POLYNOMIAL TIME GRAPH FAMILIES FOR ARC KAYLES

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Abstract

Arc Kayles is a combinatorial game played on a graph. We give nim-sequences for Arc Kayles on varying classes of graphs including: equimatchable graphs, path graphs, cycle graphs, and wheel graphs. Lastly, we provide an automatic periodicity check for generalized star graphs with three rays (two fixed) and conjecture about the period for generalized star graphs with one ray fixed to two vertices, a second ray fixed to n vertices (in an equivalence class modulo 34) and the third ray varying to infinity.

1. Introduction

Graph theoretic combinatorial games became of interest within the last several decades as a means to model complex networks. One well studied graph game is called Node Kayles. Node Kayles is a combinatorial game on a graph. Players take turns choosing a single vertex such that it does not repeat, and is not adjacent to, any previously chosen vertices. The last player to move wins. Together, players are forming a maximal independent set. Node Kayles is well studied in [4, 6, 8, 14], including analysis on varying graph classes as well as complexity results.

A natural extension is to consider the edge counterpart of this game which is called Arc Kayles.

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Definition 1. ARC KAYLES is a combinatorial game on a graph. Players take turns choosing a single edge such that it does not repeat, and is not adjacent to, any previously chosen edges. The last player to move wins.

Choosing an edge is the same as deleting that edge and its incident vertices since adjacent edges cannot be chosen during game play. Throughout this paper, we consider edge deletion to make the remaining options for the next player clear.

ARC KAYLES was first introduced by Schaefer in 1978 [14] and to date very little is known about the game (see [12] for some discussion about complexity). Here, we analyze ARC KAYLES on the wheel graph and generalized star graphs, determining their Grundy values and nim-sequences. For the generalized star graphs we present an automatic periodicity check, motivated by octal games described in [1].

Definition 2. A *maximal matching* is a set of edges, M, from a graph, Γ, in which the following two properties hold:

1. no two edges share a vertex; and
2. no additional edges from Γ can be added to the set without violating the first condition.

The game of ARC KAYLES ends when there are no more moves. Hence the proof of Lemma 1 is immediate.

Lemma 1. *The end state of a game of* ARC KAYLES *is a maximal matching.*

The paper proceeds as follows. In Section 2 we begin by giving necessary background. In Section 3 we examine ARC KAYLES on equimatchable graphs, path graphs, and cycle graphs; determining outcome classes and nim-sequences. In Section 4, we take a non-standard approach to solving ARC KAYLES played on the wheel graph. Next, in Section 5 we examine a generalized star as motivated by [6] and present an automatic periodicity check motivated by [1]. We conclude by discussing future directions for research.

2. Background

A *path* is a connected, acyclic graph in which all vertices have degree at most two. We denote a path on n vertices by P_n. A *cycle* is a connected graph where every vertex has degree 2. We denote a cycle on n vertices by C_n.

ARC KAYLES is an *impartial* game since both players have the same options from all positions. Impartial games have two outcome classes: the game is a $\mathcal{P}$-*position* if the second (previous) player can force a win; otherwise, it is an $\mathcal{N}$-*position*, which means the next (first) player can force a win.

The *minimum excluded value* (mex) of a set of non-negative integers is the least non-negative integer which does not occur in the set. For example, $\text{mex}\{1, 3, 4\} = 0$. An *option* H of a game G is a subgame of G which can be reached in exactly one move. The *value* (or nim-value or Grundy value) of an impartial game, denoted by $\mathcal{G}$, is determined by the mex of the values of its options. If the next player does not have a move, the game position has value 0. There is a natural connection between the impartial game outcome classes and their $\mathcal{G}$-values: if the $\mathcal{G}$-value for a game G is 0, then G is a previous player win (or a $\mathcal{P}$-position); otherwise, it is an $\mathcal{N}$-position.

Sometimes, making a move in a game splits the game into disjoint boards. Such a game is the disjunctive sum of two (or more) games. Consider two games G and H, their disjunctive sum is denoted by $G + H$. The value of a disjunctive sum of impartial games is the sum of the values of each component, in binary, without carrying. This is denoted by $\oplus$ (XOR in computer science). For example, suppose we were playing the game $G + H$, where G and H are games. Then $\mathcal{G}(G + H) = \mathcal{G}(G) \oplus \mathcal{G}(H)$.

Fixing all parameters of a game G and letting one vary, we calculate $\mathcal{G}(G(n))$, for all $n \geq 0$. Arranging these $\mathcal{G}$-values in a sequence for a particular game G is called the *nim-sequence* of G. Throughout this paper we write $\mathcal{G}(\Gamma)$ or $\mathcal{G}(\Gamma_n)$ to represent the value of the game of Arc Kayles played on a specific graph Γ, and if there is a parameter varying, this is specified by n.

3. Well-behaved Graph Classes

A graph, Γ, is *equimatchable* if every maximal matching, M, is maximum. Hence, all maximal matchings for an equimatchable graph have the same size. The winner of a game of Arc Kayles on such a graph is therefore determined by the parity of a maximal matching in Γ.

Theorem 1. *The value of a game of* Arc Kayles *played on an equimatchable graph* Γ*, where* M *is the maximal matching at the end state of the game, is given by*

$$\mathcal{G}(\Gamma) = \begin{cases} 1 & \text{if } |\text{M}| \equiv 1 \pmod 2 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Follows directly from Lemma 1 and Definition 2. □

If a graph has the property that all maximal matchings have the same parity then its Grundy value is simple to calculate. The following proposition shows that these graphs are all, in fact, equimatchable. This contrasts with the case of independent sets. Graphs where all maximal independent sets have the same size are called *well-covered* [10]. Graphs where all maximal independent sets have the same parity are called *parity graphs* and are not equivalent to well-covered graphs [2, 5]. We outline the proof of our proposition, the details are left to the reader.

Proposition 1. *Let Γ be a graph such that all maximal matchings have the same parity. Then Γ is equimatchable.*

Sketch of proof. Let Γ be a graph and Γ has the property that all maximal matchings have the same parity. Suppose M_1 and M_2 are maximal matchings and without loss of generality $|M_1| < |M_2|$. Consider the symmetric difference of the matchings, $M_1 \triangle M_2 = \{e \in E(\Gamma) : e \text{ is in only one of the matchings}\}$. All connected components of $M_1 \triangle M_2$ are either vertices, paths or cycles and edges in the paths and cycles alternate between the two matchings. An augmenting path is a path component in $M_1 \triangle M_2$. Since $|M_1| < |M_2|$, an augmenting path, P, must exist, which begins and ends with edges from M_2. Use this to augment matching M_1 to produce $M_1' = M_1 \triangle P$. Since M_1 and M_2 were maximal, M_1' is maximal too but has parity opposite to M_1 and M_2. The result follows. □

Complete graphs, K_n, and complete bipartite graphs $K_{m,n}$ are both equimatchable classes of graphs. For characterizations of equimatchable graphs, see [7, 11, 13, 15]. For other games which utilize equimatchable graph theory, see [9].

Arc Kayles played on path graphs results in several different options: removing end edges results in a path on $n-2$ vertices, while removing intermediate edges results in a disjunctive sum of paths.

A *line graph* of a graph Γ is the graph produced from switching all edges of Γ to vertices and these vertices are connected by an edge if the corresponding edges were adjacent in Γ. Choosing an edge in Γ corresponds exactly to choosing a vertex in the line graph of Γ.

Arc Kayles on paths has the same nim-sequence as that of Berlekamp et al. [3] for Node Kayles on paths. Indeed, while playing Node Kayles players are deleting a vertex and its neighbours, while in Arc Kayles players are deleting an edge and its incident vertices. This is the same as playing Node Kayles on the line graph. This proves the next theorem.

Theorem 2. Arc Kayles *on* Γ *is* Node Kayles *on the line graph of* Γ*.*

Corollary 1. Arc Kayles *played on paths, P_n, has pre-period length* 53*, period length* 34*.*

The nim-sequence for Arc Kayles on paths is shown in Table 1.

Up to isomorphism, there is one move from C_n; to P_{n-2}. The value for C_n is 1 if $\mathcal{G}(\mathrm{P_{n-2}}) = 0$ and 0 otherwise. The positions with value 1 of the nim-sequence for C_n within the pre-period are $n \in \{3, 7, 11, 17, 23, 27, 31, 37\}$ with pre-period length 38. The period length is 34. Positions with value 1 within the periodic portion of the nim-sequence are

	t→	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
s ↓																		
0		0	0	1	1	2	0	3	1	1	0	3	3	2	2	4	0	5
17		2	2	3	3	0	1	1	3	0	2	1	1	0	4	5	2	7
34		4	0	1	1	2	0	3	1	1	0	3	3	2	2	4	4	5
51		5	2	**3**	**3**	**0**	**1**	**1**	**3**	**0**	**2**	**1**	**1**	**0**	**4**	**5**	**3**	**7**
68		**4**	**8**	**1**	**1**	**2**	**0**	**3**	**1**	**1**	**0**	**3**	**3**	**2**	**2**	**4**	**4**	**5**
85		**5**	**9**	3	3	0	1	1	3	0	2	1	1	0	4	5	3	7

Table 1: Nim-sequence for ARC KAYLES on paths, P_n, where $n = s + t$ [3].

$$n \in \{41 + 34i, 45 + 34i, 57 + 34i, 61 + 34i, 65 + 34i,\ i \in \mathbb{N}\}.$$

This proves the following result.

Proposition 2. *Let $n \geq 3$. The nim-sequence for cycles is given by*

$$\mathcal{G}\left(\mathrm{C}_n\right) = \begin{cases} 1 & \text{if } n \in S \\ 0 & \text{otherwise} \end{cases}$$

where,

$$S = \{3, 7, 11, 17, 23, 27, 31, 37, 41 + 34i, 45 + 34i, 57 + 34i, 61 + 34i, 65 + 34i, i \in \mathbb{N}\}.$$

4. Wheel Graphs

A *wheel* is a connected graph with one distinguished center node, surrounded by a cycle where every node on the cycle is connected by one edge to the center node.

There are two moves for a general wheel graph, W_n, where n is the number of vertices in the cycle surrounding the distinguished node and $n \geq 3$. One move is to choose a spoke edge, which leaves a path on $n-1$ vertices. The other move is to choose a rim edge which leaves a fan graph with $n-2$ vertices on the rim. We refer to the fan graph as a *pizza graph*[3], Pz_n, since further breakdown of options by removing more rim edges from a fan resembles a partially eaten pizza. Note: the values for $\mathcal{G}\left(\mathrm{W}_0\right), \mathcal{G}\left(\mathrm{W}_1\right)$ and $\mathcal{G}\left(\mathrm{W}_2\right)$ are 0 since we are not allowing multi-edges; the graphs have no edges and the next player does not have a move.

Lemma 2. *Let $n \geq 3$, then $\mathcal{G}\left(\mathrm{Pz}_{n-2}\right) \geq \mathcal{G}\left(\mathrm{P}_{n-1}\right)$.*

Proof. We begin by breaking down the options of each game. From P_{n-1}, a player may remove any edge. The set of moves are summarized as the set of disjunctive sums of paths $\{\mathrm{P}_{n-3-i} + \mathrm{P}_i\}$, for $0 \leq i \leq n-3$. From Pz_{n-2} there are two possible

[3]Some other authors have used the term *Fan-Star* to describe the same graph.

moves: a player may remove a spoke edge and obtain the same set of disjunctive sums of paths as from P_{n-1}: $\{\mathrm{P}_{n-3-i}+\mathrm{P}_i\}$, for $0 \le i \le n-3$. Otherwise, a player may remove a rim edge and we obtain another pizza with one or two separate smaller fans joined at the distinguished center node. This set of positions looks like the following set: $\{\mathrm{Pz}_{n-i-4,i}\}$, for $0 \le i \le n-4$, where $n-i-4$ and i represent the sizes of the rim edge paths of the fans. Since the set of options of Pz_{n-2} contains the set of options of P_{n-1}, all these options are also contributing to the calculation of the mex of Pz_{n-2}. Therefore, $\mathcal{G}(\mathrm{Pz}_{n-2})$ is at least as large as $\mathcal{G}(\mathrm{P}_{n-1})$. □

Theorem 3. *Let $n \ge 3$, then*

$$\mathcal{G}(\mathrm{W}_n) = \begin{cases} 1 & \text{if } \mathcal{G}(\mathrm{P_{n-1}}) = 0 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Up to isomorphism there are only two options from any wheel graph and hence we are only taking the mex of two values: $mex\{\mathcal{G}(\mathrm{Pz}_{n-2}), \mathcal{G}(\mathrm{P}_{n-1})\}$. By Lemma 2, we know that $\mathcal{G}(\mathrm{Pz}_{n-2}) \ge \mathcal{G}(\mathrm{P}_{n-1})$. There are three possibilities:

1. $\mathcal{G}(\mathrm{Pz}_{n-2}) \ge \mathcal{G}(\mathrm{P}_{n-1}) > 0$ implying that $\mathcal{G}(\mathrm{W}_n) = 0$.
2. $\mathcal{G}(\mathrm{Pz}_{n-2}) > 1 > \mathcal{G}(\mathrm{P}_{n-1}) = 0$ or $\mathcal{G}(\mathrm{Pz}_{n-2}) = \mathcal{G}(\mathrm{P}_{n-1}) = 0$, both implying that $\mathcal{G}(\mathrm{W}_n) = 1$.
3. $\mathcal{G}(\mathrm{Pz}_{n-2}) = 1, \mathcal{G}(\mathrm{P}_{n-1}) = 0$ implying that $\mathcal{G}(\mathrm{W}_n) = 2$.

We show that Case 3 cannot occur. If $\mathcal{G}(\mathrm{P}_{n-1}) = 0$ we know all of its options have $\mathcal{G}$-value greater than 0. Recall that the options of P_{n-1} are also options of Pz_{n-2}. If we can show that all the paths which have value 0 will have an option with value equal to 1, we will have completed the proof. The path sequence is ultimately periodic as shown in Section 3, so we only need to check the zero valued paths within the pre-period and first period of the path nim-sequence; games contributing to subsequent period values will, by definition, have the same values contributing to their mex calculation as did its corresponding position (modulo the period length) within the first period. Hence, there are a finite number of zero valued paths ($n = 6, 10, 16, 22, 26, 30, 36, 40, 44, 56, 60, 64, 74, 78$) that we need to check. Recall n denotes the number of vertices on the rim of the wheel, the number of vertices on the corresponding path is one less. We know that the options from P_{n-1} are $\{\mathrm{P}_i + \mathrm{P}_{n-i-3}\}$. For $0 \le i \le 6$, the Grundy values of P_i are $(0, 0, 1, 1, 2, 0, 3)$. The Grundy values required for P_{n-i-3} to yield a sum of 1 are therefore $(1, 1, 0, 0, 3, 1, 2)$ for i in the same range. For $n \in (6, 10, 16, 22, 26, 30, 36, 40, 44, 56, 60, 64, 74, 78)$, the values of i that establish that P_{n-1} has an option with value 1 are therefore $i = (0, 0, 5, 6, 0, 0, 5, 0, 0, 6, 0, 0, 0, 0)$ respectively. This concludes the proof. □

5. Generalized Star Graphs

A *star* is a connected acyclic graph that has one distinguished center node with degree $n \geq 1$ and all other vertices with degree 1. We call a graph a *generalized star* if we allow for the possibility of all non-center vertices having degree at most 2.

We denote a generalized star graph with three rays as $S_{f,g,n}$, where f, g and n are the number of vertices on each ray. When allowing one ray, n, to vary we denote the class of generalized stars by $S_{f,g}(n)$ or simply $S_{f,g}$ to mean the nim-sequence $\{\mathcal{G}(S_{f,g,n}) : n = 1, 2, \ldots\}$. Playing Arc Kayles on a generalized star graph with three rays, we have the following decomposition.

1. A player may remove an edge incident to the center node. This can happen in three ways and produces the following three outcomes:
   (a) $\mathrm{P}_{f-2} + \mathrm{P}_{g-1} + \mathrm{P}_{n-1}$
   (b) $\mathrm{P}_{f-1} + \mathrm{P}_{g-2} + \mathrm{P}_{n-1}$
   (c) $\mathrm{P}_{f-1} + \mathrm{P}_{g-1} + \mathrm{P}_{n-2}$.
2. A player may remove the second edge from the center vertex on any of the rays. These will give the following three outcomes:
   (a) $\mathrm{P}_{f-3} + \mathrm{P}_{g+n-1}$
   (b) $\mathrm{P}_{g-3} + \mathrm{P}_{f+n-1}$
   (c) $\mathrm{P}_{n-3} + \mathrm{P}_{f+g-1}$.
3. A player may remove any other edge from any of the rays. These will leave a disjunctive sum of a generalized star and a path:
   (a) $S_{f-i-2,g,n} + \mathrm{P}_i, 0 \leq i \leq f-2$
   (b) $S_{f,g-i-2,n} + \mathrm{P}_i, 0 \leq i \leq g-2$
   (c) $S_{f,g,n-i-2} + \mathrm{P}_i, 0 \leq i \leq n-2$.

The decomposition shows that there are many values contributing to the mex calculation of the current game position. We now present a method to implement an automatic check for periodicity for this graph class. This check is motivated by work presented in [1] and [3] with regards to Subtraction and octal games. For notational simplicity, $\widehat{S}$ is the expected pre-period associated with the current generalized star $S_{f,g,n}$ and $\overline{S}$ is its expected period. Similarly we reserve $\widehat{D}$ for the pre-period and $\overline{D}$ for the period for other nim-sequences with respect to a class of graphs denoted by D.

Theorem 4 (Automatic check for periodicity). *Let f and g be positive integers. Suppose $S_{f',g'}$ is periodic, where $f' \leq f$, $g' \leq g$ and $f' + g' < f + g$. If $\mathcal{G}(S_{f,g,n}) = \mathcal{G}\left(S_{f,g,n+\overline{S}}\right)$ for all $\widehat{S} \leq n \leq \widehat{I} + 2\overline{I} + 2$, then $S_{f,g}$ is periodic with*

$$\widehat{I} = max\left\{\widehat{S} + \widehat{P}_{f,g,n}, \left(\widehat{S}_{f-j-2,g,n}\right)_{j=0}^{f-2}, \left(\widehat{S}_{f,g-j-2,n}\right)_{j=0}^{g-2}, \widehat{P}_{f,g-j-2,n} - \overline{P}, \widehat{P}\right\}$$

and

$$\overline{I} = LCM\left\{\overline{S}, \left(\overline{S}_{f,g-2-j}\right)_{j=0}^{g-2}, \left(\overline{S}_{f-2-j,g}\right)_{j=0}^{f-2}, \overline{P}\right\}.$$

Proof. Recall that the path nim-sequence is periodic (presented in Section 3). The value of a generalized star $S_{f,g,n}$ is determined by the minimum excluded value of its options. We want to show that if n is large enough and $\mathcal{G}(S_{f,g,n}) = mex(T)$ and $\mathcal{G}\left(S_{f,g,n+\overline{S}}\right) = mex(S)$, then $T = S$. Hence the nim-sequences of the following options need to be periodic:

1. $\mathrm{P}_{f-2} + \mathrm{P}_{g-1} + \mathrm{P}_{n-1}$
2. $\mathrm{P}_{f-1} + \mathrm{P}_{g-2} + \mathrm{P}_{n-1}$
3. $\mathrm{P}_{f-1} + \mathrm{P}_{g-1} + \mathrm{P}_{n-2}$
4. $\mathrm{P}_{f-3} + \mathrm{P}_{g+n-1}$
5. $\mathrm{P}_{g-3} + \mathrm{P}_{f+n-1}$
6. $\mathrm{P}_{n-3} + \mathrm{P}_{f+g-1}$
7. $S_{f-i-2,g,n} + \mathrm{P}_i, 0 \leq i \leq f-2$
8. $S_{f,g-i-2,n} + \mathrm{P}_i, 0 \leq i \leq g-2$
9. $S_{f,g,n-i-2} + \mathrm{P}_i, 0 \leq i \leq n-2$

Their common period will be $\overline{S}$.

We now check that each option has a periodic nim-sequence. First consider options $1, 2, 3$ as listed above. These options are paths where two of the summands of paths are fixed and the third (involving n) is varying. This is equivalent to taking the nim-sequence of the path (which we know to be periodic) and adding the same constant to every entry. If n is larger than $\widehat{P} + \overline{P} + 1$, then $\mathcal{G}(\mathrm{P}_{n-1}) = \mathcal{G}\left(\mathrm{P}_{n+\overline{S}-1}\right)$, and so $\mathcal{G}(\mathrm{P}_{n-1}) \oplus \mathcal{G}(\mathrm{P}_{f-2}) \oplus \mathcal{G}(\mathrm{P}_{g-1}) = \mathcal{G}\left(\mathrm{P}_{n+\overline{S}-1}\right) \oplus \mathcal{G}(\mathrm{P}_{f-2}) \oplus \mathcal{G}(\mathrm{P}_{g-1})$. These nim-sequences are periodic with period length 34. The same holds for option 2 (indices are switched and the proof is the same). For option 3, the only difference is that n must be larger than $\widehat{P} + \overline{P} + 2$. This shows that when n is sufficiently large, the Grundy values of the first three positions are all contained in both T and in S.

The nim-sequence for options 4, 5 involving $n+f-1$ and $n+g-1$, is a path nim-sequence shifted by $f-1$ and $g-1$ respectively. The second component of the nim-addition is a fixed length path and so the same explanation as the first set $\{1,2,3\}$ of path options applies. Lastly, the sixth option consists of a path nim-sequence (shifted by three) being nim-added to a constant. This also reduces to the earlier cases with the slight modification that n must be as large as $\widehat{P}+\overline{P}+3$. All three options are periodic with period length 34. Once again, n must be larger than $\widehat{P}+\overline{P}+3$. And so, the Grundy values of the three options $\{4,5,6\}$ are in both T and S.

The seventh and eighth options listed rely on the breakdown of the fixed rays f and g respectively. We examine the seventh option in detail, the eighth is symmetric. There are finitely many options of the form $\{S_{f-j-2,g,n}+P_j\}_{j=0}^{f-2}$. The generalized stars considered in this set rely on different generalized star classes with respect to g which we consider to have already been determined. Hence, in order to determine the periodicity of the current class, we need to be large enough with respect to all of the previously calculated classes already attaining periodicity and the periodicity of all classes involved have to be synchronized. And so, we need

$$n \geq max\left\{\left(\widehat{S}_{f-j-2,g,n}\right)_{j=0}^{f-2}\right\}+LCM\left\{\left(\overline{S}_{f-j-2,g,n}\right)_{j=0}^{f-2}\right\}.$$

If n is sufficiently large then the Grundy values will be contained in both T and S.

Lastly, the first terms of the options $S_{f,g,n-j-2}+P_j, 0\leq j\leq n-2$, remain within the same generalized star class and we look back at previously calculated positions and nim-add a path of the appropriate length. We consider two cases:

Case 1: Suppose $n-j-2>\widehat{S}+\overline{I}$. Since $n-j-2>\widehat{S}+\overline{I}$, $n-j-2-\overline{I}>\widehat{S}$, the nim-sequence of $S_{f,g,n}$ is already showing signs of periodicity. And so, $\mathcal{G}\left(S_{f,g,n-j-2+\overline{S}}\right)=\mathcal{G}\left(S_{f,g,n-j-2}\right)$. This implies that adding a constant to both positions will give the same value. Hence $\mathcal{G}\left(S_{f,g,n-j-2+\overline{S}}+\mathrm{P}_j\right)=\mathcal{G}\left(S_{f,g,n-j-2}+\mathrm{P}_j\right)$. Thus the Grundy values of these positions appear in both T and S.

Case 2: Suppose $n-j-2\leq\widehat{S}+\overline{I}$. If $n>\widehat{I}+2\overline{I}+2$, then $j>\widehat{P}+\overline{I}$ and $\mathcal{G}\left(\mathrm{P}_{j+\overline{S}}\right)=\mathcal{G}\left(\mathrm{P}_j\right)$. Then $\mathcal{G}\left(S_{f,g,n-j-2}+\mathrm{P}_{j+\overline{S}}\right)=\mathcal{G}\left(S_{f,g,n-j-2}+\mathrm{P}_j\right)$. Thus the Grundy values of these positions appear in both T and S.

When $n>\widehat{I}+2\overline{I}+2$, we conclude that $S=T$, where

$$\widehat{I}=max\left\{\widehat{S}+\widehat{P}_{f,g,n},\left(\widehat{S}_{f-j-2,g,n}\right)_{j=0}^{f-2},\left(\widehat{S}_{f,g-j-2,n}\right)_{j=0}^{g-2},\widehat{P}_{f,g-j-2,n}-\overline{P},\widehat{P}\right\}$$

and

$$\overline{I}=LCM\left\{\overline{S},\left(\overline{S}_{f,g-j-2,n}\right)_{j=0}^{g-2},\left(\overline{S}_{f-j-2,g,n}\right)_{j=0}^{f-2},\overline{P}\right\}.$$

□

We have investigated the nim-sequences when $f = 2$, g fixed and the third ray varying off to infinity. Based on our preliminary results it appears as though all classes $g \pmod{34}$ have a stabilizing pre-period (see Table 2) meaning that each class $\pmod{34}$ eventually has a stable period also.

The *quasi-pre-period* for a generalized star with a ray $g \pmod{34}$ is the value of g at which the pre-period for that class of generalized stars becomes stable in n.

Conjecture 1. *For all $g >$ quasi-pre-period, and $n >$ pre-period length shown in Table 2, the $\mathcal{G}$-value of* Arc Kayles *played on $S_{2,g,n}$ is equal to the value shown in Table 5, where g and n are modulo 34 in the lookup table.*

Let us look at an example to understand how the lookup tables (Table 2 and 5) work. Consider $S_{2,750,673}$. This is a generalized star with one ray fixed with 2 vertices, a second ray fixed with 750 vertices and a third ray fixed with 673 vertices. Now, $750 \equiv 2 \pmod{34}$ and $673 \equiv 27 \pmod{34}$. In Table 2 we see that $g = 750$ is beyond the value where this class modulo 34 becomes stable in n; it is far enough, because in that column $g = 274$ is the value at which this class is stable for n. Also, our value for n, namely $n = 673$, is beyond this last irregularity since for this stabilized class $h = 459$ was the last irregularity. Proceed to Table 5 and look up the value of $S_{2,750,673}$ which will be found at the intersection of row 2 and column 27. Hence, we conjecture that $\mathcal{G}(S_{2,750,673}) = 47$.

If Conjecture 1 is true, it means that there is indeed an ultimate generalized star $S_{2,g,n}$ which is completely determined by smaller generalized star classes (as was hoped for Node Kayles in [6]) for Arc Kayles. At this time, it is unclear how to prove this conjecture as even the first step of proving boundedness a priori of the nim-sequence is hard. This will be further discussed in the future directions section.

	t→	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
s↓																		
0		52	52	52	196	52	156	349	232	301	191	249	181	190	383	232	188	388
34		349	249	249	249	215	245	216	250	165	256	250	388	258	248	249	250	261
68		279	249	283	204	241	260	383	301	165	261	459	388	301	232	283	279	244
102		336	320	283	204	283	260	349	248	285	299	388	388	294	232	249	250	272
136		313	**388**	312	228	283	396	349	313	279	283	388	388	260	232	343	333	271
170		313	**388**	459	221	283	396	349	345	313	283	388	388	396	232	299	252	373
204		316	**388**	459	**340**	261	275	349	388	313	283	275	388	**459**	232	303	461	280
238		333	**388**	326	**340**	261	**338**	349	**459**	301	283	**388**	320	**459**	232	367	461	340
272		299	**388**	**459**	**340**	**283**	**338**	349	**459**	313	**351**	**388**	**349**	**459**	374	316	337	408
306		299	**388**	**459**	**340**	**283**	**338**	349	**459**	313	**351**	**388**	**349**	**459**	340	303	**461**	**340**
340		299	**388**	**459**	**340**	**283**	**338**	**392**	**459**	313	**351**	**388**	**349**	**459**	374	303	**461**	**340**
374		**461**	**388**	**459**	**340**	**283**	**338**	**392**	**459**	**301**	**351**	**388**	**349**	**459**	374	303	**461**	**340**
408		**461**	**388**	**459**	**340**	**283**	**338**	**392**	**459**	**301**	**351**	**388**	**349**	**459**	**340**	**329**	**461**	**340**
442		**461**	**388**	**459**	**340**	**283**	**338**	**392**	**459**	**301**	**351**	**388**	**349**	**459**	**340**	**329**	**461**	**340**
476		**461**	**388**	**459**	**340**	**283**	**338**	**392**	**459**	**301**	**351**	**388**	**349**	**459**	**340**	**329**	**461**	**340**
510		**461**	**388**	**459**	**340**	**283**	**338**	**392**	**459**	**301**	**351**	**388**	**349**	**459**	**340**	**329**	**461**	**340**
544		**461**	**388**	**459**	**340**	**283**	**338**	**392**	**459**	**301**	**351**	**388**	**349**	**459**	**340**	**329**	**461**	**340**
578		**461**	**388**	**459**	**340**	**283**	**338**	**392**	**459**	**301**	**351**	**388**	**349**	**459**	**340**	**329**	**461**	**340**
612		**461**	**388**	**459**	**340**	**283**	**338**	**392**	**459**	**301**	**351**	**388**	**349**	**459**	**340**	**329**	**461**	**340**

	t→	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
s↓																		
0		190	283	170	250	261	349	125	349	249	162	191	249	177	303	178	245	232
34		197	249	245	388	246	249	383	349	249	241	204	284	212	245	250	349	245
68		349	388	258	260	283	245	383	349	388	260	301	349	246	250	333	314	264
102		316	283	258	245	280	280	250	396	388	294	459	349	333	250	314	349	320
136		306	303	272	250	280	301	250	396	388	273	459	303	459	298	300	349	388
170		459	283	459	333	333	396	**388**	396	286	340	461	383	459	250	**333**	358	459
204		374	283	258	301	280	260	**388**	349	255	336	461	383	459	298	**333**	**459**	367
238		326	341	258	301	283	459	**388**	**459**	337	329	**388**	383	301	298	**333**	**459**	**459**
272		348	341	256	**392**	**261**	**443**	**388**	**459**	303	341	**388**	383	301	298	**333**	**459**	**459**
306		**313**	341	256	**392**	**261**	**443**	**388**	**459**	**255**	374	**388**	**303**	**392**	**284**	**333**	**459**	**459**
340		**313**	**443**	**309**	**392**	**261**	**443**	**388**	**459**	**255**	341	**388**	**303**	**392**	**284**	**333**	**459**	**459**
374		**313**	**443**	**309**	**392**	**261**	**443**	**388**	**459**	**255**	**374**	**388**	**303**	**392**	**284**	**333**	**459**	**459**
408		**313**	**443**	**309**	**392**	**261**	**443**	**388**	**459**	**255**	**374**	**388**	**303**	**392**	**284**	**333**	**459**	**459**
442		**313**	**443**	**309**	**392**	**261**	**443**	**388**	**459**	**255**	**374**	**388**	**303**	**392**	**284**	**333**	**459**	**459**
476		**313**	**443**	**309**	**392**	**261**	**443**	**388**	**459**	**255**	**374**	**388**	**303**	**392**	**284**	**333**	**459**	**459**
510		**313**	**443**	**309**	**392**	**261**	**443**	**388**	**459**	**255**	**374**	**388**	**303**	**392**	**284**	**333**	**459**	**459**
544		**313**	**443**	**309**	**392**	**261**	**443**	**388**	**459**	**255**	**374**	**388**	**303**	**392**	**284**	**333**	**459**	**459**
578		**313**	**443**	**309**	**392**	**261**	**443**	**388**	**459**	**255**	**374**	**388**	**303**	**392**	**284**	**333**	**459**	**459**
612		**313**	**443**	**309**	**392**	**261**	**443**	**388**	**459**	**255**	**374**	**388**	**303**	**392**	**284**	**333**	**459**	**459**

Table 2: Pre-period lengths for generalized star graphs $S_{f,g,n}$; $f = 2$, $g = s + t$ and n goes to infinity.

	$n' \rightarrow$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$g' \downarrow$																							
0		6	8	8	1	26	4	28	3	13	29	41	42	27	13	33	12	8	1	45	29	3	24
1		8	1	1	9	0	23	8	26	0	4	23	2	6	22	15	5	5	9	3	3	34	5
2		8	1	1	2	0	37	1	43	0	28	52	20	18	4	38	5	43	39	3	50	27	1
3		1	9	2	3	14	26	2	4	8	14	5	20	4	8	5	18	41	3	8	5	13	3
4		26	0	0	14	1	2	4	14	14	5	43	4	16	5	31	4	4	24	2	19	36	0
5		4	23	37	26	2	4	12	6	13	7	4	42	5	9	33	7	19	2	9	4	16	28
6		28	8	1	2	4	12	1	29	7	4	8	2	17	4	4	21	5	51	3	15	37	5
7		3	26	43	4	14	6	29	4	12	46	14	44	7	58	49	13	43	43	54	44	37	29
8		13	0	0	8	14	13	7	12	8	5	26	23	3	5	5	4	4	12	2	6	9	4
9		29	4	28	14	5	7	4	46	5	5	53	7	24	1	48	4	0	31	1	17	26	7
10		41	23	52	5	43	4	8	14	26	53	3	11	5	34	15	12	2	13	22	4	7	27
11		42	2	20	20	4	42	2	44	23	7	11	1	18	19	0	29	1	35	4	0	42	6
12		27	6	18	4	16	5	17	7	3	24	5	18	4	19	14	2	39	40	4	41	27	40
13		13	22	4	8	5	9	4	58	5	1	34	19	19	2	2	26	4	12	14	6	9	3
14		33	15	38	5	31	33	4	49	5	48	15	0	14	2	3	3	21	1	1	30	4	7
15		12	5	5	18	4	7	21	13	4	4	12	29	2	26	3	1	1	43	38	7	8	9
16		8	5	43	41	4	19	5	43	4	0	2	1	39	4	21	1	1	25	4	7	42	1
17		1	9	39	3	24	2	51	43	12	31	13	35	40	12	1	43	25	7	12	41	54	40
18		45	3	3	8	2	9	3	54	2	1	22	4	4	14	1	38	4	12	5	1	9	3
19		29	3	50	5	19	4	15	44	6	17	4	0	41	6	30	7	7	41	1	26	32	3
20		3	34	27	13	36	16	37	37	9	26	7	42	27	9	4	8	42	54	9	32	3	31
21		24	5	1	3	0	28	5	29	4	7	27	6	40	3	7	9	1	40	3	3	31	1
22		50	21	23	28	33	13	6	18	8	36	6	27	7	8	50	5	40	12	8	3	1	19
23		26	3	41	39	2	13	4	4	31	14	13	0	23	39	5	38	7	41	2	13	0	12
24		48	16	36	5	38	4	31	5	1	18	4	26	5	5	16	4	39	19	9	22	4	28
25		15	2	13	14	4	8	19	9	10	32	20	5	13	7	7	15	44	10	30	0	12	6
26		54	6	51	4	24	20	1	52	7	20	25	5	26	12	42	50	2	16	4	19	34	26
27		14	18	47	7	6	5	9	52	8	6	47	43	4	19	32	13	0	12	20	6	5	26
28		10	4	4	5	5	10	0	17	5	5	35	3	8	1	2	21	12	28	6	6	7	4
29		12	0	4	6	9	7	8	46	5	38	12	3	14	13	0	0	28	48	13	31	4	4
30		4	8	5	14	4	8	8	1	9	7	7	36	14	0	4	36	2	6	26	4	21	5
31		38	5	40	15	2	42	5	48	11	35	19	1	36	4	17	9	6	40	4	41	5	9
32		5	21	7	39	6	5	40	48	22	13	14	23	0	2	30	3	47	35	29	5	14	20
33		30	19	4	18	5	40	15	20	13	6	0	4	38	45	2	34	46	37	5	5	43	3

	$n' \rightarrow$	22	23	24	25	26	27	28	29	30	31	32	33
$g' \downarrow$													
0		50	26	48	15	54	14	10	12	4	38	5	30
1		21	3	16	2	6	18	4	0	8	5	21	19
2		23	41	36	13	51	47	4	4	5	40	7	4
3		28	39	5	14	4	7	5	6	14	15	39	18
4		33	2	38	4	24	6	5	9	4	2	6	5
5		13	13	4	8	20	5	10	7	8	42	5	40
6		6	4	31	19	1	9	0	8	8	5	40	15
7		18	4	5	9	52	52	17	46	1	48	48	20
8		8	31	1	10	7	8	5	5	9	11	22	13
9		36	14	18	32	20	6	5	38	7	35	13	6
10		6	13	4	20	25	47	35	12	7	19	14	0
11		27	0	26	5	5	43	3	3	36	1	23	4
12		7	23	5	13	26	4	8	14	14	36	0	38
13		8	39	5	7	12	19	1	13	0	4	2	45
14		50	5	16	7	42	32	2	0	4	17	30	2
15		5	38	4	15	50	13	21	0	36	9	3	34
16		40	7	39	44	2	0	12	28	2	6	47	46
17		12	41	19	10	16	12	28	48	6	40	35	37
18		8	2	9	30	4	20	6	13	26	4	29	5
19		3	13	22	0	19	6	6	31	4	41	5	5
20		1	0	4	12	34	5	7	4	21	5	14	43
21		19	12	28	6	26	26	4	4	5	9	20	3
22		7	28	6	9	42	4	15	5	5	51	7	50
23		28	5	6	29	3	8	2	9	3	3	36	5
24		6	6	7	4	36	5	9	34	3	32	6	5
25		9	29	4	8	5	9	32	12	34	13	13	12
26		42	3	36	5	6	4	7	33	5	48	38	56
27		4	8	5	9	4	4	31	13	25	43	41	38
28		15	2	9	32	7	31	1	26	9	37	42	14
29		5	9	34	12	33	13	26	1	15	2	2	10
30		5	3	3	34	5	25	9	15	2	6	13	28
31		51	3	32	13	48	43	37	2	6	10	4	4
32		7	36	6	13	38	41	42	2	13	4	8	35
33		50	5	5	12	56	38	14	10	28	4	35	5

Table 3: Grundy values for generalized star graphs $S_{f,g,n}$, with $f = 2$, $g >$ quasi-pre-period, $n >$ pre-period length shown in Table 2, where $g \equiv g'$ (mod 34) and $n \equiv n'$ (mod 34).

6. Future Directions

Some games have clear bounds on the possible $\mathcal{G}$-values for the nim-sequence (see [1]). Depending on the underlying graph, ARC KAYLES could potentially have $\mathcal{G}$-values growing without bound as the number of vertices increases. However, as observed with path graphs and generalized star graphs, even though they could grow without bound, in these cases they do not. It would be interesting to understand why this is the case. Future work involves solving the conjecture presented in Section 5. It would be a great feat to develop methods of determining a priori bounds on $\mathcal{G}$-values for impartial games. Alternatively, a general theorem on periodic behaviour of games would help this analysis.

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