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FURTHER MULTIVARIATE GENERALIZATIONS OF EULER'S PENTAGONAL NUMBER THEOREM AND THE ROGERS-RAMANUJAN IDENTITIES

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Abstract

We use the idea of index invariance under the Franklin mapping to prove higher power generalizations of two results discovered by M. V. Subbarao. We then apply similar ideas to a two-variable generalization of the Rogers-Ramanujan identities due to G. E. Andrews.

1. Introduction

In 1970, M. V. Subbarao published a paper [5] providing combinatorial proofs of the following two generalizations of Euler's celebrated pentagonal number theorem:

$$\sum_{k=1}^{\infty} a^k(-aq^k)(aq;q)_{k-1} = \sum_{k=1}^{\infty}(-1)^k\left(a^{3k-1}q^{k(3k-1)/2} + a^{3k}q^{k(3k+1)/2}\right); \tag{1.1}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^k a^{2k} q^{k(k+1)/2}}{(aq;q)_k} = \sum_{k=1}^{\infty}(-1)^k\left(a^{3k-1}q^{k(3k-1)/2} + a^{3k}q^{k(3k+1)/2}\right), \tag{1.2}$$

where $(a;q)_n$ is the q-shifted factorial defined as follows for $n \geq 0$:

$$(a;q)_n = \prod_{k=0}^{n-1}\left(1 - aq^k\right). \tag{1.3}$$

Both L. J. Rogers and N. J. Fine had proven before Subbarao that these three series are equal [1]. However, these proofs are both analytic. Subbarao's fundamental observation was that the Franklin mapping leaves the sum of the largest part and number of parts in the Ferrers diagram unchanged. (We refer the reader to [4] for complete details of Franklin's proof of the pentagonal number theorem). Then upon noticing that the exponent of a in (1.1) records this quantity, which we will henceforth refer to as the *index*, we immediately see that the left-hand side equals

$$\sum_{k,r\geq 1} \big(S_{\mathrm{e}}(k,r) - S_{\mathrm{o}}(k,r)\big) a^r q^k. \tag{1.4}$$

Following Subbarao, $S_{\mathrm{e}}(n,m)$ $\big(S_{\mathrm{o}}(n,m)\big)$ denotes the number of partitions of n with index m into an even (odd) number of parts. After calculating the index for certain partitions of the pentagonal numbers for which the mapping fails, the desired result quickly follows.

At the end of his paper, Subbarao noted that invariance of the index under the Franklin mapping implies the invariance of real-valued functions of the index (such as the square of the index). He proceeded to ask whether or not one could obtain identities like (1.1) and (1.2) by taking advantage of the invariance of this more general quantity. We answer this question in the affirmative by rewriting the left-hand sides of (1.1) and (1.2), and using Franklin's combinatorial methods to deduce the following pair of identities:

$$\sum_{r,k\geq 1} (-1)^r q^{r(r-1)/2+k} a_{k+r} \begin{bmatrix} k-1 \\ r-1 \end{bmatrix} = \sum_{k=1}^{\infty} (-1)^k \big(a_{3k-1} q^{\frac{k(3k-1)}{2}} + a_{3k} q^{\frac{k(3k+1)}{2}}\big); \tag{1.5}$$

$$\sum_{r,k,j\geq 1} (-1)^k q^{rk+k(k-1)/2+j-1} a_{2k+r+j-2} \begin{bmatrix} k+j-3 \\ j-1 \end{bmatrix} = \sum_{k=1}^{\infty} (-1)^k \big(a_{3k-1} q^{k(3k-1)/2} + a_{3k} q^{k(3k+1)/2}\big). \tag{1.6}$$

Here the a_n are indeterminates and the $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ are q-binomial coefficients defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} \tag{1.7}$$

for $0 \leq k \leq n$, and 0 otherwise. Equation (1.6) is a rewriting of (1.5) so that it is revealed as a companion to (1.2). Equations (1.1) and (1.2) follow from (1.5) and (1.6) upon setting $a_n = a^n$ and invoking the q-binomial theorem.

We note in passing an interesting identity that arises upon setting $a_n = n$ in (1.5) and using Zagier's identity [7, Theorem 2] to rewrite the subsequent right-hand side:

$$\sum_{r,k\geq 1} (-1)^r q^{r(r-1)/2+k} (k+r) \begin{bmatrix} k-1 \\ r-1 \end{bmatrix} = \sum_{n=0}^{\infty} [(q;q)_\infty - (q;q)_n] - (q;q)_\infty \sum_{k=1}^{\infty} \frac{q^k}{1-q^k}. \tag{1.8}$$

Zagier introduced the series

$$F_1(q) := \sum_{n=1}^{\infty} n (q;q)_{n-1} q^n, \tag{1.9}$$

which he showed is equal [7] to

$$F_2(q) := \sum_{n=0}^{\infty} [(q;q)_n - (q;q)_\infty]. \tag{1.10}$$

As a power series in $\zeta - q$ for any root of unity ζ, $F_1(q)$ equals the Kontsevich function

$$F(q) := \sum_{n=0}^{\infty} (q)_n, \tag{1.11}$$

and Zagier used this fact to describe the expansion of $F(q)$ for q near roots of unity. (Indeed, $F(q)$ only makes sense as a complex function of q at roots of unity, for the series does not converge anywhere else in $\mathbb{C}$). $F(q)$ was further studied in [8] as an example of a quantum modular form.

The key to proving (1.5) and (1.6) is nothing more than a few simple combinatorial observations. However, their relevance lies in the fact that similar techniques allow us to generalize Andrews' analytic version of Schur's combinatorial proof of the Rogers-Ramanujan identities [2]. Thus the second objective of this paper is to prove the following two theorems, where the D_n are the classical Schur polynomials which admit simple closed form expressions as given in [2]:

Theorem 1. *If $D_{-1} = D_0 = 1$, $D_n = D_{n-1} + q^n D_{n-2}$ for $n > 0$ and y_n is a sequence of indeterminates, then*

$$\sum_{k,n\geq 1} (-1)^k q^{\binom{k}{2}+2n} D_{n-2}\, y_{k+2n} \left(\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} - q^{2-k} \begin{bmatrix} n \\ k-2 \end{bmatrix} \right)$$
$$= \sum_{m=1}^{\infty} (-1)^m \big(y_{5m-2} q^{m(5m-1)/2} + y_{5m-1} q^{m(5m+1)/2} \big). \tag{1.12}$$

Theorem 2. *If $D^*_{-1} = 0$, $D^*_0 = 1$, $D^*_n = D^*_{n-1} + q^n D^*_{n-2}$ for $n > 0$ and y_n is a sequence of indeterminates, then*

$$\sum_{k,n\geq 1} (-1)^k q^{\binom{k}{2}+2n} D_{n-2}\, y_{k+2n} \left(\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} - q^{2-k} \begin{bmatrix} n \\ k-2 \end{bmatrix} \right)$$
$$= \sum_{m=1}^{\infty} (-1)^m \big(y_{5m} q^{m(5m+3))/2} - y_{5m+2} q^{(m+1)(5m+1)/2} \big). \tag{1.13}$$

In Section 2 we will deduce (1.5) and (1.6) using Franklin's combinatorial methods and prove these results using recurrence-based arguments in Section 3. In Section 4 we will prove Theorems 1 and 2 and obtain the Rogers-Ramanujan identities as special cases. In Section 5 we will make some concluding remarks.

2. Combinatorial Arguments for (1.5) and (1.6)

We begin by studying the summand on the left-hand side of (1.1). For a particular k, the coefficient of q^n is a polynomial in a where each term is of the form $(-1)^r a^{k+r}$. Here, r designates the number of parts in a particular partition of n. Let us now fix a particular $r = r_0$ and extract, from each polynomial coefficient of each q^n that arises in the summand, the terms of the form $(-1)^{r_0} a^{k+r_0}$. If we do this for each r from 1 to k, then we may reformulate the summand as

$$\sum_{r=1}^{k} p_{k,r}(q)(-1)^r a^{k+r}, \tag{2.1}$$

where $p_{k,r}(q)$ is the generating function for partitions into r distinct parts with largest part exactly k. Hence $p_{k,r}(q)$ is simply the coefficient of z^r in

$$zq^k(-zq;q)_{k-1} = \sum_{r=0}^{k-1} \begin{bmatrix} k-1 \\ r \end{bmatrix} z^{r+1} q^{r(r+1)/2+k}. \tag{2.2}$$

By the q-binomial theorem [3, Eq. (3.3.6)],

$$(z;q)_N \quad = \quad \sum_{j=0}^{N} \begin{bmatrix} N \\ j \end{bmatrix} (-1)^j z^j q^{j(j-1)/2}, \tag{2.3}$$

this coefficient is found to be

$$\begin{bmatrix} k-1 \\ r-1 \end{bmatrix} q^{\binom{r}{2}+k}. \tag{2.4}$$

Now, because a is a parameter that explicitly records the index in this generating function, we may invoke the invariance of the square of the index under the Franklin mapping to see that (1.5) holds in the case $a_n = a^{n^2}$. The argument in fact reveals that (1.5) holds generally for $a_n = a^{n^u}$, where u is any real number, and so we may replace a^{k+r} in (2.1) with any real-valued function of $k+r$, from which (1.5) easily follows. Note that the combinatorial argument is essential in passing from a^{k+r} to a_{k+r}. As mentioned before, (1.5) is trivially true in the case $a_{k+r} = a^{k+r}$, but without knowing a priori that the identity resulting from comparing coefficients of a^n holds (which we analytically deduce in the next section), one cannot say immediately that the statement is true for general a_n.

For (1.6), we first prove the following lemma. The argument is similar to the one offered by Subbarao for (1.2), using the same notation and paraphrasing in a little more detail.

Lemma 1. *For $|a|, |q| < 1$,*

$$\sum_{r,k\geq 1} \frac{(-1)^r a^{k+2r-1} q^{rk+\binom{r}{2}}}{(aq;q)_{r-1}} = \sum_{k=1}^{\infty} (-1)^k \left(a^{3k-1} q^{k(3k-1)/2} + a^{3k} q^{k(3k+1)/2}\right). \tag{2.5}$$

Proof. We note that a typical term in the summand of the left-hand side of (2.4) is of the form $u(n)q^n$, where

$$u(n)q^n = (-1)^r a^{k+2r-1} q^{rk+\binom{r}{2}} (aq)^{b_1} (aq^2)^{b_2} \cdots (aq^{r-1})^{b_{r-1}} \tag{2.6}$$

with $b_i \geq 1$. Clearly the exponent on a is $k + 2r - 1 + b_1 + \cdots + b_{r-1}$, and n equals

$$rk + \binom{r}{2} + \sum_{i=1}^{r-1} i b_i. \tag{2.7}$$

We rewrite (2.7) as a sum of r distinct positive integers $c_1 + \cdots + c_r$, with $c_j = k + j + b_{r-1} + \cdots + b_{r-j}$ for $1 \leq j \leq r-1$ and $c_r = k$. The desired result then follows upon invoking the Franklin mapping. ☐

Alternatively, one may note that the sum indexed by k is a geometric series whose sum equals $aq^r/(1 - aq^r)$. Thus the left-hand side of (2.4) equals the left-hand side of (1.2), which in turn equals the right-hand side of (2.4). Hence the lemma is proven.

The q-binomial theorem in the following form [3, Eq. (3.3.7)]:

$$\frac{1}{(z;q)_N} = \sum_{j=0}^{\infty} \begin{bmatrix} N+j-1 \\ j \end{bmatrix} z^j \tag{2.8}$$

applied to $\frac{1}{(aq;q)_{r-1}}$ then results in (1.6), after employing the same logic used to prove (1.5).

3. Analytic Proofs of (1.5) and (1.6)

The fact that the a terms are isolated on either side of (1.5) and (1.6) suggests that we can prove the identities by comparing coefficients. We comment that our arguments here are alternatives to the classical proofs of (1.1) and (1.2), which employ the Rogers-Fine identity. After shifting $k \to k - r$ in the left-hand side of (1.5) and interchanging the order of summation, we obtain the following identity upon comparing coefficients:

$$q^{k-1} \sum_{r=1}^{k-1} (-1)^r q^{r(r-3)/2+1} \begin{bmatrix} k-r-1 \\ r-1 \end{bmatrix} = \begin{cases} (-1)^k q^{\frac{k(3k+1)}{2}} & \text{if } k \equiv 0 \pmod 3, \\ 0 & \text{if } k \equiv 1 \pmod 3, \\ (-1)^k q^{\frac{k(3k-1)}{2}} & \text{if } k \equiv -1 \pmod 3. \end{cases} \tag{3.1}$$

Dividing both sides by q^{k-1}, shifting $r \to r + 1$, and substituting $k - 2$ for n, (3.1) becomes

$$\sum_{r=0}^{n} (-1)^r q^{r(r-1)/2} \begin{bmatrix} n-r \\ r \end{bmatrix} = \begin{cases} (-1)^k q^{\frac{k(3k-1)}{2}} & \text{if } n = 3k, \\ (-1)^k q^{\frac{k(3k+1)}{2}} & \text{if } n = 3k+1, \\ 0 & \text{if } n = 3k+2. \end{cases} \tag{3.2}$$

Let $S(n)$ denote the left-hand side of (3.2), $R(n)$ the right-hand side, and

$$T(n) = \sum_{r=0}^{n} (-1)^r q^{r(r+1)/2} \begin{bmatrix} n-r \\ r \end{bmatrix}. \tag{3.3}$$

Then

$$\begin{aligned} T(n) - T(n-1) &= \sum_{r=0}^{n} (-1)^r q^{r(r+1)/2} \left(\begin{bmatrix} n-r \\ r \end{bmatrix} - \begin{bmatrix} n-r-1 \\ r \end{bmatrix} \right) \\ &= \sum_{r=0}^{n-1} (-1)^r q^{r(r+1)/2+n-2r} \begin{bmatrix} n-r-1 \\ r-1 \end{bmatrix} \\ &= -q^{n-1} \sum_{r=0}^{n-2} (-1)^r q^{r(r-1)/2} \begin{bmatrix} n-r-2 \\ r \end{bmatrix} \\ &= -q^{n-1} S(n-2). \end{aligned} \tag{3.4}$$

Similarly we can prove $T(n-1) - T(n-2) = S(n)$, which implies that $S(n) = -q^{n-2} S(n-3)$. A quick check that $R(n)$ satisfies this same recurrence proves (1.5). Warnaar presents a different proof of (3.2) in [6].

In (1.6), we first shift $j \to j - 2r + 2$ and interchange the j and r sums to obtain

$$\sum_{k,j,r \geq 1} (-1)^r q^{kr + r(r-1)/2 + j - 2r + 1} a_{k+j} \begin{bmatrix} j-r-1 \\ r-2 \end{bmatrix}. \tag{3.5}$$

Now we shift $j \to j - k$ and interchange the j and k sums to get

$$\sum_{j=1}^{\infty} a_j \sum_{k=1}^{j-1} \sum_{r=1}^{\lfloor (j-k+1)/2 \rfloor} (-1)^r q^{kr + r(r-1)/2 + j - k - 2r + 1} \begin{bmatrix} j-k-r-1 \\ r-2 \end{bmatrix}. \tag{3.6}$$

We study the inner two sums:

$$\begin{aligned} &\sum_{k=1}^{j-1} \sum_{r=1}^{\lfloor (j-k+1)/2 \rfloor} (-1)^r q^{kr + r(r-1)/2 + j - k - 2r + 1} \begin{bmatrix} j-k-r-1 \\ r-2 \end{bmatrix} \\ &\qquad = \sum_{r=1}^{\lfloor j/2 \rfloor} (-1)^r q^{r(r-1)/2 + (j-2r+1)r} \sum_{k=0}^{j-2r} q^{k(1-r)} \begin{bmatrix} k+r-2 \\ k \end{bmatrix} \\ &\qquad = \sum_{r=1}^{j-1} (-1)^r q^{r(r-1)/2 + j - r} \begin{bmatrix} j-r-1 \\ r-1 \end{bmatrix}, \end{aligned} \tag{3.7}$$

where the first step involves an interchange of sums followed by the substitution $k \to j - 2r + 1 - k$, and the second uses the identity

$$\sum_{k=0}^{K} q^{km} \begin{bmatrix} k-m-1 \\ k \end{bmatrix} = q^{Km} \begin{bmatrix} K+m \\ K \end{bmatrix}, \tag{3.8}$$

which can be proven by induction. The result then follows upon comparison with (3.1).

4. Proofs of Theorems 1 and 2

Andrews' original left-hand side (as presented in Theorems 1 and 2 of [2]) associated with our Theorem 1 is

$$-\sum_{n=1}^{\infty}(yq;q)_{n-1}y^{2n+1}q^{2n}D_{n-2}-\sum_{n=1}^{\infty}(yq;q)_{n}y^{2n+2}q^{2n+1}D_{n-2}. \tag{4.1}$$

The proof proceeds in much the same way as that of (1.5), the only difference being that we now extract terms of the form $(-1)^{k_0}y^{2n+k_0}$ and $(-1)^{k_0}y^{2n+1+k_0}$. (See [2] for details as to why $2n+k_0$ and $2n+1+k_0$ are the invariant quantities under Schur's transformations). One obtains two simplified series that have the same form as (2.1), namely they each involve $p_{n,k}(q)$ summed against a polynomial in y. Substituting in (2.4) for $p_{n,k}(q)$, with k replaced by n, yields Theorem 1. Theorem 2 is just the identity that results from the same procedure applied to

$$-\sum_{n=1}^{\infty}(yq;q)_{n-1}y^{2n+1}q^{2n}D^*_{n-2}-\sum_{n=1}^{\infty}(yq;q)_{n}y^{2n+2}q^{2n+1}D^*_{n-2}. \tag{4.2}$$

We now deduce the first of the Rogers-Ramanujan identities [3] from Theorem 1, which states that

$$(q;q)_\infty \lim_{n\to\infty} D_n=(q;q)_\infty\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=1+\sum_{n=1}^{\infty}(-1)^n\left(q^{\frac{n(5n-1)}{2}}+q^{\frac{n(5n+1)}{2}}\right), \tag{4.3}$$

and leave out a proof for the second as the treatment would be nearly identical. Adding 1 to the left-hand side of (1.12) and setting y_n to be the constant sequence 1 yields

$$1+\sum_{n,k\geq 1}(-1)^k q^{k(k-1)/2+2n}D_{n-2}\left(\begin{bmatrix}n-1\\k-1\end{bmatrix}+q\begin{bmatrix}n\\k-1\end{bmatrix}\right). \tag{4.4}$$

Using (2.3), (4.4) becomes

$$1-\sum_{n=1}^{\infty}q^nD_{n-2}\left(q^n(q;q)_{n-1}+q^{n+1}(q;q)_n\right)=(q;q)_\infty\lim_{n\to\infty}D_n,$$

by [2, Lemma 1]. Clearly 1 added to the the right-hand side of (1.12) equals

$$1+\sum_{n=1}^{\infty}(-1)^n\left(q^{n(5n-1)/2}+q^{n(5n+1)/2}\right), \tag{4.5}$$

and (4.3) is proven.

5. Concluding Remarks

It would be highly desirable to obtain simple q-hypergeometric proofs of Theorems 1 and 2 in this paper or the corresponding theorems in Andrews' paper. A shift from $k \to k-2n$ on the left-hand side of (1.12), followed by an interchange of sums, yields the following identity after comparing coefficients of y_k:

$$S(k) = \sum_{n=1}^{\lfloor (k-1)/2 \rfloor} (-1)^k q^{(k-2n-1)(k-2n-2)+2n+1D_{n-2}} \times \left(q^{k-2n-2} \begin{bmatrix} n-1 \\ k-2n-1 \end{bmatrix} - \begin{bmatrix} n \\ k-2n-2 \end{bmatrix} \right) := T(k), \tag{5.1}$$

where

$$S(k) = \begin{cases} (-1)^k q^{k(5k-1)/2} & \text{if } k \equiv 3 \pmod 5, \\ (-1)^k q^{k(5k+1)/2} & \text{if } k \equiv 4 \pmod 5, \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

Christoph Koutschan has used his package HolonomicFunctions to provide a computer proof of this equality. His package generates the simple recurrence relation $T(k+5) = -q^{k+4}T(k)$, and it is not difficult to check that $S(k)$ satisfies this as well.

Also interesting would be further analytic or partition theoretic generalizations of this type.

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