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AN ASYMPTOTIC ROBIN INEQUALITY

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Abstract

The conjectured Robin inequality for an integer $n > 7!$ is $\sigma(n) < e^{\gamma} n \log\log n$, where γ denotes the Euler constant, and $\sigma(n) = \sum_{d|n} d$. Robin proved that this conjecture is equivalent to the Riemann hypothesis (RH). Writing $D(n) = e^{\gamma} n \log\log n - \sigma(n)$, and $d(n) = \frac{D(n)}{n}$, we prove unconditionally that $\liminf_{n\to\infty} d(n) = 0$. The main ingredients of the proof are an estimate for the Chebyshev summatory function, and an effective version of Mertens' third theorem due to Rosser and Schoenfeld. A new criterion for RH depending solely on $\liminf_{n\to\infty} D(n)$ is derived.

1. Introduction

1.1. History

The conjectured Robin's inequality for an integer $n > 7! = 5040$ is $\sigma(n) < e^{\gamma} n \log\log n$, where $\gamma \approx 0.577\cdots$ denotes the Euler constant, and σ is the sum-of-divisors function $\sigma(n) = \sum_{d|n} d$. This inequality has been shown to hold unconditionally for $7! < n \leq N$ with $N \approx e^{e^{26}}$ [1], and for infinite families of integers that are

- odd and greater than 9 [4];
- square-free and greater than 30 [4];
- a sum of two squares and greater than 720 [2];
- not divisible by the fifth power of a prime [4];
- not divisible by the seventh power of a prime [11];
- not divisible by the eleventh power of a prime [3].

Ramanujan showed that the Riemann Hypothesis implies Robin's inequality for n large enough [8]. Robin proved the converse statement [9], thus making that conjecture a criterion for RH. This criterion was made popular by [6] which derives an alternate criterion involving harmonic numbers.

1.2. Contribution

Denote the difference between the right-hand side and the left-hand side of Robin's inequality by $D(n) = e^{\gamma} n \log\log n - \sigma(n)$. Let $d(n) = \frac{D(n)}{n}$. The main result of this note is

Theorem 1. *We have* $\liminf_{n\to\infty} d(n) = 0$.

The proof of Theorem 1 will depend on the following intermediate result.

Theorem 2. *For large* n, *the quantity* $\liminf_{n\to\infty} d(n)$ *is finite and nonnegative.*

The main ingredients of the proof of Theorem 2 are a combinatorial inequality between arithmetic functions (Lemma 1), an effective version of Mertens' third theorem due to Rosser and Schoenfeld (Lemma 2), and an asymptotic estimate of Chebyshev's first summatory function (Lemma 4). Also needed is a result of Ramanujan of 1915, first published in 1997 [8].

We also study the asymptotic behavior of $D(n)$. Recall that a number is *colossally abundant* (CA for short) if it is a left-to-right maximum for the function with domain the set of integers $x \mapsto \frac{\sigma(x)}{x^{1+\epsilon}}$, where ϵ is a real parameter. Thus n is CA if $m < n$ entails $\frac{\sigma(m)}{m^{1+\epsilon}} < \frac{\sigma(n)}{n^{1+\epsilon}}$.

Theorem 3. *We have the following limits when n ranges over the set of CA numbers:*

- *If RH is false then* $\liminf_{n\to\infty} D(n) = -\infty$
- *If RH is true then* $\lim_{n\to\infty} D(n) = \infty$.

This result constitutes a new criterion for RH. Its proof will depend, for the RH false part, on an oscillation theorem of Robin [9], modelled after and depending upon an oscillation theorem of Nicolas [7] for the Euler totient function. For the RH true case, we use a result of Ramanujan from 1915, first published in 1997 [8].

1.3. Organization

The material is arranged as follows. The next section contains the proof of Theorem 1, Section 3 that of Theorem 2, and Section 4 that of Theorem 3. Section 5 concludes and gives some open problems.

2. Proof of Theorem 1

The result will follow from Theorem 2 if we exhibit a sequence of integers n_m with $\lim_{m\to\infty} D(n_m) = 0$. We follow the approach of [4, §4, proof of Lemma 4.1, 1), p. 366]. Consider n of the shape $n = \prod_{p\le x} p^{t-1}$, with $t > 1$ integer and x real, both going to infinity, and to be specified later. By this reference, we have

$$d(n) = e^{\gamma} \log\log n \left(1 - \frac{1}{\zeta(t)} + o_t(1)\right),$$

with ζ denoting the Riemann zeta function. The error term can be made effective as follows. By [10, (3.28),(3.30)] we have

$$e^{\gamma} \log x(1 - \frac{1}{2\log^2 x}) \le \prod_{p\le x}(1 - \frac{1}{p})^{-1} \le e^{\gamma} \log x(1 + \frac{1}{\log^2 x}).$$

From the Euler product of ζ and [4, Lemma 6.4] we derive

$$\frac{1}{\zeta(t)} \le \prod_{p\le x}(1 - \frac{1}{p^t}) \le \frac{\exp(\frac{tx^{1-t}}{t-1})}{\zeta(t)}.$$

Combining these four bounds together we can take $o_t(1) = O\big(\exp(\frac{tx^{1-t}}{t-1}) - 1\big) = O(x^{1-t})$. Now it is elementary to show that for an integer $t > 1$ we have $\zeta(t) = 1 + h(t)$, with $h(t) = O(1/2^t)$. Indeed

$$\frac{1}{2^t} \le \zeta(t) - 1 \le \sum_{m=1}^{\infty} \frac{1}{2^{mt}} = \frac{2^{-t}}{1 - 2^{-t}}.$$

Thus, summarizing, we get

$$d(n) = e^{\gamma} \log\log n \left(O(1/2^t) + O(x^{1-t})\right).$$

To achieve $d(n) \to 0$, we need both $\log\log n \ll 2^t$, and $\log\log n \ll x^{t-1}$, where $\ll$ stands for $o()$ ("little-oh") notation. This is ensured if we take $x = p_m$, and $t = m + 1$. In that case we have $\log\log n = \log m + \log\theta(p_m)$. By Lemma 4 below, $\log\theta(p_m) \sim \log p_m$. On the other hand, $p_m \sim m\log m$ as is well-known (see e.g. [5]). Combining the last two estimates we see that $\log\log n \sim 2\log m << 2^m$. Similarly, $\log\log n << p_m^m$.

3. Proof of Theorem 2

If $\liminf_{n\to\infty} d(n) = \infty$ then $\lim_{n\to\infty} D(n) = \infty$, and, by Robin's criterion, RH holds. We know then by [8, p.25] that the sequence $d(n)\sqrt{\log n}$ admits finite upper and lower limits when n ranges over the set of CA numbers (see §4), which is a contradiction.

Assume therefore that $\liminf_{n\to\infty} d(n)$ is finite, and let us show that it is non-negative. For any integer n write its decomposition into prime powers as

$$n = \prod_{i=1}^{m} q_i^{a_i},$$

where the q_i's are prime numbers, indexed by increasing order, and a_i's are positive integers. Denote by p_i the i^{th} prime number, and for any integer n, let

$$\bar{n} = \prod_{i=1}^{m} p_i^{a_i}.$$

Note that, by definition, for each $i = 1, 2, \cdots, m$ we have $q_i \geq p_i$, and that, therefore, $n \geq \bar{n}$. With this notation observe that

$$\sigma(\bar{n}) = \prod_{i=1}^{m} \frac{p_i^{a_i+1} - 1}{p_i - 1} = \bar{n} \prod_{i=1}^{m} \frac{p_i - p_i^{-a_i}}{p_i - 1}.$$

In particular

$$d(\bar{n}) \leq \prod_{i=1}^{m} \frac{p_i}{p_i - 1} \leq 2^m,$$

and, likewise, $\frac{\sigma(n)}{n} \leq 2$. Thus, if m is bounded and $n \to \infty$, we see that $d(n) \to \infty$. We can thus assume when considering $\liminf_{n\to\infty} d(n)$ that $m \to \infty$.

We prepare for the proof of Theorem 2 by a series of Lemmas.

Lemma 1. *For any integer $n \geq 1$, we have $d(n) \geq d(\bar{n})$.*

Proof. Let $d(n) = f_1(n) - f_2(n)$, with $f_1(n) = e^\gamma \log\log n$, and $f_2(n) = \frac{\sigma(n)}{n}$. The monotonicity of the log and $n \geq \bar{n}$ yields $f_1(n) \geq f_1(\bar{n})$. Write $f_2(n) = \prod_{i=1}^{m} g(a_i, q_i)$, where $g(a, x) = \frac{x - x^{-a}}{x-1}$. Writing

$$g(a, x) = \frac{1 + x + \cdots + x^a}{x^a} = \sum_{i=0}^{a} \frac{1}{x^i},$$

we see that, for fixed a, the function $x \mapsto g(a, x)$ is nonincreasing in x. This implies that $g(a_i, q_i) \leq g(a_i, p_i)$ for each $i = 1, 2, \cdots, m$ and, therefore, multiplying m inequalities between nonnegative numbers, that $f_2(n) \leq f_2(\bar{n})$. The result follows then by $d(n) = f_1(n) - f_2(n)$. $\square$

Lemma 2. *For any n large enough we have* $\frac{\sigma(\bar{n})}{\bar{n}} < e^\gamma \log p_m (1 + \frac{1}{\log^2 p_m})$.

Proof. Note that, with the notation of the proof of Lemma 1, we have $g(a,x) \leq \frac{x}{x-1}$, for $x \geq 2$ and $a \geq 1$, and, therefore

$$f_2(n) = \prod_{i=1}^{m} g(a_i, q_i) \leq \prod_{i=1}^{m} \frac{p_i}{p_i - 1}.$$

The result follows then by [10, Th. 8, (39)]. □

Recall the Chebyshev summatory function $\vartheta(x) = \sum_{p \leq x} \log(p)$.

Lemma 3. *For all $n \geq 1$, we have* $\log \bar{n} \geq \vartheta(p_m)$.

Proof. By definition

$$\log \bar{n} = \sum_{i=1}^{m} a_i \log p_i \geq \sum_{i=1}^{m} \log p_i = \vartheta(p_m).$$

□

A classical result, related to the Prime Number Theorem, is

Lemma 4. *For large x, we have* $\vartheta(x) = x + O(\frac{x}{\log x})$.

Proof. An effective version is in [10, Th. 4]. See for instance [5, Th. 4.7] for a sharper error term in $O(x \exp(-\frac{\sqrt{\log x}}{15}))$. □

We are now ready for the proof of Theorem 1.

Proof. By Lemma 1 $d(n) \geq d(\bar{n})$. By Lemma 2 we have

$$-\frac{\sigma(\bar{n})}{\bar{n}} > -e^\gamma \log p_m (1 + \frac{1}{\log^2 p_m}). \tag{1}$$

By Lemma 3 and 4 we have

$$e^\gamma \log\log \bar{n} \geq e^\gamma \log \vartheta(p_m) = e^\gamma \log\left(p_m + O(\frac{p_m}{\log p_m})\right) = \tag{2}$$

$$e^\gamma \left(\log p_m + \log(1 + O(\frac{1}{\log p_m}))\right) = e^\gamma \left(\log(p_m) + O(\frac{1}{\log p_m})\right), \tag{3}$$

where the last equality results from $\log(1+u) \sim u$ for $u \to 0$. Adding up inequalities 1 and 3, after cancellation of the terms in $\log p_m$, we obtain the inequality

$$d(\bar{n}) = e^\gamma \log\log \bar{n} - \frac{\sigma(\bar{n})}{\bar{n}} \geq O(\frac{1}{\log p_m}) - \frac{e^\gamma}{\log p_m},$$

the right hand side of which goes to zero for large n. □

4. Proof of Theorem 3

Recall the standard notation for oscillation theorems [5, p. 194]. If f, g are two real valued functions of a real variable x, with $g > 0$, then we write

- $f(x) = \Omega_+(g(x))$, if $\limsup_{x\to\infty} f(x)/g(x) > 0$
- $f(x) = \Omega_-(g(x))$, if $\liminf_{x\to\infty} f(x)/g(x) < 0$
- $f(x) = \Omega_\pm(g(x))$, if both $f(x) = \Omega_+(g(x))$, and $f(x) = \Omega_-(g(x))$ hold

By [9, Proposition,§4] if RH is false then, for CA numbers we have

$$D(n) = \Omega_\pm(\frac{n \log\log n}{(\log n)^b}),$$

for some $b \in (0, 1)$. This would imply, using the infinitude of CA numbers [9], that $\liminf_{n\to\infty} D(n) = -\infty$.

If RH holds then by [8, p.25] the sequence $\frac{D(n)\sqrt{\log n}}{n}$ admits upper and lower limits for colossally abundant n that are finite and greater than 0. Thus there are reals greater than 0 say A, and B such that

$$A\frac{n}{\log n} \le D(n) \le B\frac{n}{\log n},$$

when n is CA. Therefore $\lim_{n\to\infty} D(n) = \infty$.

5. Conclusion and Open Problems

In this note we have studied the quantity $D(n)$ which is the difference between the two sides of Robin's inequality, and its normalization $d(n) = \frac{D(n)}{n}$. While the asymptotic behavior of $d(n)$ can be determined unconditionally (Theorem 1), that of $D(n)$ depends crucially on the truth of RH (Theorem 3). It would be desirable to extend Theorem 3 to integers that are not CA. It seems impossible to use Theorem 1 and Theorem 3 together to prove that RH holds. For instance, one cannot rule out the case that $D(n)$ behaves like $-\sqrt{n}$ when $n \to \infty$, which would not contradict the fact that $\liminf_{n\to\infty} d(n) = 0$.

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