EQUAL SUMS OF LIKE POWERS WITH MINIMUM NUMBER OF TERMS

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Abstract

This paper is concerned with the diophantine system, $\sum_{i=1}^{s_1} x_i^r = \sum_{i=1}^{s_2} y_i^r,\ r = 1, 2, \ldots, k$, where s_1 and s_2 are integers such that the total number of terms on both sides, that is, s_1+s_2, is as small as possible. We define $\beta(k)$ to be the minimum value of s_1+s_2 for which there exists a nontrivial solution of this diophantine system. We show that $\beta(k) \geq 2k$ for any arbitrary positive integer k. We also find several nontrivial solutions of the aforementioned diophantine system and thereby prove that $\beta(k) = 2k$ when $k = 2$, 3, 4 or 5.

1. Introduction

The Tarry-Escott problem of degree k consists of finding two distinct sets of integers $\{x_1, x_2, \ldots, x_s\}$ and $\{y_1, y_2, \ldots, y_s\}$ such that

$$\sum_{i=1}^{s} x_i^r = \sum_{i=1}^{s} y_i^r, \quad r = 1, 2, \ldots, k. \tag{1}$$

It is well-known that for a non-trivial solution of (1) to exist, we must have $s \geq (k+1)$ [12, p. 616]. Solutions of (1) with the minimum possible value of s, that is, with $s = k+1$, are known as ideal solutions of the problem.

This paper is concerned with finding solutions in integers of the related diophantine system,

$$\sum_{i=1}^{s_1} x_i^r = \sum_{i=1}^{s_2} y_i^r, \quad r = 1, 2, \ldots, k, \tag{2}$$

where s_1 and s_2 are integers such that the total number of terms on both sides, that is, $s_1 + s_2$, is minimum.

Without loss of generality, we may take $s_1 \leq s_2$. A solution of the system of equations (2) will be said to be trivial if $y_i = 0$ for $s_2 - s_1$ values of i and the remaining integers y_i are a permutation of the integers x_i.

We define $\beta(k)$ to be the minimum value of $s_1 + s_2$ for which there exists a nontrivial solution of the diophantine system (2).

According to a well-known theorem of Frolov [12, p. 614], if $x_i = a_i,\ y_i = b_i,\ i = 1, 2, \ldots, s$, is any nontrivial solution of the diophantine system (1), and d is an arbitrary integer, then $x_i = a_i + d,\ y_i = b_i + d,\ i = 1, 2, \ldots, s$, is also a solution of (1). Taking $d = -a_1$, we immediately get a solution of (2) with $s_1 = s - 1,\ s_2 = s$. Thus, if an ideal solution of (1) is known for any specific value of k, then $\beta(k) \leq 2k + 1$.

Ideal solutions of (1) are known for $k = 2, 3, \ldots, 9$ ([1], [3], [4], [5], [9, pp. 52, 55-58], [11], [13, pp. 41-54], [14], [15]) as well as for $k = 11$ [6]. Thus, for these values of k, we have $\beta(k) \leq 2k + 1$.

In Section 2 of this paper, we show that $\beta(k) \geq 2k$ for any arbitrary positive integer k. In Section 3 we find several nontrivial solutions of (2) when $2 \leq k \leq 5$ and show that $\beta(k) = 2k$ when $k = 2, 3, 4$ or 5.

2. A Lower Bound for $\beta(k)$

Theorem 1. *For any arbitrary positive integer k, we have $\beta(k) \geq 2k$.*

Proof. We first note that if there is a nontrivial solution of the diophantine system (2) in which the sum $s_1 + s_2$ is minimum, then all the integers x_i and y_i must be nonzero and the sets $\{x_i\}$ and $\{y_i\}$ must be disjoint since any zero terms and any terms common to both sides can simply be excluded to obtain another nontrivial solution of the diophantine system (2) with a smaller value of $s_1 + s_2$.

We also note that if there exists a nontrivial solution of the diophantine system (2) with $s_1 = s_2 = s$, Frolov's theorem immediately gives another nontrivial solution of (2) with $s_1 = s - 1,\ s_2 = s$. Thus, if $s_1 + s_2$ is to be minimum, there is no loss of generality in taking $s_1 < s_2$. Accordingly, we will henceforth always consider the diophantine system (2) with $s_1 < s_2$.

It is trivially true that $\beta(1) = 3$. Thus, the theorem is true for $k = 1$. We now show that if there exists a nontrivial solution of the diophantine system (2) with $k \geq 2$ and $s_1 < s_2$ and such that all of the integers x_i, y_i are nonzero, then

$$s_2 \geq k + 1, \tag{3}$$

and

$$s_1 \geq k - 1. \tag{4}$$

We will consider any solution of the diophantine system (2) with $s_1 < s_2$ as a solution of the diophantine system,

$$\sum_{i=1}^{s_2} x_i^r = \sum_{i=1}^{s_2} y_i^r, \quad r = 1, 2, \ldots, k, \tag{5}$$

in which the $s_2 - s_1$ terms $x_{s_1+1}, x_{s_1+2}, \ldots, x_{s_2}$ are all 0.

Thus, any solution of the diophantine system (2) with $s_2 < k+1$, yields a solution of (5) where $s_2 < k+1$, and by a theorem of Bastien (as quoted by Dickson [10, p. 712]), such a solution is necessarily trivial. It follows that for a nontrivial solution of (2) to exist, we must have the relation (3).

We will now prove the relation (4). It is obvious that for a nontrivial solution of (2) to exist, we must have $s_1 \geq 1$, so the relation (4) is true when $k = 2$. We now proceed to establish the relation (4) when $k \geq 3$.

We assume that there exists a solution of (2) with $s_1 < k - 1$ and $s_2 \geq k + 1$ where $k \geq 3$ and such that all the integers x_i and y_i are nonzero. We will use the elementary symmetric functions in the variables $x_1, x_2, \ldots, x_{s_2}$ as well as in the variables $y_1, y_2, \ldots, y_{s_2}$ defined by the following relations:

$$\begin{aligned}
p_1 &= x_1 + x_2 + \cdots + x_{s_2} = \sum_{u=1}^{s_2} x_u, &\qquad q_1 &= y_1 + y_2 + \cdots + y_{s_2} = \sum_{u=1}^{s_2} y_u,\\
p_2 &= \sum_{u<v} x_u x_v, & q_2 &= \sum_{u<v} y_u y_v,\\
p_3 &= \sum_{u<v<w} x_u x_v x_w, & q_3 &= \sum_{u<v} y_u y_v y_w,\\
&\vdots & &\\
p_{s_2} &= x_1 x_2 \cdots x_{s_2}, & q_{s_2} &= y_1 y_2 \cdots y_{s_2}.
\end{aligned}$$

It follows from Newton's theorem [2, p. 297] on sums of powers of the roots of an equation that the sums of powers $\sum_{i=1}^{s_2} x_i^r$, $r = 1, 2, \ldots, k$, can be expressed in terms of the elementary symmetric functions p_r, $r = 1, 2, \ldots, k$, and vice versa. There exist similar relations between the sums of powers $\sum_{i=1}^{s_2} y_i^r$, $r = 1, 2, \ldots, k$, and the elementary symmetric functions q_r, $r = 1, 2, \ldots, k$. Thus, the diophantine system (5) is equivalent to the following diophantine system:

$$p_r = q_r, \quad r = 1, 2, \ldots, k. \tag{6}$$

Our assumption that there exists a solution of (2) with $s_1 < k-1$ and $s_2 \geq k+1$, implies the existence of a solution of (5) in which $x_{s_1+1}, x_{s_1+2}, \ldots, x_{s_2}$, are all 0. This in turn implies the existence of a solution of (6) in which $x_{s_1+1}, x_{s_1+2}, \ldots, x_{s_2}$, are all 0. When $r > s_1$, each summand of the elementary symmetric function p_r contains 0 as a factor and is thus necessarily 0, and hence we get $p_r = 0$. Specifically, we get $p_{k-1} = 0$ and $p_k = 0$. It now follows from (6) that $q_{k-1} = 0$ and $q_k = 0$. Thus, the nonzero integers y_i, $i = 1, 2, \ldots, s_2$, satisfy an equation of the type,

$$y^{s_2} - q_1 y^{s_2-1} + q_2 y^{s_2-2} + \cdots + (-1)^{s_2} q_{s_2} = 0, \tag{7}$$

in which at least two consecutive coefficients, namely the coefficients of y^k and y^{k-1} are 0. This is, however, impossible in view of De Gua's rule [2, p. 90] according to which a polynomial equation of type (7) with two or more consecutive zero coefficients must have at least two imaginary roots.

It follows that we cannot have a nontrivial solution of (2) with $s_2 \geq k+1$ and $s_1 < k-1$. Thus, for a nontrivial solution of (2) to exist when $k \geq 3$, we must have $s_2 \geq k+1$ and $s_1 \geq k-1$. This proves the relation (4) when $k \geq 3$.

We have now proved the relations (3) and (4) when $k \geq 2$. Combining these two relations, we get $\beta(k) \geq 2k$ when $k \geq 2$. As we have already noted that the theorem is true when $k = 1$, the theorem is proved for any arbitrary positive integer k. $\square$

3. Determination of $\beta(k)$

In the next four subsections, we will find solutions of the diophantine system (2) with $s_1 = k-1,\ s_2 = k+1$ when $k = 2$, 3, 4 and 5 respectively, and thus prove that $\beta(k) = 2k$ for these values of k.

We note that all the equations of the diophantine system (2) are homogeneous, and therefore, any solution of (2) in rational numbers may be multiplied through by a suitable constant to obtain a solution of (2) in integers.

3.1.

It follows from Theorem 1 that $\beta(2) \geq 4$. We will show that $\beta(2) = 4$ by solving the diophantine system (2) with $s_1 = 1,\ s_2 = 3$ and $k = 2$. On eliminating x_1 from the two equations of this diophantine system, we get,

$$y_1y_2 + y_2y_3 + y_3y_1 = 0. \tag{8}$$

The complete solution of Equation (8) is readily obtained and this immediately yields the following simultaneous identities:

$$(p^2+pq+q^2)^r = (p^2+pq)^r + (pq+q^2)^r + (-pq)^r, \quad r = 1,\, 2, \tag{9}$$

where p and q are arbitrary parameters. This shows that $\beta(2) = 4$.

3.2.

It follows from Theorem 1 that $\beta(3) \geq 6$. We will prove that $\beta(3) = 6$ by obtaining nontrivial solutions of the diophantine system (2) with $s_1 = 2,\ s_2 = 4$ and $k = 3$, that is, of the system of equations,

$$x_1 + x_2 = y_1 + y_2 + y_3 + y_4, \tag{10}$$
$$x_1^2 + x_2^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2, \tag{11}$$
$$x_1^3 + x_2^3 = y_1^3 + y_2^3 + y_3^3 + y_4^3. \tag{12}$$

If $(a,\, b,\, c)$ is any Pythagorean triple satisfying the relation $a^2 + b^2 = c^2$, it is easily seen that a solution in integers of the simultaneous equations (10), (11) and (12) is given by $(x_1,\, x_2,\, y_1,\, y_2,\, y_3,\, y_4) = (c,\, -c,\, a,\, -a,\, b,\, -b)$.

Next we obtain a more general parametric solution of the simultaneous equations (10), (11) and (12). We will use a parametric solution of the simultaneous diophantine equations,

$$\begin{aligned} x + y + z &= u + v + w, \\ x^3 + y^3 + z^3 &= u^3 + v^3 + w^3, \end{aligned} \tag{13}$$

given by Choudhry [7, Theorem 1, p. 61]. From this solution, on writing $x_1 = x,\, x_2 = y,\, y_1 = u,\, y_2 = v,\, y_3 = w,\, y_4 = -z$, we immediately derive the following solution of the simultaneous equations (10) and (12) in terms of arbitrary parameters $p,\, q,\, r$ and s:

$$\begin{aligned} x_1 &= pq - pr + qr - (p - q - r)s, \\ x_2 &= -pq + pr + qr + (p - q + r)s, \\ y_1 &= pq + pr - qr + (p - q + r)s, \\ y_2 &= pq - pr + qr + (p + q - r)s, \\ y_3 &= -pq + pr + qr - (p - q - r)s, \\ y_4 &= -pq - pr + qr - (p + q - r)s. \end{aligned} \tag{14}$$

Substituting the above values of $x_i,\ y_i$ in (11), we get, after necessary transpositions, the following quadratic equation in s:

$$\begin{aligned} 2(p + q - r)^2 s^2 + (12p^2q - 4p^2r - 4pq^2 - 4pqr \\ + 4pr^2 + 4q^2r - 4qr^2)s + 2(pq + pr - qr)^2 = 0. \end{aligned} \tag{15}$$

On taking $r = p + q$, the coefficient of s^2 in Equation (15) vanishes, and we can then readily solve Equation (15) for s, and we thus obtain the following solution of the simultaneous equations (10), (11) and (12) in terms of arbitrary parameters p and q:

$$\begin{aligned} x_1 &= (3p^4 - 2p^3q - p^2q^2 + q^4)q, & x_2 &= (p^4 - p^2q^2 - 2pq^3 + 3q^4)p, \\ y_1 &= (p^4 - p^2q^2 + 2pq^3 - q^4)p, & y_2 &= 2pq(p - q)(p^2 - pq - q^2), \\ y_3 &= -(p^4 - 2p^3q + p^2q^2 - q^4)q, & y_4 &= 2pq(p - q)(p^2 + pq - q^2). \end{aligned} \tag{16}$$

As a numerical example, taking $p = 2,\ q = 1$, we get the solution,

$$29^r + 22^r = 30^r + 4^r + (-3)^r + 20^r, \quad r = 1,\, 2,\, 3.$$

We note that more parametric solutions of the system of equations (10), (11) and (12) can be obtained by solving Equation (15) in different ways, for example, by choosing p, q, r such that the constant term in Equation (15) vanishes and then solving this equation for s, or by choosing p, q, r such that the discriminant of Equation (15), considered as a quadratic equation in s, becomes a perfect square, and then solving this equation for s.

As we have obtained nontrivial solutions of the system of equations (10), (11) and (12), we get $\beta(3) \leq 6$, and on combining this result with the relation $\beta(3) \geq 6$ obtained from Theorem 1, we get $\beta(3) = 6$.

3.3.

We will now obtain parametric solutions of the diophantine system,

$$\sum_{i=1}^{3} x_i^r = \sum_{i=1}^{5} y_i^r, \quad r = 1,\, 2,\, 3,\, 4. \tag{17}$$

We write,

$$\begin{aligned} x_1 &= 4uv + w + 1, & x_2 &= -4uv + w - 1, \\ x_3 &= -8u^2 + 8uv + 4u - 2, & y_1 &= 4u - 2, \\ y_2 &= -4u, & y_3 &= 4uv + w - 1, \\ y_4 &= -4uv + w + 1, & y_5 &= -8u^2 + 8uv + 4u, \end{aligned} \tag{18}$$

where u, v, w are arbitrary parameters.

It is readily verified that the values of x_i, y_i given by (18) satisfy Equation (17) when $r = 1$ and $r = 2$. Further, on substituting these values of x_i, y_i in Equation (17) and taking $r = 3$, we get the condition,

$$4u^3 - 8u^2v + 4uv^2 - 4u^2 + 4uv - vw + u - v = 0. \tag{19}$$

On solving Equation (19), we get,

$$w = (4u^3 - 8u^2v + 4uv^2 - 4u^2 + 4uv + u - v)/v. \tag{20}$$

Finally, we substitute the values of x_i, y_i given by (18) in Equation (17), and take $r = 4$, and use the value of w given by (20) to get the condition,

$$u^2(2u - 1)^2\{24uv^2 - 2(4u - 1)^2v + 3u(2u - 1)^2\} = 0. \tag{21}$$

While equating the first two factors of Equation (21) to 0 leads to trivial results, on equating the last factor to 0, we get a quadratic equation in v which will have a rational solution if its discriminant $4(-32u^4 + 32u^3 + 24u^2 - 16u + 1)$ becomes a perfect square. We thus have to solve the diophantine equation,

$$t^2 = -32u^4 + 32u^3 + 24u^2 - 16u + 1. \tag{22}$$

Now Equation (22) is a quartic model of an elliptic curve, and we use the birational transformation given by,

$$\begin{aligned} t &= (X^3 - 36X^2 + 36X - 72Y + 432)/(4X + Y - 12)^2, \\ u &= (X - 12)/(4X + Y - 12), \end{aligned} \tag{23}$$

and,

$$\begin{aligned} X &= (4u^2 - 8u + t + 1)/(2u^2), \\ Y &= (8u^3 + 12u^2 - 4ut - 12u + t + 1)/(2u^3), \end{aligned} \tag{24}$$

to reduce Equation (22) to the Weierstrass model which is given by the cubic equation,

$$Y^2 = X^3 - 36X. \tag{25}$$

It is readily seen from Cremona's well-known tables [8] that (25) is an elliptic curve of rank 1 and its Mordell-Weil basis is given by the rational point P with coordinates $(X, Y) = (-3, 9)$. There are thus infinitely many rational points on the elliptic curve (25) and these can be obtained by the group law. Using the relations (23), we can find infinitely many rational solutions of Equation (22) and thus obtain infinitely many integer solutions of the diophantine system (17).

While the point P leads to a trivial solution of the diophantine system (17), the point $2P$ yields the solution,

$$(-74)^r + 124^r + 78^r = 126^r + (-72)^r + (-20)^r + 70^r + 24^r, \quad r = 1, 2, 3, 4,$$

and the point $3P$ leads to the solution,

$$\begin{aligned} (-40573)^r + 66494^r + 118981^r = (-15181)^r + 119510^r + 63756^r \\ + (-37835)^r + 14652^r, \quad r = 1, 2, 3, 4. \end{aligned}$$

In view of the above solutions of the diophantine system (17), it follows that $\beta(4) \leq 8$, and on combining with the result $\beta(4) \geq 8$ which follows from Theorem 1, we get $\beta(4) = 8$.

3.4.

We will now obtain parametric solutions of the diophantine system,

$$\sum_{i=1}^{4} x_i^r = \sum_{i=1}^{6} y_i^r, \quad r = 1, 2, 3, 4, 5. \tag{26}$$

We will use a parametric solution of the diophantine system,

$$\sum_{i=1}^{6} X_i^r = \sum_{i=1}^{6} Y_i^r, \quad r = 1, 2, 3, 4, 5, \tag{27}$$

to obtain two parametric solutions of the diophantine system (26).

A solution of the simultaneous equations $\sum_{i=1}^{3} X_i^r = \sum_{i=1}^{3} Y_i^r,\ r = 2,\,4$, in terms of arbitrary parameters m, n, x and y, given by Choudhry [5, p. 102], is as follows:

$$\begin{aligned} X_1 &= (m+2n)x-(m-n)y, & X_2 &= -(2m+n)x-(m+2n)y,\\ X_3 &= (m-n)x+(2m+n)y, & Y_1 &= (m-n)x-(m+2n)y,\\ Y_2 &= -(2m+n)x-(m-n)y, & Y_3 &= (m+2n)x+(2m+n)y, \end{aligned} \tag{28}$$

It immediately follows that a parametric solution of the diophantine system (27) is given by (28) and

$$\begin{aligned} X_4 &= -X_3, & X_5 &= -X_2, & X_6 &= -X_1,\\ Y_4 &= -Y_3, & Y_5 &= -Y_2, & Y_6 &= -Y_1. \end{aligned} \tag{29}$$

We choose the parameters x and y such that $X_3 = 0$ and immediately obtain the following parametric solution of the diophantine system (26):

$$\begin{aligned} x_1 &= m^2+mn+n^2, & x_2 &= m^2+mn+n^2,\\ x_3 &= -m^2-mn-n^2, & x_4 &= -m^2-mn-n^2,\\ y_1 &= m^2-n^2, & y_2 &= -m^2-2mn,\\ y_3 &= 2mn+n^2, & y_4 &= -2mn-n^2,\\ y_5 &= m^2+2mn, & y_6 &= -m^2+n^2, \end{aligned} \tag{30}$$

where m and n are arbitrary parameters.

To obtain a second solution of the diophantine system (26), we again use the parametric solution of the diophantine system (27) given by (28) and (29). We now choose the parameters x, y such that we get $X_2 = X_3$, and then apply the theorem of Frolov mentioned in the Introduction, taking $d = -X_3$. We thus get a solution of the diophantine system (26) which may be written as follows:

$$\begin{aligned} x_1 &= 3m^2+3mn+3n^2, & x_2 &= 2m^2+2mn+2n^2,\\ x_3 &= -m^2-mn-n^2, & x_4 &= 2m^2+2mn+2n^2,\\ y_1 &= 3m^2+3mn, & y_2 &= -3mn,\\ y_3 &= 3mn+3n^2, & y_4 &= 2m^2-mn-n^2,\\ y_5 &= 2m^2+5mn+2n^2, & y_6 &= -m^2-mn+2n^2, \end{aligned} \tag{31}$$

where m and n are arbitrary parameters.

As a numerical example, taking $m = 2, n = 1$ in (31), we get the solution,

$$21^r + 14^r + (-7)^r + 14^r = 18^r + (-6)^r + 9^r + 5^r + 20^r + (-4)^r, \quad r = 1,\,2,\,3,\,4,\,5.$$

The two parametric solutions (30) and (31) of the diophantine system (26) are rather special since in both of them, the ratios x_i/x_j of the integers on the left-hand side are all fixed. We now show that there exist infinitely many other solutions of the diophantine system (26) that are not generated by these parametric solutions.

We write,

$$\begin{aligned} x_1 &= uv^2 + (6u^3 - 12u^2 + 32u - 32)v \\ &\quad + 9u^5 - 36u^4 - 336u^2 + 96u^3 + 240u, \\ x_2 &= (2u-2)v^2 + (12u^3 - 48u^2 + 40u - 16)v \\ &\quad + 18u^5 - 126u^4 - 288u^2 + 264u^3 + 96, \\ x_3 &= -x_1, \quad x_4 = -x_2, \end{aligned} \tag{32}$$

and

$$\begin{aligned} y_1 &= (2u-2)v^2 + (12u^3 - 48u^2 + 48u)v \\ &\quad + 18u^5 - 126u^4 - 144u^2 + 288u^3 + 96u - 96, \\ y_2 &= uv^2 + (6u^3 - 12u^2 - 32u + 32)v \\ &\quad + 9u^5 - 36u^4 + 240u^2 - 96u^3 - 144u, \\ y_3 &= 2v^2 + (24u^2 - 40u + 16)v \\ &\quad + 54u^4 - 264u^3 + 96u + 192u^2 - 96, \\ y_4 &= -y_3, \quad y_5 = -y_2, \quad y_6 = -y_3. \end{aligned} \tag{33}$$

With these values of x_i, y_i, it is readily seen that (26) is identically satisfied for $r = 1, 3, 5$. Further,

$$\sum_{i=1}^{4} x_i^2 - \sum_{i=1}^{6} y_i^2 = -8(9u^4 - 72u^3 + 24u^2 + 96u - 48 - v^2)^2, \tag{34}$$

$$\sum_{i=1}^{4} x_i^4 - \sum_{i=1}^{6} y_i^4 = -8(9u^4 - 72u^3 + 24u^2 + 96u - 48 - v^2)^4. \tag{35}$$

It follows that a solution of the diophantine system (26) will be given by (32) and (33) if we choose u, v such that

$$v^2 = 9u^4 - 72u^3 + 24u^2 + 96u - 48. \tag{36}$$

Now Equation (36) represents the quartic model of an elliptic curve, and the birational transformation given by

$$\begin{aligned} u &= (6X + 2Y - 12)/(3X - 24), \\ v &= (4X^3 - 96X^2 + 84X - 144Y + 832)/\{3(X-8)^2\}, \end{aligned} \tag{37}$$

and,

$$\begin{aligned} X &= (9u^2 - 36u + 3v + 4)/8, \\ Y &= (27u^3 - 162u^2 + 9uv + 36u - 18v + 72)/16, \end{aligned} \tag{38}$$

reduces Equation (36) to the Weierstrass form of the elliptic curve which is as follows:

$$Y^2 = X^3 - 21X - 20. \tag{39}$$

We again refer to Cremona's database of elliptic curves, and find that (39) represents an elliptic curve of rank 1 and its Mordell-Weil basis is given by the rational point P with co-ordinates $(X, Y) = (-3, 4)$. There are thus infinitely many rational points on the elliptic curve (25) and these can be obtained by the group law. Using the relations (37), we can find infinitely many rational solutions of Equation (36) and thus obtain infinitely many solutions of the diophantine system (26). While the point P leads to a trivial solution of the diophantine system (26), the point 2P yields the solution,

$$\begin{aligned} 241^r + 218^r + (-241)^r + (-218)^r = 266^r + 143^r + 120^r \\ + (-266)^r + (-143)^r + (-120)^r, \quad r = 1, 2, 3, 4, 5. \end{aligned}$$

The solutions of the diophantine system (26) obtained in this Section show that $\beta(5) \leq 10$, and on combining with the result $\beta(5) \geq 10$ which follows from Theorem 1, we get $\beta(5) = 10$.

4. Concluding Remarks

We have shown that $\beta(k) = 2k$ when $k = 2, 3, 4$ or 5. When $k \geq 6$, we have noted in the Introduction that the existence of ideal solutions of (1) for $k = 6, 7, 8, 9$ and 11 implies that $\beta(k) \leq 2k + 1$ for these values of k. Combining this with the result of Theorem 1, we get

$$2k \leq \beta(k) \leq 2k + 1 \quad \text{when } k = 6, 7, 8, 9 \text{ or } 11.$$

It would be of interest to determine the precise values of $\beta(k)$ for these values of k.

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