Journal of Applied Analysis
Vol. 1, No. 2 (1995), pp. 159–172

GLOBAL EXISTENCE OF SOLUTIONS FOR DIRICHLET PROBLEM TO NONLINEAR DIAGONAL PARABOLIC SYSTEM WITH MAXIMAL GROWTH CONDITIONS

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Abstract. The Dirichlet problem to nonlinear diagonal parabolic system with some special right–hand sides but still satisfying the maximum growth conditions is considered. First applying the idea of Stampacchia a L_∞ – estimate in terms of data for a priori bounded weak solutions is found. Next following the methods of Ladyzhenskaya and Uraltseva the Hölder continuity with some exponent is proved. Next applying the results of Amann global existence of solutions is shown.

1991 *Mathematics Subject Classification.* 35B50, 35D10, 35K50, 35K65.
Key words and phrases. Nonlinear diagonal parabolic system, regularity of solutions, global existence.

1. Introduction. In this paper we consider the Dirichlet problem for the following diagonal parabolic system

$$
\begin{aligned}
&u_{it} - \nabla\cdot(a_i(x,t,u,\nabla u)\cdot\nabla u_i) = b_i(x,t,u,\nabla u) && \text{in } \Omega^T=\Omega\times(0,T)\,, \qquad (1.1)\\
&u_i|_{t=0} = u_{0i} && \text{in } \Omega\,,\\
&u_i = u_{bi} && \text{on } S^T = S\times(0,T)\,,
\end{aligned}
$$

where $i=1,\dots,m$, $\Omega\subset\mathbb{R}^n$ is a bounded domain with a sufficiently smooth boundary S, $\nabla u_i = (\partial_{x_1}u_i,\dots,\partial_{x_n}u_i)$, $u=(u_1,\dots,u_m)\in\mathbb{R}^m$, $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, and dot denotes the scalar product in $\mathbb{R}^n$.

The aim of this paper is to prove some estimates for solutions of (1.1) and next to show existence. Therefore, we assume some growth conditions. First we impose that

$$a_i:\Omega^T\times\mathbb{R}^m\times\mathbb{R}^{mn}\to\mathbb{R}^{n^2}\,,\quad i=1,\dots,m\,,$$

satisfy the Caratheodory condition where

$$
\begin{aligned}
\alpha_0|\nabla u_i|^2 - \phi_{1i}(x,t) &\le a_i\cdot\nabla u_i\cdot\nabla u_i = \qquad (1.2)\\
&= \sum_{\alpha,\beta=1}^n a_i^{\alpha\beta}\cdot\partial_{x_\alpha}u_i\,\partial_{x_\beta}u_i \le \gamma_0|\nabla u_i|^2\,,
\end{aligned}
$$

and α_0, γ_0 are positive constants and ϕ_{1i}, $i=1,\dots,m$, are some positive functions.

Moreover,

$$b_i:\Omega^T\times\mathbb{R}^m\times\mathbb{R}^{mn}\to\mathbb{R}\,,\quad i=1,\dots,m\,,$$

satisfy the Caratheodory condition and are such that

$$b_i(x,t,u,\nabla u)\le\beta_0|\nabla u_i|^2+\phi_{2i}(x,t)\,,\quad i=1,\dots,m\,, \qquad (1.3)$$

where β_0 is a positive constant and ϕ_{2i} are some positive functions.

Definition 1.1. We call $u=(u_1,\dots,u_m)$ to be a weak bounded solution to problem (1.1) if

$$u\in C([0,T];L_2(\Omega;\mathbb{R}^m))\cap L_2(0,T;W_2^1(\Omega;\mathbb{R}^m))\,,$$

u is bounded and satisfy the following integral identity

$$\sum_{i=1}^m\int_{\Omega^{T-h}}(u_{iht}\phi_i+(a_i\nabla u_i)_h\cdot\nabla\phi_i-b_{ih}\phi_i)\,dx\,dt=0\,, \qquad (1.4)$$

where $u_h = \frac{1}{h}\int_t^{t+h} u(x,\tau)\,d\tau$ is the Steklov average, which holds for any $\phi \in L_2(0,T; \overset{\circ}{W}{}_2^1(\Omega; I\!R^m)) \cap L_\infty(\Omega^T)$.

Our aim is to prove global existence of solutions to problem (1.1). First we find L_∞ – estimate. To show the estimate the idea of Stampacchia has been used (see [8]). It can be formulated as follows. Assuming that solutions of (1.1) are qualitatively bounded we find a quantitative $L_\infty(\Omega^T)$ estimate depending only upon the data. The method of Stampacchia is used by DiBenedetto to obtain L_∞ – estimate for solutions of parabolic equations (see [2, Ch.5, Sect.17]). Next dropping the dependence on ∇u in a_i, $i = 1,\dots,m$, and assuming some special dependence on $u_1,\dots,u_m$ in a_i, $i = 1,\dots,m$ we apply the result of Amann, Theorem 3 from [1] to prove global existence. On the other hand using the L_∞ – estimate and applying the results of Giaquinta – Struwe [4], [5] and Struwe [9] the Hölder continuity of solutions of (1.1) can be proved in the case where the matrices a_i, $i = 1,\dots,m$, do not depend on ∇u and the following restriction holds

$$\alpha_0 > 2\beta_0 \max_{i\in\{1,\dots,m\}} \| u_i \|_{L_\infty(\Omega^T)} . \tag{1.5}$$

Then using another result of Amann, Theorem 2 from [1], we prove global existence of solutions of (1.1), where a_i, $i = 1,\dots,m$, do not depend on ∇u, but there is no restrictions on dependence on u.

Moreover, to find L_∞ – estimate we generalize on the one hand the results of Ladyzhenskaya and Uraltseva (see [6, Ch.7, Th.7.1]) because the coefficient β_0 in (1.3) need not to be small but on the other hand we have more restrictive form of the r.h.s. (see also (1.3)).

By $W_p^1(\Omega; I\!R^m)$, $L_p(\Omega; I\!R^m)$, $C^\alpha(\Omega; I\!R^m)$ we denote commonly used the Sobolev and Hölder spaces for functions with values in $I\!R^m$. By $C^{\alpha,\alpha/2}(\Omega^T)$ we mean the anisotropic Hölder space (see [6, Ch.1]) and

$$L_{p,q}(\Omega^T) = L_q(0,T; L_p(\Omega)),$$

$$\overset{\circ}{W}{}_p^1(\Omega; I\!R^m) = \left\{ u \in W_p^1(\Omega; I\!R^m) : u|_{\partial\Omega} = 0 \right\}.$$

Finally we introduce that

$$\| u \|_{L_p(Q)} = |u|_{p,Q}, \quad \text{where } Q \text{ is a domain in } I\!R^s, \quad s \ge 1 .$$

We denote by $[\alpha]$ the integer part of α.

2. L_∞ **– estimate.** To obtain the estimate we follow with some modifications the proof of DiBenedetto from [2, Ch. 5, Sect. 17].

Lemma 2.1. *Let* $u = (u_1, \dots, u_m)$ *be a qualitatively bounded weak solution of (1.1) in* Ω^T. *Let* $u_0 \in L_\infty(\Omega; \mathbb{R}^m)$, $u_b \in L_\infty(S^T; \mathbb{R}^m)$ *and* $u \in L_p(\Omega^T; \mathbb{R}^m)$, $p \geq 1$. *Let* $\phi_i \in L_{p_i}(\Omega^T; \mathbb{R}^m)$, $i = 1, 2$, $p_1 = \frac{n+2}{2-n\kappa_0}$, $p_2 = \frac{2(n+2)}{4-n\kappa_0}$, $\kappa_0 > 0$, *where* κ_0 *is such that* $p_i > \frac{n+2}{2}$, $i = 1, 2$.

Then there exists a constant c_i *that can be determined quantitatively only in terms of the data, such that*

$$|u_i|_{\infty,\Omega^T} \leq c_i \max\left\{|u_{0i}|_{\infty,\Omega}, |u_{bi}|_{\infty,\Omega}, |u_i|_{p,\Omega^T}\right\}, \tag{2.1}$$

where $i = 1, \dots, m$, $c_i = f(n, \alpha_0, \kappa_0, p)\left(|\phi_{1i}|_{p_1,\Omega^T}^{\frac{n+2}{n\kappa_0 p}} + |\phi_{2i}|_{p_2,\Omega^T}^{\frac{2(n+2)}{n\kappa_0 p}}\right)$,

Proof. In view of the structure conditions the proof can be done for each component u_i, $i = 1, \dots, m$, separately. Assume that u_i, $i = 1, \dots, m$, is non–negative. Assume that M is the essential supremum of u_i, $i = 1, \dots, m$, in Ω^T, which is such that $|u_{0i}|_{\infty,\Omega} < \frac{1}{2}M$, $|u_{bi}|_{\infty,S^T} < \frac{1}{2}M$, $i = 1, \dots, m$. Let $k \in \mathbb{R}_+$ be such a number that

$$|u_{bi}|_{\infty,S^T} \leq k < M, \quad |u_{0i}|_{\infty,\Omega} \leq k < M.$$

Inserting $\phi_i = (u_{hi} - k)_+ = \max\{u_{hi} - k, 0\}$, $\phi_j = 0$ for $j \neq i$ into (1.4), performing the calculations in the first integral, passing with h to zero and using the structure conditions (1.2) and (1.3) we obtain

$$\begin{aligned} &\frac{1}{2}\int_\Omega (u_i - k)_+^2\,dx + \alpha_0 \int_{\Omega^t} |\nabla(u_i - k)_+|^2\,dx\,dt \leq \\ &\leq \beta_0 \int_{\Omega^t} |\nabla(u_i - k)_+|^2 (u_i - k)_+\,dx\,dt + \\ &+ \int_{\Omega^t} [\phi_{1i}\chi(u_i > k) + \phi_{2i}(u_i - k)_+]\,dx\,dt, \quad i = 1, \dots, m, \end{aligned} \tag{2.2}$$

where $\chi(u_i > k)$ is the characteristic function of the set

$$\left\{(x,t) \in \Omega^T : u_i(x,t) > k\right\}.$$

In view of the Hölder inequality we get

$$\begin{aligned} &\frac{1}{2}\int_\Omega (u_i - k)_+^2\,dx + \alpha_0 \int_{\Omega^t} |\nabla(u_i - k)_+|^2\,dx\,dt \leq \\ &\leq \beta_0 \int_{\Omega^t} |\nabla(u_i - k)_+|^2 (u_i - k)_+\,dx\,dt + \\ &+|\phi_{1i}|_{p_1,\Omega^T}|A_{k,i}^+|^{1/p_1'} + |\phi_{2i}|_{p_2,\Omega^T}\left(\int_{\Omega^t} (u_i - k)_+^{p_2'}\,dx\,dt\right)^{1/p_2'}, \end{aligned} \tag{2.3}$$

where $\frac{1}{p_i} + \frac{1}{p_i'} = 1$, $i = 1, 2$, and $|A^+_{k,i}| = \text{meas}\Big\{(x,t) \in \Omega^T : u_i(x,t) > k\Big\}$.

Now, we examine the last term on the r.h.s. of (2.3). In view of the Hölder inequality it is bounded by

$$|\phi_{2i}|_{p_2,\Omega^T} |A^+_{k,i}|^{1/\lambda_1 p_2'} \left(\int_{\Omega^t} (u_i - k)_+^{\lambda_2 p_2'}\, dx\, dt\right)^{1/\lambda_2 p_2'} \equiv I\,,$$

where $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = 1$. Assuming now $\lambda_1 = \frac{2(n+2)}{np_2'}$ and using the imbedding theorem for the parabolic norm with a constant γ (see [3, Ch.1, Sect.3]) we have

$$I \leq \gamma |\phi_{2i}|_{p_2,\Omega^T} |A^+_{k,i}|^{1/\lambda_1 p_2'} \left[\left(\text{ess}\ \ \sup_{t\in[0,T]} \int_\Omega (u_i - k)_+^2\, dx\right)^{1/2} + \right.$$

$$\left. + \left(\int_{\Omega^t} |\nabla (u_i - k)_+|^2\, dx\, dt\right)\right] \leq$$

$$\leq \frac{\varepsilon_1}{2} \text{ess}\ \ \sup_{t\in[0,T]} \int_\Omega (u_i - k)_+^2\, dx + \frac{\gamma^2}{2\varepsilon_1} |\phi_{2i}|^2_{p_2,\Omega^T} |A^+_{k,i}|^{2/p_2' - \frac{n}{n+2}} +$$

$$+ \frac{\varepsilon_2}{2} \int_{\Omega^t} |\nabla (u_i - k)_+|^2\, dx\, dt + \frac{\gamma^2}{2\varepsilon_2} |\phi_{2i}|^2_{p_2,\Omega^T} |A^+_{k,i}|^{2/p_2' - \frac{n}{n+2}}\,,$$

where the Young inequality and the relation $\frac{1}{\lambda_1} = 1 - \frac{np_2'}{2(n+2)}$ were used.

Taking $\varepsilon_1 = \frac{1}{2}$, $\varepsilon_2 = \alpha_0$ and using (2.3) yields

$$\frac{1}{2} \text{ess}\ \ \sup_{t\in[0,T]} \int_\Omega (u_i - k)_+^2\, dx + \alpha_0 \int_{\Omega^t} |\nabla (u_i - k)_+|^2\, dx\, dt \leq \tag{2.4}$$

$$\leq 2\beta_0 \int_{\Omega^t} |\nabla (u_i - k)_+|^2 (u_i - k)_+\, dx\, dt +$$

$$+ 2|\phi_{1i}|_{p_1,\Omega^T} |A^+_{k,i}|^{1/p_1'} + 2\left(1 + \frac{1}{\alpha_0}\right) \gamma^2 |\phi_{2i}|^2_{p_2,\Omega^T} |A^+_{k,i}|^{2/p_2' - \frac{n}{n+2}}\,.$$

Since we assumed that there exists a number M such that $u_i \leq M$, $i = 1, \dots, m$, we can choose $k = M - 2\varepsilon$, where $\varepsilon \in (0,1)$ is so small that $M - 2\varepsilon \geq \max\{|u_{0i}|_{\infty,\Omega}, |u_{bi}|_{\infty,S^T}\}$ and $2\varepsilon \leq \frac{\alpha_0}{4\beta_0}$. Then the first term on the r.h.s. of (2.4) can be estimated by the second term on the l.h.s. of (2.4). Therefore, for such a k the inequality (2.4) takes the form

$$\text{ess}\ \ \sup_{t\in[0,T]} \int_\Omega (u_i - k)_+^2\, dx + \alpha_0 \int_{\Omega^t} |\nabla (u_i - k)_+|^2\, dx\, dt \leq \tag{2.5}$$

$$\leq c_1 |A^+_{k,i}|^{1/p_1'} + c_2 |A^+_{k,i}|^{2/p_2' - \frac{n}{n+2}}\,,$$

where $c_1 = 4|\phi_{1i}|_{p_1,\Omega^T}$, $c_2 = 4\left(1+\frac{1}{\alpha_0}\right)\gamma^2|\phi_{2i}|^2_{p_2,\Omega^T}$.

Consider now the sequence of increasing levels

$$k_s = M - \varepsilon - \frac{\varepsilon}{2^s}, \quad s = 0,1,\dots, \tag{2.6}$$

and the corresponding family of sets

$$A^+_{k_s,i} = \left\{(x,t)\in\Omega^T : u_i(x,t) > k_s\right\}. \tag{2.7}$$

Using (2.6) and (2.7) in (2.5) yields

$$\text{ess }\sup_{t\in[0,T]}\int_\Omega (u_i-k_s)^2_+\,dx + \alpha_0\int_{\Omega^t}|\nabla(u_i-k_s)_+|^2\,dx\,dt \le \tag{2.8}$$
$$\le c_1|A^+_{k_s,i}|^{1/p_1'} + c_2|A^+_{k_s,i}|^{2/p_2'-\frac{n}{n+2}}.$$

By the multiplicative inequality (3.1) from [2, Ch.1, Sect.3] and (2.8) we obtain

$$\left(\frac{\varepsilon}{2^{s+1}}\right)^{\frac{2(n+2)}{n}}|A^+_{k_{s+1},i}| \le \tag{2.9}$$
$$\le \int_{A^+_{k_{s+1},i}}(u_i-k_s)_+^{\frac{2(n+2)}{n}}\,dx\,dt \le \int_{\Omega^T}(u_i-k_s)_+^{\frac{2(n+2)}{n}}\,dx\,dt \le$$
$$\le \gamma^{\frac{n+2}{n}}\left(\text{ess }\sup_{t\in[0,T]}\int_\Omega(u_i-k_s)^2_+\,dx\right)^{2/n}\int_{\Omega^T}|\nabla(u_i-k_s)_+|^2\,dx\,dt \le$$
$$\le \frac{2}{\alpha_0}\gamma^{\frac{n+2}{n}}\left(c_1^{1+\frac{2}{n}}|A^+_{k_s,i}|^{\frac{1}{p_1'}(1+\frac{2}{n})} + c_2^{1+\frac{2}{n}}|A^+_{k_s,i}|^{\frac{2(n+2)}{p_2'n}-1}\right),$$

where γ is from the inequality.

To prove the required result we have to find such restrictions on p_1 and p_2 that the exponents on the r.h.s. of (2.9) are larger than 1. Assume that they are equal to $1+\kappa_0$, where $\kappa_0>0$. Then $p_1 = \frac{n+2}{2-n\kappa_0}$, $p_2 = \frac{2(n+2)}{4-n\kappa_0}$. Since $\kappa_0 = \frac{2}{n}-\frac{n+2}{np_1}>0$, $\kappa_0 = \frac{4}{n}-\frac{2(n+2)}{np_2}>0$ we have that $p_i > \frac{n+2}{2}$, $i=1,2$. Let $c_4 = \frac{2}{\alpha_0}\gamma^{\frac{n+2}{n}}(c_1^{\frac{n+2}{n}}+c_2^{\frac{n+2}{n}})$. Then (2.9) takes the form

$$\left(\frac{\varepsilon}{2^{s+1}}\right)^{\frac{2(n+2)}{n}}|A^+_{k_{s+1},i}| \le c_4|A^+_{k_s,i}|^{1+\kappa_0}. \tag{2.10}$$

Therefore, we get

$$|A^+_{k_{s+1},i}| \le c_5 b^s\varepsilon^{-\frac{2(n+2)}{n}}|A^+_{k_s,i}|^{1+\kappa_0}, \quad i=1,\dots,m, \tag{2.11}$$

where $c_5 = 2^{\frac{2(n+2)}{n}} c_4$, $b = 2^{\frac{2(n+2)}{n}}$.

From (2.11) it follows that $|A^+_{k_{s+1},i}| \to 0$ as $s \to \infty$ if

$$|A^+_{k_0,i}| \le \gamma_* \equiv \left(\frac{\varepsilon^{\frac{2(n+2)}{n}}}{c_5} \right)^{1/\kappa_0} b^{-1/\kappa_0^2}\,, \tag{2.12}$$

see either Lemma 4.1 [2], Ch.1, or Lemma 5.6 [6], Ch.2, or Lemma 4.7 [7], Ch.2.

In this case we have that

$$u_i \le M - \varepsilon \quad \text{a.e.} \quad \text{in} \quad \Omega^T\,,$$

which contradicts the definition of M.

Since $k_s = M - \varepsilon - \frac{\varepsilon}{2^s}$, $s \ge 0$, we have $k_0 = M - 2\varepsilon$ and we can take ε so small that $k_0 > \frac{M}{2}$. Then we have

$$\left(\frac{M}{2}\right)^p |A^+_{k_0,i}| \le \left(\frac{M}{2}\right)^p |A^+_{\frac{M}{2},i}| \le \int_{\Omega^T} |u_i|^p \, dx\, dt\,.$$

Hence,

$$|A^+_{k_0,i}| \le \left(\frac{2}{M}\right)^p \int_{\Omega^T} |u_i|^p \, dx\, dt\,. \tag{2.13}$$

If the r.h.s. is less than γ_* we have a contradiction. Thus,

$$\text{ess } \sup_{\Omega^T} u_i \le 2\gamma_*^{-1/p} \left(\int_{\Omega^T} |u_i|^p \, dx\, dt \right)^{1/p}\,, \tag{2.14}$$

$$i = 1, \dots, m\,, \quad p \ge 1\,.$$

If $u_i < 0$ we have to introduce the cut–off function

$$(u_i - k)_- = \max\{-(u_i - k), 0\}\,, \qquad k < 0\,.$$

Then we obtain a similar estimate from below. This concludes the proof. $\square$

Finally we have to obtain an estimate for $|u_i|_{p,\Omega^T}$, $i = 1, \dots, m$, $p \ge 1$. Therefore, we have

Lemma 2.2. *Let* $u_0 \in L_\infty(\Omega; I\!R^m)$, $u_b \in L_\infty(S^T; I\!R^m)$, $\phi_i \in L_\lambda(\Omega^T; I\!R^m)$, $\lambda > \frac{n+2}{2}$, $i = 1, 2$.
Then the following estimate holds

$$|u_i|_{\frac{n+2}{2},\Omega^T} \le g\left(|\phi_{1i}|_{\lambda,\Omega^T}, |\phi_{2i}|_{\lambda,\Omega^T} \right) \cdot \tag{2.15}$$

$$\cdot \left[1 + |\phi_{1i}|_{1,\Omega^T} + |\phi_{2i}|_{1,\Omega^T} + |u_{0i}|_{\infty,\Omega} + |u_{bi}|_{\infty,S^T} \right]\,,$$

where g is an increasing positive function.

Proof. Let $k_* = \max\{\text{ess} \quad \sup_\Omega |u_{0i}|, \text{ess} \quad \sup_{S^T} |u_{bi}|\}$, $i = 1, \ldots, m$. Let us assume that u_i is positive. Let $\phi_j = 0$ for $j \neq i$, $\phi_i = (u_{ih} - k_*)_+ e^{\alpha(u_{ih}-k_*)_+}$, $\alpha > 0$. Inserting ϕ_i into (1.4) yields

$$\int_{\Omega^t} u_{iht}(u_{ih} - k_*)_+ e^{\alpha(u_{ih}-k_*)_+} \, dx \, dt +$$

$$+ \int_{\Omega^t} (a_i(x,t,u,\nabla u)\nabla u_i)_h \nabla \left((u_{ih} - k_*)_+ e^{\alpha(u_{ih}-k_*)_+} \right) dx \, dt =$$

$$= \int_{\Omega^t} (b_i(x,t,u,\nabla u))_h (u_{ih} - k_*)_+ e^{\alpha(u_{ih}-k_*)_+} \, dx \, dt.$$

Performing calculations in the first integral, passing with h to zero and using the structure conditions we get

$$\begin{aligned} &\frac{1}{\alpha} \int_\Omega \left[(u_i - k_*)_+ - \frac{1}{\alpha} \right] e^{\alpha(u_i-k_*)_+} \, dx + \\ &+\alpha_0 \int_{\Omega^t} |\nabla(u_i - k_*)_+|^2 \left(1 + \alpha(u_i - k_*)_+\right) e^{\alpha(u_i-k_*)_+} \, dx \, dt \leq \\ &\leq \int_{\Omega^t} \phi_{1i} \left(1 + \alpha(u_i - k_*)_+\right) e^{\alpha(u_i-k_*)_+} \, dx \, dt + \\ &+\beta_0 \int_{\Omega^t} |\nabla(u_i - k_*)_+|^2 (u_i - k_*)_+ e^{\alpha(u_i-k_*)_+} \, dx \, dt + \\ &+ \int_{\Omega^t} \phi_{2i} (u_i - k_*)_+ e^{\alpha(u_i-k_*)_+} \, dx \, dt \, , \end{aligned} \tag{2.16}$$

where we used that

$$\left[(u_i - k_*)_+ - \frac{1}{\alpha} \right] e^{\alpha(u_i-k_*)_+} \Big|_{t=0} = -\frac{1}{\alpha} < 0 \, .$$

Assuming that $\alpha > \frac{2\beta_0}{\alpha_0}$ we get from (2.16)

$$\begin{aligned} &\frac{1}{\alpha} \int_\Omega \left[(u_i - k_*)_+ - \frac{1}{\alpha} \right] e^{\alpha(u_i-k_*)_+} \, dx + \\ &+\alpha_0 \int_{\Omega^t} |\nabla(u_i - k_*)_+|^2 \left(1 + \frac{\alpha}{2}(u_i - k_*)_+\right) e^{\alpha(u_i-k_*)_+} \, dx \, dt \leq \\ &\leq \int_{\Omega^t} \left[\phi_{1i} \left(1 + \alpha(u_i - k_*)_+\right) + \phi_{2i}(u_i - k_*)_+ \right] e^{\alpha(u_i-k_*)_+} \, dx \, dt \, . \end{aligned} \tag{2.17}$$

To simplify notation we introduce the functions $v_i = (u_i - k_*)_+ > 0$ and $\phi_{0i} = \max\{\phi_{1i}, \phi_{2i}\}$. Then we write (2.17) in the following form

$$\frac{1}{\alpha}\int_\Omega \left(v_i - \frac{1}{\alpha}\right) e^{\alpha v_i}\, dx + \alpha_0 \int_{\Omega^t} |\nabla v_i|^2 \left(1 + \frac{\alpha}{2} v_i\right) e^{\alpha v_i}\, dx\, dt \le \tag{2.18}$$

$$\le (1+\alpha)\int_{\Omega^t} (1+\phi_{0i})(1+v_i) e^{\alpha v_i}\, dx\, dt\,, \quad i = 1, \dots, m\,.$$

Let

$$\omega_i = (v_i - 1)_+^{1/2} e^{\frac{\alpha}{2}(v_i - 1)_+}\,, \quad i = 1, \dots, m\,. \tag{2.19}$$

Let $\alpha > 1$ and $v_i > 1$. For $v_i \le 1$ we have the sup estimate for v_i so there is nothing to prove. Then we have

$$\int_\Omega \left(v_i - \frac{1}{\alpha}\right) e^{\alpha v_i}\, dx = \int_\Omega (v_i - 1)\, e^{\alpha v_i}\, dx + \left(1 - \frac{1}{\alpha}\right)\int_\Omega e^{\alpha v_i}\, dx \ge \tag{2.20}$$

$$\int_\Omega (v_i - 1)\, e^{\alpha v_i}\, dx \ge \int_\Omega (v_i - 1)\, e^{\alpha (v_i - 1)}\, dx = \int_\Omega \omega_i^2\, dx\,.$$

Since $v_i > 1$ we have that $\omega_i = (v_i - 1)^{1/2} e^{\frac{\alpha}{2}(v_i - 1)}$, so

$$\nabla \omega_i = \frac{1}{2}\left[(v_i - 1)^{-1/2} + \alpha (v_i - 1)^{1/2}\right] \nabla v_i e^{\frac{\alpha}{2}(v_i - 1)}\,,$$

and

$$|\nabla \omega_i|^2 \le c_1 (1 + v_i) |\nabla v_i|^2 e^{\alpha v_i}\,, \tag{2.21}$$

where in reality we used that $v_i \ge c > 1$.
Finally $\omega_i^2 = (v_i - 1) e^{\alpha(v_i - 1)} \ge c_2 e^{\alpha v_i}$, and also $\omega_i^2 \ge e^{-\alpha}\left[v_i e^{\alpha v_i} - e^{\alpha v_i}\right]$, so $v_i e^{\alpha v_i} \le e^{\alpha}\omega_i^2 + e^{\alpha v_i} \le c_3 \omega_i^2$.
Therefore, we have proved

$$(1 + v_i) e^{\alpha v_i} \le c_4 \omega_i^2\,. \tag{2.22}$$

Using (2.19)–(2.22) in (2.18) implies

$$\text{ess } \sup_{t \in [0,T]} \int_\Omega \omega_i^2\, dx + \int_{\Omega^T} |\nabla \omega_i|^2\, dx\, dt \le \tag{2.23}$$

$$\le c_5 \int_{\Omega^T} (1 + \phi_{0i})(1 + \omega_i^2)\, dx\, dt\,.$$

Using the imbedding (3.1) from [2], Ch.1, Sect.3 for the space $V_0^2(\Omega^T) = L_\infty(0,T; L_2(\Omega)) \cap L_2(0,T; \overset{\circ}{W}{}_2^1(\Omega))$ and the Hölder inequality on the r.h.s.

of (2.23) we get

$$|\omega_i|_{\frac{2(n+2)}{n},\Omega^T} \le c_6 + c_5^{1/2}\left(\int_{\Omega^T}(1+\phi_{0i})^{\lambda_1}\,dx\,dt\right)^{1/2\lambda_1} \tag{2.24}$$

$$\cdot\left(\int_{\Omega^T}|\omega_i|^{2\lambda_2}\,dx\,dt\right)^{1/2\lambda_2} \le c_6 + c_7|\Omega^T|^{\frac{1}{2\lambda_2}-\frac{n}{2(n+2)}}|\omega_i|_{\frac{2(n+2)}{n},\Omega^T},$$

where $c_6=\gamma c_5^{1/2}\left(\int_{\Omega^T}(1+\phi_{0i})\,dx\,dt\right)^{1/2}$, $c_7=\gamma c_5^{1/2}\left(\int_{\Omega^T}(1+\phi_{0i})^{\lambda_1}\,dx\,dt\right)^{1/2\lambda_1}$, $\frac{1}{\lambda_1}+\frac{1}{\lambda_2}=1$, $\lambda_2<\frac{n+2}{n}$, $\lambda_1>\frac{n+2}{n}$, and γ is the constant from the imbedding.

If T is so small that

$$c_7|\Omega^T|^{\frac{1}{2}\left(\frac{1}{\lambda_2}-\frac{n}{n+2}\right)} \le \frac{1}{2}$$

we obtain the estimate

$$|\omega_i|_{\frac{2(n+2)}{n},\Omega^T} \le 2c_6\,. \tag{2.25}$$

For arbitrary T the argument can be repeated up to covering the whole $[0,T]$ in a finite number of steps.

From (2.19) and the definition of v_i we have that

$$((u_i-k_*)_+-1)_+^{1/2} = \omega_i e^{-\frac{\alpha}{2}(v_i-1)_+}\,.$$

Hence, either

$$u_i \le k_*+1 \tag{2.26}$$

or $u_i>k_*+1$. In the second case we have the estimate

$$u_i^{1/2} \le c_8\left[(k_*+1)^{1/2}+\omega_i\right]\,.$$

Therefore

$$|u_i|_{\frac{n+2}{n},\Omega^T} \le c_9\left[(k_*+1)+|\omega_i|^2_{\frac{2(n+2)}{n},\Omega^T}\right] \le \tag{2.27}$$

$$\le c_9(k_*+1+4c_6^2)\,.$$

From (2.26) and (2.27) we obtain (2.15). This concludes the proof. □

From Lemmas 2.1, 2.2 we have

Theorem 2.3. *Let the assumptions of Lemmas 2.1, 2.2 be satisfied. Then a qualitatively bounded weak solution of (1.1) is bounded in terms of data (see the inequalities (2.1) and (2.15)).*

3. Hölder continuity. Using the methods of Ladyzhenskaya and Uraltsewa (see [6, Ch.5, Sect.1]) we obtain

Lemma 3.1. *Let the structure conditions (1.2), (1.3) hold.*
Let $u = (u_1, \dots, u_m)$, be bounded, so $|u_i| \le M$, $i = 1, \dots, m$, where M is a positive constant. Let $\phi_1 = (\phi_{11}, \dots, \phi_{1m})$, $\phi_2 = (\phi_{21}, \dots, \phi_{2m})$ be such that $\chi_i = |\phi_{1i}| + |\phi_{2i}|^2 \in L_{q,r}(\Omega^T)$, $\frac{1}{r} + \frac{n}{2q} = 1 - \frac{n\kappa}{2}$, $\frac{n\kappa}{2} \in (0,1)$, $n \ge 2$, $i = 1, \dots, m$. Moreover, let $u_0 \in C^\alpha(\Omega, I\!\!R^m)$, $u_b \in C^\alpha(S^T, I\!\!R^m)$, $\alpha \in (0,1)$.

Then $u_i \in C^{\alpha,\alpha/2}(\Omega^T \cup S^T \cup \Omega \times \{0\})$, where $\alpha = \min\left\{-\log_4\left(1 - \frac{1}{2^s}\right), \frac{n\kappa}{2}\right\}$, $s = \left[\frac{2M}{\delta}\right] + 4 + s_0$, $s_0 > 0$,$\delta = \frac{\alpha_0}{4\beta_0}$, $i = 1, \dots, m$.

Proof. Putting $\phi_i = (u_{ih} - k)_+ \zeta^2(x,t)$ and $\phi_j = 0$ for $j \ne i$ into (1.4) we get

$$\int_0^t \int_\Omega \Big[u_{iht}(u_{ih} - k)_+ \zeta^2 + (a_i \nabla u_i)_h \nabla \left((u_{ih} - k)_+ \zeta^2 \right) + \tag{3.1}$$
$$- b_{ih}(u_{ih} - k)_+ \zeta^2 \Big] \, dx \, dt = 0 \,,$$

where $\zeta(x,t)$ is an arbitrary, nonnegative, continuous piece–wise smooth function vanishing on the boundary. In the case when ζ does not vanish near the boundary we assume that $k > \max_i \operatorname{ess} \ \sup_{S^T} |u_{bi}(x,t)|$.
Performing the calculations in the first term, passing with h to zero and using that $k > \max_i \operatorname{ess} \ \sup_\Omega |u_{0i}|$ we get

$$\frac{1}{2} \int_\Omega (u_{ih} - k)_+^2 \zeta^2 \, dx - \int_{\Omega^t} (u_i - k)_+^2 \zeta \zeta_t \, dx \, dt + \tag{3.2}$$
$$+ \int_{\Omega^t} \Big(a_i \cdot \nabla(u_i - k)_+ \cdot \nabla(u_i - k)_+ \zeta^2 \, dx \, dt +$$
$$+ 2 a_i \cdot u_i (u_i - k)_+ \zeta \nabla \zeta \Big) \, dx \, dt +$$
$$- \int_{\Omega^t} b_{ih}(u_{ih} - k)_+ \zeta^2 \, dx \, dt = 0 \,.$$

Using the structure conditions (1.2), (1.3), we obtain the inequality

$$\frac{1}{2} \int_\Omega (u_{ih} - k)_+^2 \zeta^2 \, dx + \frac{\alpha_0}{2} \int_{\Omega^t} |\nabla(u_i - k)_+|^2 \zeta^2 \, dx \, dt \le \tag{3.3}$$

$$\leq \beta_0 \int_{\Omega^t} |\nabla(u_i-k)_+|^2 \zeta^2 (u_i-k)_+ \,dx\,dt +$$

$$+ \int_{\Omega^t} (u_{ih}-k)_+^2 \left(\zeta|\zeta_t| + \frac{2\gamma_0^2}{\alpha_0}\zeta_x^2 \right) dx\,dt +$$

$$+ \int_{\Omega^t} \left(\phi_{1i}\chi_{A^+_{k,i,\rho}} + \phi_{2i}(u_{ih}-k)_+ \right) \zeta^2 \,dx\,dt\,,$$

where χ_Q is the characteristic function of Q, $A^+_{k,i,\rho} = A^+_{k,i} \bigcap B_\rho(x_0)$, $B_\rho(x_0) = \{x : |x-x_0| < \rho\}$, $A^+_{k,i}(t) = \{x : u_i(x,t) - k > 0\}$. Moreover, we introduce cylinders $Q(\rho,\tau) = B_\rho(x_0) \times (t_0, t_0+\tau)$, and assume that supp $\zeta \subset Q(\rho,\tau)$ and $\zeta(x,t) = 1$ for $(x,t) \in Q(\rho - \sigma_1\rho, \tau - \sigma_2\tau)$, for some σ_1, σ_2 less than 1.

Taking now k such that

$$(u_i - k)_+ \leq \delta = \frac{\alpha_0}{4\beta_0}$$

and using the properties of $\zeta(x,t)$ we obtain from (3.3),

$$\sup_{t\in[0,T]} |(u_i-k)_+|^2_{2,B_{\rho-\sigma_1\rho}} + |\nabla(u_i-k)_+|^2_{2,Q(\rho-\sigma_1\rho,\tau-\sigma_2\tau)} \leq \tag{3.4}$$

$$\leq \gamma_1 \left[\left(\frac{1}{(\sigma_1\rho)^2} + \frac{1}{\sigma_2\tau} \right) |(u_i-k)_+|^2_{2,Q(\rho,\tau)} + \int_0^t dt \int_{A^+_{k,i,\rho}} \chi_i \,dx\,dt \right],$$

where $\gamma_1 = c_1\left(1 + \frac{\gamma_0^2}{\alpha_0}\right)$, $\chi_i = |\phi_{1i}| + |\phi_{2i}|^2$.

The last term on the r.h.s. of (3.4) we estimate by (see [6, Ch.3, Sect.10])

$$\| \chi_i \|_{L_{q,r}(Q_{k,i}(\rho,\tau))} \, \mu_i^{\frac{2(1+\kappa)}{\hat r}} \left(k, \frac{\hat r}{\hat q}, \rho, \tau \right),$$

where

$$\frac{1}{r} + \frac{n}{2q} = 1 - \kappa_1\,, \quad \kappa_1 \in (0,1) \quad \text{for } n \geq 2\,,$$

$$Q_{k,i}(\rho,\tau) = \{(x,t) \in Q(\rho,\tau) : u_i(x,t) > k\}\,,$$

$$\mu_i\left(k, \frac{\hat r}{\hat q}, \rho, \tau\right) = \int_{t_0}^{t_0+\tau} \text{meas}^{\frac{\hat r}{\hat q}} A_{k,i,\rho}(t)\,dt\,,$$

$$\hat{q} = \bar{q}(1+\kappa)\,, \quad \hat{r} = \bar{r}(1+\kappa)\,,$$

$$\bar{q} = \frac{2q}{q-1}\,, \quad \bar{r} = \frac{2r}{r-1}\,, \quad \kappa = \frac{2\kappa_1}{n}\,, \quad \frac{1}{\bar{r}} + \frac{n}{2\bar{q}} = \frac{n}{4} + \frac{\kappa_1}{2}\,,$$

$$\frac{1}{\hat{r}} + \frac{n}{2\hat{q}} = \frac{n}{4}\,, \qquad \text{for } n \geq 2\,.$$

Therefore, $u_i \in \mathcal{B}_2(\Omega^T, M, \gamma_1, \hat{r}, \delta, \kappa)$ (see [6, Ch.2, Sect.7]). Hence, in view of Theorem 7.1 from [6], Ch.2, Sect.7, and Theorem 8.1 from [6], Ch.2, Sect.8, we have that

$$u_i \in C^{\alpha,\alpha/2}(\Omega^T \cup S^T \cup \Omega \times \{0\})\,,$$

where $\alpha = \min\{-\log_4(1-\frac{1}{2^s}), \frac{n\kappa}{2}\}$, $s = \left[\frac{2M}{\delta}\right]+4+s_0$ and s_0 is some positive constant.

Since $s > 4$ we see that α is a very small positive number. This concludes the proof. □

4. Global existence. To prove global existence of solution to problem (1.1) we apply Theorems 2 and 3 from [1]. In the case of Theorem 2 we need Hölder continuity of solutions with $\alpha > \frac{n}{n+1}$. Therefore Theorem 3.1 can not be used because there α is too small. Then the result of Struwe form [9] must be applied, where however very strong structure condition (1.5) must be imposed. To apply Theorem 3 we need only boundedness of solutions, so Theorem 2.3 is sufficient, but then time independent boundary conditions and some additional strong structural restrictions must be added on the form of system (1.1).

Summarizing we have

Theorem 4.1. *Let the assumptions of Theorem 2.3 and Theorem 3 from* [1] *be satisfied. Then there exists a global solution to problem (1.1) in* $C([0,T]; C^s(\bar{\Omega} \times \mathbb{R}^m))$, $s < 1$.

Theorem 4.2. *Let the assumptions of Theorem 2.3, Theorem 2 from* [1], *and Theorem 2 from* [9] *be satisfied. Then there exists a qualitatively bounded solution to problem (1.1) in* $C^{\varepsilon}([0,T]; C^{\theta}(\bar{\Omega}; \mathbb{R}^m))$, *where* $\varepsilon > 0$, $\theta > \frac{n}{n+1}$.

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