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UPPER BOUNDS FOR $B_H[G]$-SETS WITH SMALL H

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Abstract

For $g \geq 2$ and $h \geq 3$, we give small improvements on the maximum size of a $B_h[g]$-set contained in the interval $\{1, 2, \ldots, N\}$. In particular, we show that a $B_3[g]$-set in $\{1, 2, \ldots, N\}$ has at most $(14.3gN)^{1/3}$ elements. The previously best known bound was $(16gN)^{1/3}$ proved by Cilleruelo, Ruzsa, and Trujillo. We also introduce a related optimization problem that may be of independent interest.

1. Introduction

Let $A \subseteq [N] := \{1, 2, \ldots, N\}$ and let h and g be positive integers. We say that A is a $B_h[g]$-*set* if for any integer n, there are at most g distinct multi-sets $\{a_1, a_2, \ldots, a_h\} \subseteq A$ such that

$$a_1 + a_2 + \cdots + a_h = n.$$

Determining the maximum size of a $B_h[g]$-set in $A \subseteq [N]$ is a well-studied problem in number theory. Initial bounds on $B_h[g]$-sets were obtained combinatorially. Indeed, if A is a $B_h[g]$-set, then consider the $\binom{|A|+h-1}{h}$ multi-sets of size h in A. The sum of the elements in each of the multi-sets represents each integer in $\{1, 2, \ldots, hN\}$ at most g times. Therefore,

$$\binom{|A| + h - 1}{h} \leq ghN \tag{1}$$

which implies $|A| \leq (h!ghN)^{1/h}$. The breakthrough papers of Cilleruelo, Ruzsa, Trujillo [3], Cilleruelo, Jiménez-Urroz [2], and Green [4] introduced methods from analysis and probability to obtain significant improvements on (1). Several of the results in these papers have yet to be improved upon. For more on $B_h[g]$-sets, we recommend the survey papers of O'Bryant [5] and Plagne [6]. We will be concerned

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with $B_h[g]$-sets where $g \geq 2$ and $h \geq 3$. For $3 \leq h \leq 6$ and $g \geq 2$, the best known upper bound on the size of a $B_h[g]$-set $A \subseteq [N]$ is

$$|A| \leq \left(\frac{h! hgN}{1 + \cos^h(\pi/h)} \right)^{1/h} \tag{2}$$

due to Cilleruelo, Ruzsa, and Trujillo [3]. For $h \geq 7$, the best known bound is

$$|A| \leq \left(\sqrt{3h} h! gN \right)^{1/h} \tag{3}$$

which was proved by Cilleruelo and Jiménez-Urroz [2] using an idea of Alon. For $g = 1$, the best bounds can be found in [4] and [1]. In the case that $h = 2$ and $g \geq 2$, Yu [7] was able to make some improvements to the results of Green [4]. In this note we improve (2) and make a small improvement upon (3).

Theorem 1. *(i) Let $g \geq 2$ and $h \geq 4$ be integers. If $A \subseteq [N]$ is a $B_h[g]$-set, then*

$$|A| \leq (1 + o_N(1)) \left(\frac{x_h h! hgN}{\pi} \right)^{1/h}$$

where x_h is the unique real number in $(0, \pi)$ that satisfies $\frac{\sin x_h}{x_h} = \left(\frac{4}{3 - \cos(\pi/h)} - 1 \right)^h$.
(ii) If $A \subseteq [N]$ is a $B_3[g]$-set, then for large enough N,

$$|A| \leq (14.295gN)^{1/3}.$$

Our improvements for small h are contained in the following table.

h	upper bound of [3], [2]	new upper bound
3	$(16gN)^{1/3}$	$(14.295gN)^{1/3}$
4	$(76.8gN)^{1/4}$	$(71.49gN)^{1/4}$
5	$(445.577gN)^{1/5}$	$(413.07gN)^{1/5}$
6	$(3054.7gN)^{1/6}$	$(2774.16gN)^{1/6}$
7	$(23096.19gN)^{1/7}$	$(21294.74gN)^{1/7}$

Table 1: Upper bounds on $B_h[g]$-sets in $\{1, 2, \ldots, N\}$ for sufficiently large N.

By looking at Table 1, it is clear that Theorem 1 improves (2) for $3 \leq h \leq 6$. The inequality

$$\frac{\sin(\pi\sqrt{3/h})}{\pi\sqrt{3/h}} < \left(\frac{4}{3 - \cos(\pi/h)} - 1 \right)^h$$

holds for all $h \geq 3$; a fact that can be verified using Taylor series. Since $\frac{\sin x}{x}$ is decreasing on $[0, \pi]$, we must have $x_h < \pi\sqrt{3/h}$ for all $h \geq 3$ which shows that Theorem 1 improves (3). The improvement, however, is $(1 - o_h(1))$ since $\frac{x_h \sqrt{h}}{\pi\sqrt{3}} \to 1$ as $h \to \infty$.

In the next section we prove Theorem 1. Our arguments rely heavily on [3] and [4]. In Section 3 we introduce an optimization problem that is motivated by our work in Section 2.

2. Proof of Theorem 1.1

First we show how to improve (2) using the arguments of [3] and [4]. Let $A \subseteq [N]$ be a $B_h[g]$-set where $h \geq 2$. Define $f(t) = \sum_{a \in A} e^{iat}$, $t_h = \frac{2\pi}{hN}$, and

$$r_h(n) = |\{(a_1, \dots, a_h) \in A^h : a_1 + \cdots + a_h = n\}|.$$

The first lemma is a variation of inequality (40) from [4].

Lemma 1 (Green [4]). *For any* $j \in \{1, 2, \dots, hN - 1\}$,

$$|f(t_h j)| \leq (1 + o_N(1))|A| \left(\frac{\sin(\pi Q_h)}{\pi Q_h} \right)^{1/h}$$

where $Q_h = \frac{|A|^h}{h!hgN}$.

Proof. Let $j \in \{1, 2, \dots, hN - 1\}$. Define $g : [hN] \to \{0, 1, \dots\}$ by $g(n) = h!g - r_h(n)$. Following [3], we observe that

$$f(t_h j)^h = \sum_{n=1}^{hN} r_h(n) e^{\frac{2\pi i n j}{hN}} = -\sum_{n=1}^{hN} (h!g - r_h(n)) e^{\frac{2\pi i n j}{hN}}. \tag{4}$$

Let $\hat{g}$ be the Fourier transform of g so $\hat{g}(j) = \sum_{n=1}^{hN} g(n) e^{\frac{2\pi i n j}{hN}}$ for $j \in [hN]$. From (4) and the definition of g,

$$|f(t_h j)|^h = |\hat{g}(j)|. \tag{5}$$

Since A is a $B_h[g]$-set, the inequality $0 \leq g(n) \leq h!g$ holds for all n. Furthermore, $\sum_{n=1}^{hN} g(n) = h!ghN - |A|^h$. Lemma 26 of [4] gives

$$|\hat{g}(j)| \leq h!g \left| \frac{\sin(\frac{\pi}{hN}(\frac{h!hgN - |A|^h}{h!g} + 1))}{\sin(\frac{\pi}{hN})} \right| = h!g \left| \frac{\sin(\pi Q_h - \frac{\pi}{hN})}{\sin(\frac{\pi}{hN})} \right|. \tag{6}$$

By (2), the value Q_h satisfies $0 \leq Q_h \leq 1$ for all N. Therefore,

$$|\hat{g}(j)| \leq h!g(1 + o_N(1)) \frac{\sin(\pi Q_h)}{\pi / hN} = (1 + o_N(1))|A|^h \frac{\sin(\pi Q_h)}{\pi Q_h}.$$

Combining this inequality with (5), we get

$$|f(t_h j)| \leq (1 + o_N(1))|A| \left(\frac{\sin(\pi Q_h)}{\pi Q_h} \right)^{1/h}$$

which completes the proof of the lemma. □

Again following [3], we need to choose a function $F(x) = \sum_{j=1}^{hN} b_j \cos(jx)$ such that

$$\sum_{a\in A} F\left(\left(a - \frac{N+1}{2}\right) t_h\right)$$

is large and $\sum_{j=1}^{hN} |b_j|$ is small. For $h \geq 3$, the function $F(x) = \frac{1}{\cos(\pi/h)} \cos x$ gives

$$\sum_{a\in A} F\left(\left(a - \frac{N+1}{2}\right) t_h\right) \geq |A|$$

and $\sum_{j=1}^{hN} |b_j| = \frac{1}{\cos(\pi/h)}$. This is the function that is used in [3]. We will choose a different function G that does better than F and still has a simple form. Let

$$G(x) = \left(\frac{2}{3-\cos(\pi/h)}\right) \frac{1}{\cos(\pi/h)} \cos(x) - \left(1 - \frac{2}{3-\cos(\pi/h)}\right) \frac{1}{\cos(\pi/h)} \cos(hx). \tag{7}$$

The minimum value of $G(x)$ on the interval $[-\frac{\pi}{h}, \frac{\pi}{h}]$ is $\frac{1}{\cos(\pi/h)} \left(\frac{4}{3-\cos(\pi/h)} - 1\right)$ and so

$$\sum_{a\in A} G\left(\left(a - \frac{N+1}{2}\right) t_h\right) \geq \frac{1}{\cos(\pi/h)} \left(\frac{4}{3-\cos(\pi/h)} - 1\right) |A|. \tag{8}$$

Here we are using the fact that $|(a-(N+1)/2)t_h| < \frac{\pi}{h}$ for any $a \in A$. If the constants c_j are defined by $c_{hN} = 0$ and $G(x) = \sum_{j=1}^{hN} c_j \cos(jx)$, then $\sum_{j=1}^{hN} |c_j| = \frac{1}{\cos(\pi/h)}$. Using (8), we have

$$\begin{aligned}
\frac{1}{\cos(\pi/h)} \left(\frac{4}{3-\cos(\pi/h)} - 1\right) |A| &\leq \sum_{a\in A} G\left(\left(a - \frac{N+1}{2}\right) t_h\right) \\
&= \mathrm{Re}\left(\sum_{j=1}^{hN} c_j \sum_{a\in A} e^{(a-(N+1)/2)\frac{2\pi i j}{hN}}\right) \\
&\leq \sum_{j=1}^{hN} |c_j||f(t_h j)| \\
&\leq \frac{1}{\cos(\pi/h)} (1 + o_N(1))|A| \left(\frac{\sin(\pi Q_h)}{\pi Q_h}\right)^{1/h}
\end{aligned}$$

where in the last line we have used Lemma 1 and $\sum_{j=1}^{hN} |c_j| = \frac{1}{\cos(\pi/h)}$. Some rearranging gives

$$\left(\frac{4}{3-\cos(\pi/h)} - 1\right)^h \leq (1 + o_N(1)) \frac{\sin(\pi Q_h)}{\pi Q_h}. \tag{9}$$

We remark that $\frac{4}{3-\cos(\pi/h)} - 1 > \cos(\pi/h)$ is equivalent to $(1-\cos(\pi/h))^2 > 0$. The point of this is that using G defined by (7) instead of $F(x) = \frac{1}{\cos(\pi/h)} \cos x$ (which would give the value 1 on the left hand side of (9)) does lead to a better upper bound.

Recalling that $0 \le Q_h \le 1$, lower bounds on $\frac{\sin(\pi Q_h)}{\pi Q_h}$ translate to upper bounds on πQ_h. Let x_h be the unique real number in the interval $(0, \pi)$ that satisfies

$$\left(\frac{4}{3-\cos(\pi/h)} - 1\right)^h = \frac{\sin(x_h)}{x_h}.$$

Then by (9), $\pi Q_h \le (1 + o_N(1))x_h$ since the function $\frac{\sin x}{x}$ is decreasing on $[0, \pi]$. We can rewrite $\pi Q_h \le (1 + o_N(1))x_h$ as

$$|A| \le (1 + o_N(1)) \left(\frac{x_h h! h g N}{\pi}\right)^{1/h}. \tag{10}$$

The upper bounds obtained from (10) for $h \in \{4, 5, 6, 7\}$ are given in Table 1. We have chosen to round the values so that all of the bounds in Table 1 hold for sufficiently large N. In particular, (10) implies that a $B_3[g]$-set $A \subseteq [N]$ has at most $(14.65gN)^{1/3}$ elements. We can improve this bound by considering the distribution of A in the interval $[N]$.

Assume now that A is a $B_3[g]$-set. Let δ be a real number with $0 < \delta < \frac{1}{4}$ and set $l = \lfloor \frac{1}{2\delta} \rfloor$. For $1 \le k \le l$, let

$$C_k = (A \cap ((k-1)\delta N, k\delta N]) \cup (A \cap [(1-k\delta)N, (1-(k-1)\delta)N)).$$

The definition of l ensures that the sets $C_1, \ldots, C_l$ together with $A \cap (l\delta N, (1-l\delta)N)$ form a partition of A. Using the same counting argument that is used to obtain (1), we show that if some C_k contains a large proportion of A, then $|A| \le (14.295gN)^{1/3}$. To this end, define real numbers $\alpha_1(\delta), \ldots, \alpha_l(\delta)$ by

$$\alpha_k(\delta)|A| = |C_k| \tag{11}$$

for $1 \le k \le l$. The value $\alpha_k(\delta)$ represents the proportion of A that is contained in the union $((k-1)\delta N, k\delta N] \cup [(1-k\delta)N, (1-(k-1)\delta)N)$.

Lemma 2. *If $0 < \delta < \frac{1}{4}$, $l = \lfloor \frac{1}{2\delta} \rfloor$, and $\alpha_1(\delta), \ldots, \alpha_l(\delta)$ are defined by (11), then for any $N > \frac{2}{\delta}$ and $1 \le k \le l$,*

$$|A| \le \left(\frac{72g\delta N}{\alpha_k(\delta)^3}\right)^{1/3}.$$

Proof. Let $1 \le k \le l$ and consider C_k. Since C_k is a $B_3[g]$-set,

$$\binom{|C_k| + 3 - 1}{3} \le g|C_k + C_k + C_k| \tag{12}$$

where $C_k + C_k + C_k = \{a + b + c : a, b, c \in C_k\}$. The set $C_k + C_k + C_k$ is contained in the union of the intervals

$$[3(k-1)\delta N, 3k\delta N],\ [(1+(k-2)\delta)N, (1+(k+1)\delta)N],$$
$$[(2-(k+1)\delta)N, (2-(k-2)\delta)N], \text{ and } [(3-3k\delta)N, (3-3(k-1)\delta)N].$$

Each of these four intervals has length $3\delta N$ so $|C_k+C_k+C_k| \leq 12\delta N$. Combining this inequality with (12) we have $\binom{|C_k|+2}{3} \leq 12g\delta N$ which implies $\alpha_k(\delta)|A| = |C_k| \leq (3!12g\delta N)^{1/3}$. □

Now we consider two cases.

`Case 1:` For some $0<\delta<\frac{1}{4}$ and $1 \leq k \leq l = \lfloor \frac{1}{2\delta} \rfloor$, we have

$$\left(\frac{72\delta}{14.295}\right)^{1/3} < \alpha_k(\delta).$$

In this case, we apply Lemma 2 to get $|A| \leq (14.295gN)^{1/3}$ and we are done.

`Case 2:` For all $0<\delta<\frac{1}{4}$ and $1 \leq k \leq l = \lfloor \frac{1}{2\delta} \rfloor$, we have

$$\alpha_k(\delta) \leq \left(\frac{72\delta}{14.295}\right)^{1/3}. \tag{13}$$

Let $H(x) = 1.6\cos x - 0.3\cos 3x + 0.1\cos 6x$. Partition the interval $[-\pi/3, \pi/3]$ into 128 subintervals $I_1, \ldots, I_{128}$ of equal width so

$$I_j = \left[-\frac{\pi}{3} + \frac{2\pi(j-1)}{3\cdot 128}, -\frac{\pi}{3} + \frac{2\pi j}{3\cdot 128}\right]$$

for $1 \leq j \leq 128$. Let $v_j = \min_{x\in I_j} H(x)$ for $1 \leq j \leq 128$. Since H is an even function, $v_j = v_{128-j+1}$ for $1 \leq j \leq 64$. The values v_j can be approximated numerically. They satisfy

$$v_1 < v_2 < v_3 < v_4 < v_5 < v_{35} \leq v_j \tag{14}$$

for all $6 \leq j \leq 64$. The sum

$$\sum_{a\in A} H\left(\left(a - \frac{N+1}{2}\right)t_3\right) \tag{15}$$

is minimized when $J = \left\{\left(a - \frac{N+1}{2}\right)t_3 : a \in A\right\}$ contains as many elements as possible in $I_1 \cup I_2 \cup \cdots \cup I_5$ and the remaining elements of J are contained in I_{35}. This follows from (14). Furthermore, in order to minimize (15), J must intersect I_1 in as many elements as possible, and the remaining elements in J intersect I_2 in as many elements as possible, and so on. By (13) with $\delta = 1/128$,

$$\alpha_k(1/128) \leq \left(\frac{72(1/128)}{14.295}\right)^{1/3}.$$

Thus,

$$|J \cap I_1| \leq \left(\frac{72(1/128)}{14.295}\right)^{1/3} |A|.$$

Similarly, by (13) with $\delta = j/128$ for $j \in \{2,3,4,5\}$,

$$\alpha_k(j/128) \leq \left(\frac{72(j/128)}{14.295}\right)^{1/3}.$$

We conclude that

$$|J \cap (I_1 \cup I_2 \cup \cdots \cup I_j)| \leq \left(\frac{72(j/128)}{14.295}\right)^{1/3} |A|$$

for $1 \leq j \leq 5$. From this inequality and (14), we deduce that

$$\begin{aligned} \sum_{a \in A} H\left(\left(a - \frac{N+1}{2}\right) t_3\right) \quad &\geq \quad \sum_{j=1}^{5} v_j \left(\left(\frac{72(\frac{j}{128})}{14.295}\right)^{1/3} - \left(\frac{72(\frac{j-1}{128})}{14.295}\right)^{1/3}\right) |A| \\ &+ \quad v_{35} \left(1 - \left(\frac{72(5/128)}{14.295}\right)^{1/3}\right) |A| > 1.2455|A|. \end{aligned}$$

Using 1.2455 in the derivation of (9) instead of $\frac{1}{\cos(\pi/3)}\left(\frac{4}{3-\cos(\pi/3)} - 1\right)$ gives

$$1.2455|A| \leq \frac{1}{\cos(\pi/3)}(1 + o_N(1))|A| \left(\frac{\sin(\pi Q_3)}{\pi Q_3}\right)^{1/3}.$$

This inequality can be rewritten as

$$\left(\frac{1.2455}{2}\right)^3 \leq (1 + o_N(1)) \left(\frac{\sin(\pi Q_3)}{\pi Q_3}\right).$$

Recalling that $Q_3 = \frac{|A|^3}{3!3gN}$, this inequality leads to the bound $|A| < (14.295gN)^{1/3}$ for large enough N.

3. An Optimization Problem

In this section we introduce an optimization problem that is motivated by (8) from the previous section.

Given integers K and $h \geq 2$, define

$$\mathcal{F}_{K,h} = \left\{ \sum_{j=1}^{K} b_j \cos(jx) : \sum_{j=1}^{K} |b_j| = \frac{1}{\cos(\pi/h)} \right\}.$$

For $A \subseteq [N]$ and $F \in \mathcal{F}_{K,h}$, define

$$w_F(A) = \sum_{a \in A} F\left(\left(a - \frac{N+1}{2}\right)\frac{2\pi}{hN}\right)$$

and

$$\psi(N,K,h) = \min_{A \subseteq [N], A \neq \emptyset} \sup\left\{\frac{w_F(A)}{|A|} : F \in \mathcal{F}_{K,h}\right\}.$$

Our interest in $\psi(N,K,h)$ is due to the following proposition.

Proposition 1. *If $A \subseteq [N]$ is a $B_h[g]$-set and $K \leq hN$, then*

$$|A| \leq (1 + o_N(1))\left(\frac{y_h h! hgN}{\pi}\right)^{1/h}$$

where y_h is the unique real number in $[0,\pi]$ with $\frac{\sin y_h}{y_h} = (\cos(\pi/h)\psi(N,K,h))^h$.

The function G defined by (7) shows that

$$\psi(N,h,h) \geq \frac{1}{\cos(\pi/h)}\left(\frac{4}{3-\cos(\pi/h)} - 1\right).$$

When $h = 3$, this gives $\psi(N,3,3) \geq 1.2$ which implies $\psi(N,6,3) \geq 1.2$. This is because the collection of functions $\mathcal{F}_{3,3}$ is a subset of $\mathcal{F}_{6,3}$. By considering more than one function, we can improve the bound $\psi(N,6,3) \geq 1.2$. The method by which we achieve this can be stated just as easily for general K and h so we do so.

To estimate $\psi(N,K,h)$, we will consider finite subsets of $\mathcal{F}_{K,h}$. Given a subset $\mathcal{F}'_{K,h} \subseteq \mathcal{F}_{k,h}$, we obviously have

$$\sup\left\{\frac{w_F(A)}{|A|} : F \in \mathcal{F}'_{K,h}\right\} \leq \sup\left\{\frac{w_F(A)}{|A|} : F \in \mathcal{F}_{K,h}\right\} \tag{16}$$

for every $A \subseteq [N]$ with $A \neq \emptyset$. When $\mathcal{F}'_{K,h}$ is finite, then the supremum on the left hand side of (16) can be replaced with the minimum. Let m be a positive integer and partition the interval $[-\pi/h, \pi/h]$ into m subintervals $I_1^m, \dots, I_m^m$ where

$$I_j^m = \left[-\frac{\pi}{h} + \frac{2\pi(j-1)}{hm}, -\frac{\pi}{h} + \frac{2\pi j}{hm}\right]$$

for $1 \leq j \leq m$. Any $F \in \mathcal{F}_{K,h}$ is continuous and thus obtains its minimum value on I_j^m. Given $F \in \mathcal{F}_{K,h}$, define

$$v_{m,j}(F) = \min_{x \in I_j^m} F(x).$$

Given $A \subseteq [N]$, define

$$\alpha_{m,j}(A) = \frac{1}{|A|}\left|\left\{(a - \frac{N+1}{2})\frac{2\pi}{hN} : a \in A\right\} \cap I_j^m\right|.$$

With this notation, we have that for any $A \subseteq [N]$ and $F \in \mathcal{F}_{K,h}$,

$$w_F(A) \geq \sum_{j=1}^{m} \alpha_{m,j}(A)|A|v_{m,j}(F).$$

Therefore, given a finite set $\{F_1, \ldots, F_n\} \subseteq \mathcal{F}_{K,h}$,

$$\psi(N,K,h) \geq \min_{A\subseteq[N], A\neq\emptyset} \max \left\{ \sum_{j=1}^{m} \alpha_{m,j}(A)v_{m,j}(F_k) : 1 \leq k \leq n \right\}.$$

We now put the above discussion to use by proving the following result.

Theorem 2. *For sufficiently large N, the function $\psi(N,6,3)$ satisfies the estimate*

$$\psi(N,6,3) \geq 1.2228.$$

Proof. Let

$F_1(x) = 1.7\cos x - 0.3\cos 3x$, $F_2(x) = 1.6\cos x - 0.3\cos 3x + 0.1\cos 6x$,

$F_3(x) = 1.5\cos x - 0.4\cos 3x + 0.1\cos 6x$, $F_4(x) = 1.2\cos x - 0.6\cos 3x + 0.2\cos 6x$,

$F_5(x) = -2\cos 3x$,

and $\mathcal{F} = \{F_1, F_2, F_3, F_4, F_5\}$. Observe that $\mathcal{F} \subseteq \mathcal{F}_{6,3}$. We take $m = 12$ and we must compute the numbers $v_{12,j}(F_k)$ for $1 \leq j \leq 12$ and $1 \leq k \leq 5$. Since each F_k is an even function, $v_{12,j}(F_k) = v_{12,12-j+1}(F_k)$ for $1 \leq j \leq 6$. To prove Theorem 2, we will only need to estimate these values from below.

Let $A \subseteq [N]$ with $A \neq \emptyset$. We assume that no element of the form $(a - \frac{N+1}{2})\frac{2\pi}{3N}$ is contained in two of the intervals $I_1^{12}, \ldots, I_{12}^{12}$. For large A, this will not affect $|A|$, at least in an asymptotic sense. Under this assumption, the non-negative real numbers $\alpha_{12,1}(A), \ldots, \alpha_{12,12}(A)$ satisfy

$$\alpha_{12,1}(A) + \cdots + \alpha_{12,12}(A) = 1.$$

We will consider several cases which depend on the distribution of A. For notational convenience, we write α_j for $\alpha_{12,j}(A)$.

`Case 1:` $\alpha_1 + \alpha_{12} \leq 0.6$.

Here we will use the function $F_1(x)$. Lower estimates on the $v_{12,j}(F_1)$ are

$$v_{12,1}(F_1) \geq 1.15, \;\; v_{12,2}(F_1) \geq 1.3525, \;\; v_{12,3}(F_1) \geq 1.4522,$$

$$v_{12,4}(F_1) \geq 1.4474, \;\; v_{12,5}(F_1) \geq 1.4143, \;\; \text{and} \;\; v_{12,6}(F_1) \geq 1.4.$$

In fact, these values satisfy

$$v_{12,1}(F_1) \le v_{12,2}(F_1) \le v_{12,6}(F_1) \le v_{12,5}(F_1) \le v_{12,4}(F_1) \le v_{12,3}(F_1).$$

Since $\alpha_1 + \alpha_{12} \le 0.6$, we must have

$$w_{F_1}(A) \ge (0.6v_{12,1}(F_1) + 0.4v_{12,2}(F_1))|A| \ge (0.6(1.15) + 0.4(1.3525))|A| > 1.23|A|.$$

`Case 2:` $0.6 \le \alpha_1 + \alpha_{12} \le 0.7$.

Here we use the function $F_2(x)$. A close look at Case 1 shows that if $v_{12,1}(F_2)$ is one of the two smallest values in the set $\{v_{12,j}(F_2) : 1 \le j \le 6\}$, then essentially the same estimate applies. The two smallest values are $v_{12,1}(F_2) \ge 1.2$ and $v_{12,4}(F_2) \ge 1.2834$. Since $0.6 \le \alpha_1 + \alpha_{12} \le 0.7$,

$$w_{F_2}(A) \ge (0.7(1.2) + 0.3(1.2834))|A| > 1.225|A|.$$

`Case 3:` $0.7 \le \alpha_1 + \alpha_{12} \le 0.8$.

Here we use the function $F_3(x)$. In this range of $\alpha_1 + \alpha_{12}$, our estimate behaves a bit differently. Lower estimates on the $v_{12,j}(F_3)$ are

$$v_{12,1}(F_3) \ge 1.25, \quad v_{12,2}(F_3) \ge 1.299, \quad v_{12,3}(F_3) \ge 1.199,$$

$$v_{12,4}(F_3) \ge 1.1595, \;\; v_{12,5}(F_3) \ge 1.1595, \;\; \text{and} \;\; v_{12,6}(F_3) \ge 1.18.$$

In this case, $w_{F_3}(A)$ will be minimized when $\alpha_1 + \alpha_{12}$ is as small as possible. In the previous two cases, $w_{F_i}(A)$ was minimized when $\alpha_1 + \alpha_{12}$ was as large as possible. We conclude that

$$w_{F_3}(A) \ge (0.7(1.25) + 0.3(1.1595))|A| > 1.2228|A|.$$

`Case 4:` $0.8 \le \alpha_1 + \alpha_{12} \le 0.9$.

In this case we use the function $F_4(x)$. Lower estimates on the $v_{12,j}(F_4)$ are

$$v_{12,1}(F_4) \ge 1.3909, \quad v_{12,2}(F_4) \ge 1.1192, \quad v_{12,3}(F_4) \ge 0.8392,$$

$$v_{12,4}(F_4) \ge 0.7276, \;\; v_{12,5}(F_4) \ge 0.7264, \;\; \text{and} \;\; v_{12,6}(F_4) \ge 0.7621.$$

We have

$$w_{F_4}(A) \ge (0.8(1.3909) + 0.2(0.7264))|A| > 1.25|A|.$$

`Case 5:` $0.9 \le \alpha_1 + \alpha_{12} \le 1$.

Lower estimates on the $v_{12,j}(F_5)$ are

$$v_{12,1}(F_5) \ge 1.73, \quad v_{12,2}(F_5) \ge 1, \quad v_{12,3}(F_5) \ge -.01,$$

$$v_{12,4}(F_5) \ge -1, \;\; v_{12,5}(F_5) \ge -1.8, \;\; \text{and} \;\; v_{12,6}(F_5) \ge -2.$$

As in Cases 3 and 4, $w_{F_5}(A)$ is minimized when $\alpha_1 + \alpha_{12}$ is as small as possible. Hence,

$$w_{F_5}(A) \geq (0.9(1.73) + 0.1(-2))|A| > 1.35|A|.$$

In all five cases, we can find a function $F_i \in \mathcal{F}$ such that $w_{F_i}(A) > 1.2228|A|$. This completes the proof of Theorem 2. $\square$

4. Concluding Remarks

Although it is an improvement of $\psi(N, 6, 3) \geq 1.2$, Theorem 2 is not enough to prove part (ii) of Theorem 1. The improvement on $B_3[g]$-sets uses the $B_3[g]$ property to increase the 1.2 to 1.2455 which exceeds the 1.2228 provided by Theorem 2. Similar arguments can be done for $B_h[g]$-sets with $h > 3$, but the improvements in the results of Table 1 are minimal. Aside from $B_3[g]$-sets, the bounds in Table 1 come from lower bounds on $\psi(N, h, h)$ together with Lemma 1.

The function $\psi(N, K, h)$ is relevant to an inequality of Cilleruelo. Let A be a finite set of positive integers. For an integer $h \geq 2$, let

$$r_h(n) = |\{(a_1, \ldots, a_h) \in A^h : a_1 + \cdots + a_h = n\}| \text{ and } R_h(m) = \sum_{n=1}^{m} r_h(m).$$

Generalizing the argument of [3], Cilleruelo proved the following result.

Theorem 3 (Cilleruelo [1]). *Let $A \subseteq [N]$, $h \geq 2$ be an integer, and μ be any real number. For any positive integer $H = o(N)$,*

$$\sum_{n=h}^{hN+H} |R_h(n) - R_h(n-H) - \mu| \geq (L_h + o(1))H|A|^h$$

where $L_2 = \frac{4}{(\pi+2)^2}$ and $L_h = \cos^h(\pi/h)$ for $h > 2$.

By slightly modifying the argument in [1] that is used to prove Theorem 3, it is easy to prove the next proposition.

Proposition 2. *Let $A \subseteq [N]$, $h \geq 2$ be an integer, and μ be a real number. For any positive integers $H = o(N)$ and $K \leq \frac{N}{H}$,*

$$\sum_{n=h}^{hN+H} |R_h(n) - R_h(n-H) - \mu| \geq (\psi(N, K, h)^h L_h + o(1))H|A|^h$$

where $L_2 = \frac{4}{(\pi+2)^2}$ and $L_h = \cos^h(\pi/h)$ for $h > 2$.

For instance, Theorem 2 gives

$$\sum_{n=3}^{3N+H} |R_3(n) - R_3(n-H) - \mu| \geq (1.2228^3 L_3 + o(1))H|A|^3.$$

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