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CONTINUITY OF THE SUPERPOSITION OF SET–VALUED FUNCTIONS

N. MERENTES, K. NIKODEM, AND S. RIVAS

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Abstract. Let T, X, Y be topological spaces and $F: T \times X \mapsto n(Y)$ be a set–valued function. We consider the Nemytskii operator generated by F which associates with every set–valued function $G: T \mapsto n(X)$ the superposition $F(\cdot, G(\cdot)) : T \mapsto n(Y)$. Conditions under which this superposition is lower or upper semicontinuous are presented.

1. Introduction. Let T, X and Y be topological spaces, $n(Y)$ be the family of all nonempty subsets of Y and $F: T \times X \mapsto n(Y)$ be a set–valued function (s.v. function, for short). We consider the *Nemytskii operator* N generated by F, i.e. the *superposition operator* defined by

$$N(G)(t) := F(t, G(t)) := \bigcup_{x \in G(t)} F(t, x)$$

for every s.v. function $G: T \mapsto n(X)$. For single–valued functions this operator plays an important role in nonlinear analysis. Also for s.v. functions it appears in many fields of applied mathematics, e.g. in control theory and mathematical economics. In this note we present conditions under which

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the s.v. functions $F(\cdot, G(\cdot))$ are lower or upper semicontinuous. Such problem was considered in [1] in the case where F is a s.v. function and G is single–valued and in [2] in the case where F is single–valued and G is set–valued. In this paper F as well as G are s.v. functions.

Let us recall that an s.v. function $\mathcal{H} : X \mapsto n(Y)$ is *lower semicontinuous* (l.s.c.) at a point $x_0 \in X$ if for every open set $W \subset Y$ such that $\mathcal{H}(x_0) \cap W \neq \emptyset$, there exists a neighbourhood U of x_0 such that $\mathcal{H}(X) \cap W \neq \emptyset$ for all $x \in U$. $\mathcal{H}$ is *upper semicontinuous* (u.s.c.) at x_0 if for every open set $W \subset Y$ with $\mathcal{H}(x_0) \subset W$ there exists a neighbourhood U of x_0 such that $\mathcal{H}(x) \subset W$ for all $x \in U$. $\mathcal{H}$ is *continuous* at x_0 if it is l.s.c. and u.s.c. at x_0.

2. The superposition operators generated by l.s.c. and u.s.c. s.v. functions. In this section T, X and Y are topological spaces. We denote by $c(Y)$ the family of all non–empty compact subsets of Y.

Proposition 1. *If $F : T \times X \mapsto n(Y)$ is u.s.c. and $G : T \mapsto c(X)$ is u.s.c., then $F(\cdot, G(\cdot)) : T \mapsto n(Y)$ is u.s.c.*

Proof. Fix a point $t_0 \in T$ and take an open set $W \subset Y$ such that $F(t_0, G(t_0)) \subset W$. Then $F(t_0, x) \subset W$ for all $x \in G(t_0)$. For every $x \in G(t_0)$, by the u.s.c. of F at (t_0, x), there exist a neighbourghood U_x of t_0 and a neighbourghood V_x of x such that

$$F(t, x^{'}) \subset W \qquad \text{for all} \qquad (t, x') \in U_x \times V_x. \tag{1}$$

Since $G(t_0)$ is compact and $G(t_0) \subset \bigcup_{x \in G(t_0)} V_x$, there exists points $x_1, \dots, x_n \in G(x)$ such that

$$G(t_0) \subset V_{x_1} \cup \ldots \cup V_{x_n}.$$

Put

$$U := U_{x_1} \cap \ldots \cap U_{x_n} \qquad \text{and} \qquad V := V_{x_1} \cup \ldots \cup V_{x_n}.$$

By the u.s.c. of G at t_0 there exists a neighbourhood $\overline{U} \subset U$ of t_0 such that

$$G(t) \subset V \qquad \text{for all} \quad t \in \overline{U}.$$

Then, for every $t \in \overline{U}$ and $x' \in G(t)$ there is an $i \in \{1, \dots, n\}$ for which $(t, x') \in U_{x_i} \times V_{x_i}$. Using (1) we get $F(t, x') \subset W$. Thus

$$F(t, G(t)) \subset W \qquad \text{for all} \quad t \in \overline{U},$$

which proves that $F(\cdot, G(\cdot))$ is u.s.c. at t_0. □

Proposition 2. *If $F : T \times X \mapsto n(Y)$ is l.s.c. and $G : T \mapsto n(X)$ is l.s.c., then $F(\cdot, G(\cdot)) : T \mapsto n(Y)$ is l.s.c.*

Proof. Fix a point $t_0 \in T$ and take an open set $W \subset Y$ such that $F(t_0, G(t_0)) \cap W \neq \emptyset$. Then $F(t_0, x_0) \cap W \neq \emptyset$ for some $x_0 \in G(t_0)$. By the l.s.c. of F at (t_0, x_0) there exist a neighbourhood U of t_0 and a neighbourhood V of x_0 such that

$$F(t,x) \cap W \neq \emptyset \qquad \text{for all} \quad (t,x) \in U \times V. \tag{2}$$

Since $G(t_0) \cap V \neq \emptyset$, by the l.s.c. of G at t_0 there exists a neighbourhood $\overline{U} \subset U$ of t_0 such that

$$G(t) \cap V \neq \emptyset \qquad \text{for all} \quad t \in \overline{U}.$$

Fix a point $t \in \overline{U}$ and take an $x_t \in G(t) \cap V$. Then $(t, x_t) \in \overline{U} \times V$ and so, by (2), $F(t, x_t) \cap W \neq \emptyset$. Thus

$$F(t, G(t)) \cap W \neq \emptyset \qquad \text{for every} \quad t \in \overline{U},$$

which means that $F(\cdot, G(\cdot))$ is l.s.c. at t_0. □

REMARKS. An s.v. function $F: T \times X \mapsto n(Y)$ is called *superpositionally continuous* if for every continouos single–valued function $\varphi: T \mapsto X$ the superposition $F(\cdot, \varphi(\cdot)): T \mapsto n(Y)$ is continuous. As an immediate consequence of Propositions 1 and 2 we get that every continuous s.v. function $F: T \times X \mapsto n(Y)$ is superpositionally continuous. This result is formulated in [1, Lemma 2] for s.v. functions $F: \Omega \times \mathbb{R} \mapsto c(\mathbb{R})$, where Ω is a compact subset of the Euclidean space.

The superpositions $F(\cdot, G(\cdot))$ where $F: T \times X \mapsto Y$ is a single–valued continuous function and T, X, Y are metric spaces are considered in [2] as so called *parametrized set–valued maps*. In this case our Propositions 1 and 2 reduces to a result presented there (cf. [2, Prop. 1.4.14]).

3. The superposition operators generated by midconvex s.v. functions. In this section we consider the superposition operator generated by midconvex s.v. functions. Let X, Y be topological vector spaces and D be a convex subset of X. Recall that an s.v. function $\mathcal{H}: D \mapsto n(Y)$ is said to be *midconvex* if

$$\frac{\mathcal{H}(x) + \mathcal{H}(y)}{2} \subset \mathcal{H}\left(\frac{x+y}{2}\right), \qquad x, y \in D.$$

It is known that midconvex s.v. functions need not be continuous (even if the domain D is an open subset of $\mathbb{R}^n$). However their continuity can be deduced from other (much weaker) properties such as measurability, lower semicontinuity at a single point or boundedness on a set with non–empty interior (cf. eg. [4]). Using these results we can obtain various conditions under which the superposition operator generated by a midconvex s.v. function transforms the space of continuous (l.s.c., u.s.c.,) (s.v.) functions into itself. The theorem presented below gives one of such possibilities. We say that an s.v. function $F: D \mapsto n(Y)$ is *weakly bounded* on a set $A \subset D$ if there exists a bounded set $B \subset Y$ such that $F(x) \cap B \neq \emptyset$ for every $x \in A$.

Theorem 1. *Let Y be a topological vector space and $F: (a,b) \times \mathbb{R} \mapsto c(Y)$ be a midconvex s.v. function. Assume that for some continuous function $\varphi: (a,b) \mapsto \mathbb{R}$ which is not affine on any interval $I \subset (a,b)$, the superposition $F(\cdot, \varphi(\cdot)): (a,b) \mapsto c(Y)$ is weakly bounded on an interval $(\alpha, \beta) \subset (a,b)$. Then the superposition $F(\cdot, G(\cdot))$ is l.s.c. for every l.s.c. s.v. function $G: (a,b) \mapsto n(\mathbb{R})$, and it is u.s.c. for every u.s.c. s.v. function $G: (a,b) \mapsto c(\mathbb{R})$.*

The proof of this theorem is based on the following two results.

Lemma 1. [3, Thm. 2, §5, Chpt. 9] *Let $\varphi: (\alpha, \beta) \mapsto \mathbb{R}$ be a continuous not affine function. Then the set $Gr\varphi + Gr\varphi$ (the algebraic sum of the graphs of φ) has non–empty interior.*

Lemma 2. [4, Cor. 3.3 for $K = \{0\}$] *Let X, Y be topological vector spaces and $D \subset X$ be an open convex set. If a midconvex s.v. function $\mathcal{H}: D \mapsto c(Y)$ is weakly bounded on a set $A \subset D$ with non–empty interior, then it is continuous on D.*

(In fact, in [4] the continuity is understood in the sense of the Hausdorff topology in n(Y). However for s.v. functions with compact values these two concepts of continuity are equivalent).

Proof of Theorem 1. The weak boundedness of the superposition $F(\cdot, \varphi(\cdot))$ on (α, β) means that F is weakly bounded on the set $G := Gr\varphi|_{(\alpha,\beta)}$.

Therefore there exists a bounded set $B \subset Y$ such that

$$F(s,x) \cap B \neq \emptyset \qquad \text{for every} \quad (s,x) \in G.$$

Fix arbitrary (s,x), $(t,y) \in G$ and take points $u \in F(s,x) \cap B$ and $v \in F(t,y) \cap B$. Then $\frac{1}{2}(u+v) \in \frac{1}{2}(B+B)$ and, by the midconvexity of F,

$$\frac{1}{2}(u+v) \in \frac{1}{2}(F(s,x) + F(t,y)) \subset F\left(\frac{(s,x)+(t,y)}{2}\right).$$

Hence

$$F\left(\frac{(s,x)+(t,y)}{2}\right) \cap \frac{B+B}{2} \neq \emptyset.$$

Since the set $\frac{1}{2}(B+B)$ is bounded, this means that F is weakly bounded on the set $\frac{1}{2}(G+G)$. By Lemma 1 this set has non-empty interior; so, by Lemma 2, F is continuous on $(a,b) \times \mathbb{R}$. Now it is enough to apply Propositions 1 and 2. □

As a consequence of the above theorem we get the following result.

Corollary 1. *Let Y be a locally bounded topological vector space and $F : (a,b) \times \mathbb{R} \mapsto c(Y)$ be a midconvex s.v. function. Assume that for some continuous function $\varphi : (a,b) \mapsto \mathbb{R}$ which is not affine on any interval $I \subset (a,b)$, the superposition $F(\cdot, \varphi(\cdot))$ has a selection continuous at a point. Then F is superpositionally continuous.*

Proof. Assume that $g : (a,b) \mapsto \mathbb{R}$ is a selection of $F(\cdot, \varphi(\cdot))$ continuous at a point $t_0 \in (a,b)$. Fix a bounded neighbourhood W of zero in Y and take a neighbourhood (α, β) of t_0 such that

$$g(t) \in g(t_0) + W \qquad \text{for all} \quad t \in (\alpha, \beta).$$

Put $B := g(t_0) + W$. Then B is bounded and

$$F(t, \varphi(t)) \cap B \neq \emptyset, \qquad t \in (\alpha, \beta),$$

which means that F is weakly bounded on the set $Gr\varphi|_{(\alpha,\beta)}$. By Theorem 1, the superposition $F(\cdot, f(\cdot))$ is continuous for every continuous function $f : (a,b) \mapsto \mathbb{R}$, which finishes the proof. □

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NELSON MERENTES
CENTRAL UNIVERSITY
OF VENEZUELA
CARACAS, VENEZUELA

KAZIMIERZ NIKODEM
KATEDRA MATEMATYKI
FILIA POLITECHNIKI LODZKIEJ
BIELSKO–BIALA, POLAND

SERGIO RIVAS
OPEN NATIONAL UNIVERSITY
CARACAS, VENEZUELA