\\ 0 & L\end{pmatrix}\cdot\begin{pmatrix}n\\m\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix},$$

where we use the notation L for $L_{n,m}(x,y)$, L_x for $\frac{\partial}{\partial x}L_{n,m}(x,y)$, L_{xy} for $\frac{\partial^2}{\partial x\partial y}L_{n,m}(x,y)$, and so on.

Proof. The proof is a straightforward computation using our first two explicit formulas in (5) and the differential equation (7). □

3. Another Proof of the Congruence of Carlitz

In this section we prove several auxiliary results. Note that (p, q) stands for the greatest common divisor of p and q, and $(x)_n = x \cdot (x+1) \cdot \ldots \cdot (x+n-1)$ denotes the Pochhammer symbol.

Lemma 2. *Let p, q, u, v and n be integers such that either p or q is non-zero and $n \geq 0$. Then*

$$(x)_n \equiv (x + up + vq)_n \pmod{(p, q)}$$

Proof. Since (p, q) divides both p and q we have $x+i \equiv x+up+vq+i \pmod{(p, q)}$, for all integer $i \geq 0$. □

Lemma 3. *For all $m, n \in \mathbb{N}_0$, $q \in \mathbb{N}$, $p \in \mathbb{Z} \setminus \{0\}$, and $i \in \{q, \ldots, m+q\}$ we have the following congruence modulo the gcd of p and q.*

$$\binom{m}{m+q-i} \cdot (n+p+i-q+1)_{m-(i-q)} \equiv \binom{m+n}{m+q-i} \cdot (i-q-p+1)_{m-(i-q)} \quad (8)$$

Proof. Lemma 2 implies that, modulo the gcd of p and q, that

$$\binom{m}{m+q-i} \cdot (n+p+i-q+1)_{m-(i-q)} \equiv \binom{m}{m+q-i} \cdot (n+i-q+1)_{m-(i-q)},$$

and also

$$\binom{m+n}{m+q-i} \cdot (i-q+1)_{m-(i-q)} \equiv \binom{m+n}{m+q-i} \cdot (i-q-p+1)_{m-(i-q)} \pmod{(p, q)}.$$

Since

$$\binom{m+n}{m+q-i} \cdot (i-q+1)_{m-(i-q)} = \frac{(m+n)!}{(m-(i-q))!(i-q+n)!} \cdot (i-q+1) \cdot \ldots \cdot (m)$$

$$= \frac{m! \cdot (m+1) \cdot \ldots \cdot (m+n)}{(m-(i-q))!(i-q)! \cdot (i-q+1) \cdot \ldots \cdot (i-q+n)} \cdot (i-q+1)_{m-(i-q)}$$

$$= \binom{m}{m+q-i} \cdot (i-q+n+1)_{m-(i-q)},$$ we see that the lemma holds.

□

Corollary 1. *For all $m, s \in \mathbb{N}_0$, $q \in \mathbb{N}$, and $i \in \{q, \ldots, m+q\}$ we have, modulo q, that*

$$(m+q-i)! \cdot \binom{m}{m+q-i} \cdot \binom{m+s}{m+q-i} \equiv (m+q-i)! \cdot \binom{m+q}{m+q-i} \cdot \binom{m+s+q}{m+q-i}. \quad (9)$$

Proof. Using the notation from Lemma 3, and assuming that $p = -q$ and $n = s + q$ with $s \in \mathbb{N}_0$, we obtain

$$(n+p+i-q+1)_{m+q-i} = (m+q-i)! \cdot \binom{m+s}{m+q-i}, \text{ and}$$

$$(i-q-p+1)_{m+q-i} = (m+q-i)! \cdot \binom{m+q}{m+q-i}.$$

Hence we can rewrite Lemma 3, modulo q, as

$$(m+q-i)! \cdot \binom{m}{m+q-i} \cdot \binom{m+s}{m+q-i} \equiv (m+q-i)! \cdot \binom{m+q}{m+q-i} \cdot \binom{m+s+q}{m+q-i},$$

which is what we had to show. $\square$

Please notice that, for example if $m = 3,\ q = 6,\ s = 5$, and $i = 6$, then for the binomial factors from (9) we have

$$\binom{3}{3} \cdot \binom{3+5}{3} - \binom{3+6}{3} \cdot \binom{3+6+5}{3} \equiv 2 \pmod 6;$$

so the factor $(m + q - i)!$ in (9) is necessary to guarantee the equality.

Lemma 4. *For all $n, m \in \mathbb{N}_0$ and $q \in \mathbb{N}$,*

$$n!L_n^{m+q}(x) \equiv n!L_n^m(x) \pmod q.$$

Proof. Using Formula (1) we obtain

$$n!L_n^{m+q}(x) = \sum_{j=0}^{n} \frac{n!(-1)^j}{j!} \binom{n+m+q}{n-j} x^j, \quad n!L_n^m(x) = \sum_{j=0}^{n} \frac{n!(-1)^j}{j!} \binom{n+m}{n-j} x^j,$$

and since we also have

$$\frac{n!}{j!} \binom{n+m+q}{n-j} = \binom{n}{j} \cdot (m+q+j+1)_{n-j} \text{ and } \frac{n!}{j!} \binom{n+m}{n-j} = \binom{n}{j} \cdot (m+j+1)_{n-j},$$

we see that this lemma follows from Lemma 2 that implies $(m + q + j + 1)_{n-j} \equiv (m + j + 1)_{n-j} \pmod q$. $\square$

Now we give a direct proof of Carlitz's identity for the classical Laguerre polynomials.

Corollary 2. *(see [3]) For all $n, i \in \mathbb{N}_0$ and $p \in \mathbb{N}$ the following congruence holds.*

$$(n+p)!L_{n+p}^i(x) \equiv n!L_n^i(x) \cdot p!L_p^i(x) \pmod p$$

Proof. First observe that $p!L_p^i(x) \equiv (-1)^p x^p \pmod p$ (compare with (4.6) of [3]). Indeed,

$$p!L_p^i(x) = \sum_{t=0}^{p}(-1)^t \cdot \frac{p!}{t!} \cdot \binom{p+i}{p-t} x^t$$

$$= p \cdot \sum_{t=0}^{p-1}(-1)^t (t+1)_{p-1-t} \cdot \binom{p+i}{p-t} x^t + (-1)^p x^p \equiv (-1)^p x^p \pmod p.$$

Therefore it is enough to show that

$$(n+p)!L_{n+p}^i(x) \equiv (-1)^p x^p \cdot n!L_n^i(x) \pmod p. \tag{10}$$

We do it by comparing the coefficients of x^t on both sides of (10). Suppose first that $t \in \{0, \dots, p-1\}$. Then, the coefficient on the right-hand side of (10) is zero. The corresponding coefficient on the left-hand side of (10) is

$$(-1)^t \frac{(n+p)!}{t!}\binom{n+p+i}{n+p-t} = (-1)^t \binom{n+p+i}{n+p-t}\frac{(p-1)!}{t!} \cdot p \cdot \ldots \cdot (p+n) \equiv 0 \pmod p.$$

Assume now that $t \in \{p, \dots, n+p\}$. Using (1) again, we see that the coefficients of x^t on the left and right-hand sides of (10) are respectively

$$(-1)^t \cdot \frac{(n+p)!}{t!} \cdot \binom{n+p+i}{n+p-t} \quad \text{and} \quad (-1)^p \cdot (-1)^{t-p} \cdot \frac{n!}{(t-p)!} \cdot \binom{n+i}{n+p-t}. \tag{11}$$

Canceling $(-1)^t$ on both sides we can rewrite these coefficients as

$$(n+p-t)! \cdot \binom{n+p}{n+p-t} \cdot \binom{n+p+i}{n+p-t} \quad \text{and} \quad (n+p-t)! \cdot \binom{n}{n+p-t} \cdot \binom{n+i}{n+p-t}.$$

Applying Corollary 1 with $n = m$, $p = q$, $t = i$, and $i = s$ we deduce that these coefficients are congruent $\pmod p$. This finishes our proof of the Identity (2). $\square$

4. Main Theorem

Theorem 2. *For all $n, m \in \mathbb{N}_0$ and $p, q \in \mathbb{N}$ we have the following congruence modulo the gcd of p and q (compare with (2) above).*

$$(n+p)!(m+q)!L_{n+p,m+q}(x,y) \equiv n!m!L_{n,m}(x,y) \cdot p!q!L_{p,q}(x,y) \tag{12}$$

Proof. We will compare the corresponding coefficients of $x^t y^i$ on both sides of the congruence. Similarly to the one variable case we have $p!q!L_{p,q}(x,y) \equiv (-1)^{p+q}x^p y^q \pmod{(p,q)}$. Indeed, using our third formula in (5) we have

$$p!q!L_{p,q}(x,y) = \sum_{i=0}^{q}\sum_{t=0}^{p} \frac{(-1)^{i+t}p!q!}{i!t!} \cdot \binom{p+q}{q-i}\binom{p+i}{p-t} x^t y^i,$$

so if $i+t<p+q$ then $\gcd(p,q) \mid p!q!/i!t!$ since either $i<q$ or $t<p$. If $i=q$ and $t=p$, the coefficient of $x^p y^q$ equals $(-1)^{p+q}$, and hence

$$n!m!L_{n,m}(x,y)\cdot p!q!L_{p,q}(x,y) \equiv (-1)^{p+q}x^p y^q \cdot n!m!L_{n,m}(x,y) \pmod{(p,q)}. \quad (13)$$

If we take the coefficient of $x^t y^i$ in $(n+p)!(m+q)!L_{n+p,m+q}(x,y)$ with $t<p$ or $i<q$ we will have

$$(-1)^{i+t}\frac{(n+p)!(m+q)!}{t!i!}\cdot\binom{m+q+n+p}{m+q-i}\cdot\binom{n+p+i}{n+p-t},$$

and if, for example, $t<p$ then the integer $\frac{(n+p)!}{t!}$ is divisible by p, and hence by $\gcd(p,q)$. Since $\frac{(m+q)!}{i!}$, $\binom{m+q+n+p}{m+q-i}$, and $\binom{n+p+i}{n+p-t}$ are all integers, $\gcd(p,q)$ divides the coefficient of $x^t y^i$. If $t\ge p$ but $i<q$ the proof is similar since $q \mid \frac{(m+q)!}{i!}$. So to prove Theorem 2 it is enough to show that the coefficients of $x^t y^i$ on the left-hand side and the right-hand side of (12) are congruent $\pmod{(p,q)}$ for all $p\le t\le n+p$ and $q\le i\le m+q$.

Thus we assume from now on that $t\in\{p,\dots,n+p\}$ and $i\in\{q,\dots,m+q\}$. According to the first formula from our Theorem 1, the coefficient of y^i on the left-hand side of (12) is

$$(-1)^i\frac{(n+p)!(m+q)!}{i!}\binom{m+q+n+p}{m+q-i}\cdot L^i_{n+p}(x),$$

which is, due to the Identity (2),

$$\equiv(-1)^{i+p}\frac{(m+q)!}{i!}\binom{m+q+n+p}{m+q-i}\cdot n!L^i_n(x)\cdot x^p \pmod{p}.$$

Now, let us fix $j=i-q\in\{0,\dots,\ m\}$. Then the coefficient of y^{q+j} on the left-hand side of (12) is

$$\equiv(-1)^{q+j+p}\frac{(m+q)!}{(j+q)!}\binom{m+q+n+p}{m-j}\cdot n!L^{q+j}_n(x)\cdot x^p \pmod{(p,q)}.$$

Using (13), modulo (p,q), the coefficient of y^{q+j} on the right-hand side of (12) is the product of $(-1)^{p+q}\cdot x^p\cdot n!\cdot m!$ with the coefficient of y^j in $L_{n,m}(x,\ y)$. Therefore, according to Theorem 1, this coefficient of y^{q+j} is

$$(-1)^{p+q}\cdot x^p\cdot n!\cdot m!\cdot\frac{(-1)^j}{j!}\cdot\binom{m+n}{m-j}\cdot L^j_n(x).$$

By Lemma 4 then, it suffices to show that

$$\frac{(m+q)!}{(j+q)!}\binom{m+q+n+p}{m-j}\equiv\frac{m!}{j!}\binom{m+n}{m-j} \pmod{(p,q)},$$

which can be rewritten as

$$(q+j+1)_{m-j}\frac{(q+n+p+j+1)_{m-j}}{(m-j)!} \equiv (j+1)_{m-j}\frac{(n+j+1)_{m-j}}{(m-j)!} \pmod{(p,q)}.$$

To prove this last congruence we note that the product of any N consecutive integers is divisible by $N!$ so $(j+1)_{m-j}/(m-j)! \in \mathbb{Z}$ and apply Lemma 2 twice to write, modulo (p,q)

$$(q+j+1)_{m-j}\frac{(q+n+p+j+1)_{m-j}}{(m-j)!} \equiv (j+1)_{m-j}\frac{(q+n+p+j+1)_{m-j}}{(m-j)!}$$

$$= (q+n+p+j+1)_{m-j}\frac{(j+1)_{m-j}}{(m-j)!} \equiv (n+j+1)_{m-j}\frac{(j+1)_{m-j}}{(m-j)!} \pmod{(p,q)},$$

and the theorem follows. □

5. More Variables, Irreducibility, and Other Related Questions

First let us generalize our Definition 1 from Section 2 to more than two variables.

Definition 2. For all $r \in \mathbb{N}$ and $n_i \in \mathbb{N}_0$ with $i \in \{1, 2, \ldots, r\}$ let

$$L_{n_1,\ldots,n_r}(x_1,\ldots,x_r) := \left(\prod_{i=1}^{r}\frac{1}{n_i!}\right)\cdot e^{s/r}\cdot D_\partial^d\left(e^{-s/r}\prod_{i=1}^{r}(x_i)^{n_i}\right),$$

where

$$s := \sum_{i=1}^{r} x_i, \quad d := \sum_{i=1}^{r} n_i \quad \text{and} \quad D_\partial(f(x_1,\ldots,x_r)) := \sum_{i=1}^{r}\frac{\partial f(x_1,\ldots,x_r)}{\partial x_i}.$$

`Note` 2: Here we have

$$e^{s/r}\cdot D_\partial\left(e^{-s/r}f(x_1,\ldots,x_r)\right) = \sum_{i=1}^{r}\frac{\partial f(x_1,\ldots,x_r)}{\partial x_i} - f(x_1,\ldots,x_r)$$

and if it happens that $f(x_1,\ldots,x_r)$ depends on less than r variables, say only on variables $x_1,\ldots,x_{r-1}$, we obtain (for the same s as above)

$$e^{s/r}\cdot D_\partial\left(e^{-s/r}\cdot f(x_1,\ldots,x_{r-1})\right) = \sum_{i=1}^{r-1}\frac{\partial f(x_1,\ldots,x_{r-1})}{\partial x_i} - f(x_1,\ldots,x_{r-1})$$

$$= e^{(s-x_r)/(r-1)}\cdot D_\partial\left(e^{-(s-x_r)/(r-1)}\cdot f(x_1,\ldots,x_{r-1})\right),$$

and hence naturally

$$L_{n_1,\dots,n_{r-1},0}(x_1,\dots,x_r) = \left(\prod_{i=1}^{r-1}\frac{1}{n_i!}\cdot\frac{1}{0!}\right)\cdot e^{s/r}\cdot D_{\partial}^{d}\left(e^{-s/r}\cdot\prod_{i=1}^{r-1}(x_i)^{n_i}\cdot x_r^0\right)$$

$$= L_{n_1,\dots,n_{r-1}}(x_1,\dots,x_{r-1}).$$

By a similar argument for $x_1 = x_2 = \cdots = x_r = x$, we also obtain

$$L_{n_1,\dots,n_r}(x,x,\dots,x) = \left(\prod_{i=1}^{r}\frac{1}{n_i!}\right)\cdot e^{s/r}\cdot D_{\partial}^{d}\left(e^{-s/r}\cdot x^d\right) = (n_1,n_2,\dots,n_r)!\cdot L_d(x),$$

where

$$(n_1,n_2,\dots,n_r)! = \frac{(n_1+n_2+\ldots+n_r)!}{n_1!\cdot n_2!\cdot\ldots\cdot n_r!}$$

is the multinomial coefficient.

Now we discuss the irreducibility of $L_{n_1,\dots,n_r}(x_1,\dots,x_r)$.

Lemma 5. *For all $r \in \mathbb{N}$ and $n_i \in \mathbb{N}_0$ with $i \in \{1,\dots,r\}$ the polynomials $L_{n_1,\dots,n_r}(x_1,\dots,x_r)$ are irreducible over the rationals.*

Proof. We will use the strong principle of mathematical induction on the number of variables r. The base case when $r = 1$ is due to Schur (see [8]), so let's assume that the corresponding Laguerre polynomials in k variables will be irreducible over $\mathbb{Q}$ for all $k \in \{1,\dots,r-1\}$. Suppose further $\prod_{i=1}^{r} n_i \neq 0$ and $L_{n_1,\dots,n_r}(x_1,\dots,x_r) = f(x_1,\dots,x_r)\cdot g(x_1,\dots,x_r)$, where $\deg f(x_1,\dots,x_r) > 0, \deg g(x_1,\dots,x_r) > 0$, and both f and g are polynomials with rational coefficients. Then, according to our Note 2 above, substituting $x_1 = \ldots = x_r = x$ we get

$$f(x,\dots,x)\cdot g(x,\dots,x) = L_{n_1,\dots,n_r}(x,\dots,x) = (n_1,\dots,n_r)!\cdot L_d(x).$$

Since $L_d(x)$ is irreducible for all $d \in \mathbb{N}$ we must have either $\deg f(x,\dots,x) = 0$ or $\deg g(x,\dots,x) = 0$. Assuming without loss of generality that $\deg f(x,\dots,x) = 0$ we get $\deg g(x,\dots,x) = \deg L_d(x) = d$. Since $\deg g(x,\dots,x) \leq \deg g(x_1,\dots,x_r)$ we deduce from the last equality that $\deg L_{n_1,\dots,n_r}(x_1,\dots,x_r) = d \leq \deg g(x_1,\dots,x_r)$, which contradicts the assumption that $\deg f(x_1,\dots,x_r) > 0$. If one of $n_i = 0$, our polynomial $L_{n_1,\dots,n_r}(x_1,\dots,x_r)$ reduces to the one in number of variables less than r, which is irreducible by the induction assumption. $\square$

As we have mentioned in the introduction, this is the main distinction of our version of multivariable Laguerre polynomials from those considered in [2] and [4].

Laguerre polynomials in one variable have many interesting combinatorial properties. For example, Even and Gillis in 1976 showed that an integral of a product of the Laguerre polynomials and e^{-x} can be interpreted as permutations of a set of objects of different "colors" (*derangements*). Using rook polynomials $R_n(x)$, Jackson gave a shorter proof of the result Even and Gillis obtained (see [5] and [7]). These polynomials satisfy

$$R_n(x) = \sum_{k=0}^{n} r_k \cdot x^k = n!x^n \cdot L_n(-1/x),$$

where r_k stands for the rook number that counts the various ways of placing k non-attacking rooks on the full $n \times n$ board. We would like to close this paper with a general question: Do $L_{n,m}(x, y)$ have any combinatorial properties similar to those of $L_n(x)$? In particular, the two-dimensional rook numbers and their certain properties can be generalized to three and higher dimensions (see, for example, [1]), so one can ask if

$$n!x^n \cdot m!y^m \cdot L_{n,m}(-1/x, -1/y)$$

has a natural interpretation in terms of rook numbers for three-dimensional boards.

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