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Triangle-free distance-regular graphs

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Abstract Let Γ denote a distance-regular graph with diameter $d \geq 3$. By a *parallelogram of length* 3, we mean a 4-tuple $xyzw$ consisting of vertices of Γ such that $\partial(x, y) = \partial(z, w) = 1$, $\partial(x, z) = 3$, and $\partial(x, w) = \partial(y, w) = \partial(y, z) = 2$, where ∂ denotes the path-length distance function. Assume that Γ has intersection numbers $a_1 = 0$ and $a_2 \neq 0$. We prove that the following (i) and (ii) are equivalent. (i) Γ is Q-polynomial and contains no parallelograms of length 3; (ii) Γ has classical parameters (d, b, α, β) with $b < -1$. Furthermore, suppose that (i) and (ii) hold. We show that each of $b(b+1)^2(b+2)/c_2$, $(b-2)(b-1)b(b+1)/(2+2b-c_2)$ is an integer and that $c_2 \leq b(b+1)$. This upper bound for c_2 is optimal, since the Hermitian forms graph $\mathrm{Her}_2(d)$ is a triangle-free distance-regular graph that satisfies $c_2 = b(b+1)$.

Keywords Distance-regular graph · Q-polynomial · Classical parameters

1 Introduction

Let Γ denote a distance-regular graph with diameter $d \geq 3$ (see Sect. 2 for formal definitions). It is known that if Γ has classical parameters, then Γ is Q-polynomial [2, Corollary 8.4.2]. The converse is not true, since an ordinary n-gon has the Q-polynomial property, but is without classical parameters [2, Table 6.6]. Many authors prove the converse under various additional assumptions. Indeed, assume that Γ is Q-polynomial. Then Brouwer, Cohen, and Neumaier [2, Theorem 8.5.1] show that

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if Γ is a near polygon with the intersection number $a_1 \neq 0$, then Γ has classical parameters. Weng generalizes this result with a weaker assumption, without kites of length 2 or length 3 in Γ, to replace the near polygon assumption [10, Lemma 2.4]. For the complement case $a_1 = 0$, Weng shows that Γ has classical parameters if (i) Γ contains no parallelograms of length 3 and no parallelograms of length 4; (ii) Γ has the intersection number $a_2 \neq 0$; and (iii) Γ has diameter $d \geq 4$ [11, Theorem 2.11]. Our first theorem improves the above result.

Theorem 1.1 *Let Γ denote a distance-regular graph with diameter $d \geq 3$ and intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Then the following* (i)–(iii) *are equivalent*:

(i) *Γ is Q-polynomial and contains no parallelograms of length* 3.
(ii) *Γ is Q-polynomial and contains no parallelograms of any length i for $3 \leq i \leq d$.*
(iii) *Γ has classical parameters (d, b, α, β) with $b < -1$.*

Many authors study distance-regular graph Γ with $a_1 = 0$ and other additional assumptions. For example, Miklavič assumes that Γ is Q-polynomial and shows that Γ is 1-homogeneous [6]; Koolen and Moulton assume that Γ has degree 8, 9, or 10 and show that there are finitely many such graphs [5]; Jurišić, Koolen, and Miklavič assume that Γ has an eigenvalue with multiplicity equal to the valency, $a_2 \neq 0$, and the diameter $d \geq 4$ to show that $a_4 = 0$ and Γ is 1-homogeneous [4]. In the second theorem, we assume that Γ has classical parameters and obtain the following:

Theorem 1.2 *With the notation and assumptions of Theorem* 1.1, *suppose that* (i)–(iii) *hold. Then each of*

$$\frac{b(b+1)^2(b+2)}{c_2}, \qquad \frac{(b-2)(b-1)b(b+1)}{2+2b-c_2} \tag{1.1}$$

is an integer. Moreover,

$$c_2 \leq b(b+1). \tag{1.2}$$

To conclude the paper, we give a class of triangle-free distance-regular graphs, each satisfying the equality in (1.2).

2 Preliminaries

In this section, we review some definitions and basic concepts concerning distance-regular graphs. See Bannai and Ito [1] or Terwilliger [8] for more background information.

Let Γ=(X, R) denote a finite undirected connected graph without loops or multiple edges with vertex set X, edge set R, distance function ∂, and diameter $d := \max\{\partial(x, y) \mid x, y \in X\}$.

For a vertex $x \in X$ and $0 \leq i \leq d$, set $\Gamma_i(x) = \{y \in X \mid \partial(x, y) = i\}$. Γ is said to be *distance-regular* whenever for all integers $0 \leq h, i, j \leq d$ and all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = \left|\left\{z \in X \mid z \in \Gamma_i(x) \cap \Gamma_j(y)\right\}\right|$$

is independent of x, y. The constants p^h_{ij} are known as the *intersection numbers* of Γ. For convenience, set $c_i := p^i_{1\,i-1}$ for $1 \le i \le d$, $a_i := p^i_{1\,i}$ for $0 \le i \le d$, $b_i := p^i_{1\,i+1}$ for $0 \le i \le d-1$, and put $b_d := 0$, $c_0 := 0$, $k := b_0$. Note that k is called the valency of Γ. From the definition of p^h_{ij} it is immediate that $b_i \ne 0$ for $0 \le i \le d-1$ and $c_i \ne 0$ for $1 \le i \le d$. Moreover,

$$k = a_i + b_i + c_i \quad \text{for } 0 \le i \le d. \tag{2.1}$$

From now on we assume that Γ is distance-regular with diameter $d \ge 3$.

Let $\mathbb{R}$ denote the real number field. Let $\mathrm{Mat}_X(\mathbb{R})$ denote the algebra of all matrices over $\mathbb{R}$ with the rows and columns indexed by the elements of X. For $0 \le i \le d$, let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{R})$ defined by the rule

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\ 0 & \text{if } \partial(x, y) \ne i \end{cases} \quad \text{for } x, y \in X.$$

We call A_i the *distance matrices* of Γ. We have

$$A_0 = I, \tag{2.2}$$

$$A_0 + A_1 + \cdots + A_d = J, \quad \text{where } J = \text{all 1's matrix}, \tag{2.3}$$

$$A_i^{\mathrm{t}} = A_i \quad \text{for } 0 \le i \le d, \text{ where } A_i^{\mathrm{t}} \text{ means the transpose of } A_i, \tag{2.4}$$

$$A_i A_j = \sum_{h=0}^{d} p^h_{ij} A_h \quad \text{for } 0 \le i, j \le d, \tag{2.5}$$

$$A_i A_j = A_j A_i \quad \text{for } 0 \le i, j \le d. \tag{2.6}$$

Let M denote the subspace of $\mathrm{Mat}_X(\mathbb{R})$ spanned by $A_0, A_1, \ldots, A_d$. Then M is a commutative subalgebra of $\mathrm{Mat}_X(\mathbb{R})$ and is known as the *Bose–Mesner algebra* of Γ. By [2, p. 59, 64], M has a second basis $E_0, E_1, \ldots, E_d$ such that

$$E_0 = |X|^{-1} J, \tag{2.7}$$

$$E_i E_j = \delta_{ij} E_i \quad \text{for } 0 \le i, j \le d, \tag{2.8}$$

$$E_0 + E_1 + \cdots + E_d = I, \tag{2.9}$$

$$E_i^{\mathrm{t}} = E_i \quad \text{for } 0 \le i \le d. \tag{2.10}$$

The $E_0, E_1, \ldots, E_d$ are known as the *primitive idempotents* of Γ, and E_0 is known as the *trivial* idempotent. Let E denote any primitive idempotent of Γ. Then we have

$$E = |X|^{-1} \sum_{i=0}^{d} \theta^*_i A_i \tag{2.11}$$

for some $\theta^*_0, \theta^*_1, \ldots, \theta^*_d \in \mathbb{R}$ called the *dual eigenvalues* associated with E.

Set $V = \mathbb{R}^{|X|}$ (column vectors) and view the coordinates of V as being indexed by X. Then the Bose–Mesner algebra M acts on V by left multiplication. We call V

the *standard module* of Γ. For each vertex $x \in X$, set

$$\hat{x} = (0, 0, \dots, 0, 1, 0, \dots, 0)^{\mathrm{t}}, \tag{2.12}$$

where the 1 is in coordinate x. Also, let $\langle\, ,\rangle$ denote the dot product

$$\langle u, v\rangle = u^{\mathrm{t}} v \quad \text{for } u, v \in V. \tag{2.13}$$

Then referring to the primitive idempotent E in (2.11), from (2.10–2.13) we compute that, for $x, y \in X$,

$$\langle E\hat{x}, \hat{y}\rangle = |X|^{-1}\theta^*_i \tag{2.14}$$

where $i = \partial(x, y)$.

Let $\circ$ denote the entry-wise multiplication in $\mathrm{Mat}_X(\mathbb{R})$. Then

$$A_i \circ A_j = \delta_{ij} A_i \quad \text{for } 0 \le i, j \le d,$$

so M is closed under $\circ$. Thus there exists $q^k_{ij} \in \mathbb{R}$ for $0 \le i, j, k \le d$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{k=0}^{d} q^k_{ij} E_k \quad \text{for } 0 \le i, j \le d.$$

Γ is said to be *Q-polynomial* with respect to the given ordering $E_0, E_1, \dots, E_d$ of the primitive idempotents if, for all integers $0 \le h, i, j \le d$, $q^h_{ij} = 0$ (resp. $q^h_{ij} \ne 0$) whenever one of h, i, j is greater than (resp. equal to) the sum of the other two. Let E denote any primitive idempotent of Γ. Then Γ is said to be Q-polynomial with respect to E whenever there exists an ordering $E_0, E_1 = E, \dots, E_d$ of the primitive idempotents of Γ with respect to which Γ is Q-polynomial. If Γ is Q-polynomial with respect to E, then the associated dual eigenvalues are distinct [7, p. 384].

The following theorem about the Q-polynomial property will be used in this paper.

Theorem 2.1 [8, Theorem 3.3] *Assume that Γ is Q-polynomial with respect to a primitive idempotent E, and let $\theta^*_0, \dots, \theta^*_d$ denote the corresponding dual eigenvalues. Then the following* (i)–(ii) *hold*:

(i) *For all integers $1 \le h \le d$, $0 \le i, j \le d$ and for all $x, y \in X$ such that $\partial(x, y) = h$,*

$$\sum_{\substack{z\in X,\ \partial(x,z)=i\\ \partial(y,z)=j}} E\hat{z} - \sum_{\substack{z\in X,\ \partial(x,z)=j\\ \partial(y,z)=i}} E\hat{z} = p^h_{ij}\frac{\theta^*_i - \theta^*_j}{\theta^*_0 - \theta^*_h}(E\hat{x} - E\hat{y}). \tag{2.15}$$

(ii) *For an integer $3 \le i \le d$,*

$$\theta^*_{i-2} - \theta^*_{i-1} = \sigma\left(\theta^*_{i-3} - \theta^*_i\right) \tag{2.16}$$

for an appropriate $\sigma \in \mathbb{R} \setminus \{0\}$.

Γ is said to have *classical parameters* (d, b, α, β) whenever the intersection numbers of Γ satisfy

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) \quad \text{for } 0 \le i \le d, \tag{2.17}$$

$$b_i = \left(\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \quad \text{for } 0 \le i \le d, \tag{2.18}$$

where

$$\begin{bmatrix} i \\ 1 \end{bmatrix} := 1 + b + b^2 + \cdots + b^{i-1}. \tag{2.19}$$

Suppose that Γ has classical parameters (d, b, α, β). Combining (2.17–2.19) with (2.1), we have

$$\begin{aligned} a_i &= \begin{bmatrix} i \\ 1 \end{bmatrix} \left(\beta - 1 + \alpha \left(\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} - \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) \right) \\ &= \begin{bmatrix} i \\ 1 \end{bmatrix} \left(a_1 + \alpha \left(1 - \begin{bmatrix} i \\ 1 \end{bmatrix} - \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) \right) \quad \text{for } 0 \le i \le d. \end{aligned} \tag{2.20}$$

Note that if Γ has classical parameters (d, b, α, β) and $d \ge 3$, then b is an integer and $b \ne 0, -1$ [2, Proposition 6.2.1]. Γ is said to have *classical parameters* if Γ has classical parameters (d, b, α, β) for some constants d, b, α, β. It is shown that a distance-regular graph with classical parameters has the Q-polynomial property [2, Theorem 8.4.1]. Terwilliger generalizes this to the following:

Theorem 2.2 [8, Theorem 4.2] *Let Γ denote a distance-regular graph with diameter $d \ge 3$. Choose $b \in \mathbb{R} \setminus \{0, -1\}$. Then the following* (i)–(ii) *are equivalent*:

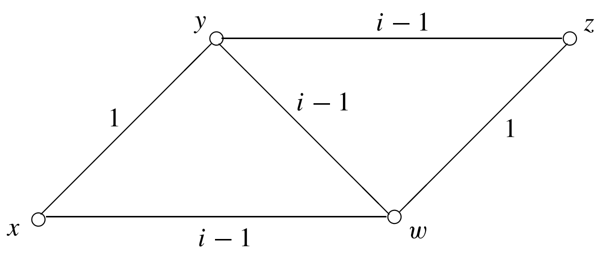
(i) *Γ is Q-polynomial with associated dual eigenvalues $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ satisfying*

$$\theta_i^* - \theta_0^* = (\theta_1^* - \theta_0^*) \begin{bmatrix} i \\ 1 \end{bmatrix} b^{1-i} \quad \textit{for } 1 \le i \le d.$$

(ii) *Γ has classical parameters (d, b, α, β) for some real constants α, β.*

Pick an integer $2 \le i \le d$. By a *parallelogram* of length i in Γ we mean a 4-tuple $xyzw$ of vertices of X such that (see Fig. 1)

Fig. 1 A parallelogram of length i



$$\partial(x,y)=\partial(z,w)=1, \qquad \partial(x,z)=i,$$
$$\partial(x,w)=\partial(y,w)=\partial(y,z)=i-1.$$

3 Proof of Theorem 1.1

In this section we prove our first main theorem. We start with a lemma.

Lemma 3.1 [6, Theorem 5.2(i)] *Let Γ denote a Q-polynomial distance-regular graph with diameter $d \geq 3$ and intersection number $a_1 = 0$. Fix an integer i for $2 \leq i \leq d$ and three vertices x, y, z such that*

$$\partial(x,y)=1, \qquad \partial(y,z)=i-1, \qquad \partial(x,z)=i.$$

Then the quantity

$$s_i(x,y,z) := \left|\Gamma_{i-1}(x)\cap\Gamma_{i-1}(y)\cap\Gamma_1(z)\right| \tag{3.1}$$

is equal to

$$a_{i-1}\frac{(\theta_0^*-\theta_{i-1}^*)(\theta_2^*-\theta_i^*)-(\theta_1^*-\theta_{i-1}^*)(\theta_1^*-\theta_i^*)}{(\theta_0^*-\theta_{i-1}^*)(\theta_{i-1}^*-\theta_i^*)}. \tag{3.2}$$

In particular, (3.1) *is independent of the choice of the vertices x, y, z.*

Proof Let $s_i(x,y,z)$ denote the expression in (3.1) and set

$$\ell_i(x,y,z) = \left|\Gamma_i(x)\cap\Gamma_{i-1}(y)\cap\Gamma_1(z)\right|.$$

Note that

$$s_i(x,y,z)+\ell_i(x,y,z)=a_{i-1}. \tag{3.3}$$

By (2.15) we have

$$\sum_{\substack{w\in X,\ \partial(y,w)=i-1\\ \partial(z,w)=1}} E\hat{w} - \sum_{\substack{w\in X,\ \partial(y,w)=1\\ \partial(z,w)=i-1}} E\hat{w} = a_{i-1}\frac{\theta_{i-1}^*-\theta_1^*}{\theta_0^*-\theta_{i-1}^*}(E\hat{y}-E\hat{z}). \tag{3.4}$$

Taking the inner product of (3.4) with $\hat{x}$ and using (2.14) and the assumption $a_1 = 0$, we obtain

$$s_i(x,y,z)\theta_{i-1}^* + \ell_i(x,y,z)\theta_i^* - a_{i-1}\theta_2^* = a_{i-1}\frac{\theta_{i-1}^*-\theta_1^*}{\theta_0^*-\theta_{i-1}^*}\left(\theta_1^*-\theta_i^*\right). \tag{3.5}$$

Solving $s_i(x,y,z)$ by using (3.3) and (3.5), we get (3.2). □

By Lemma 3.1 $s_i(x,y,z)$ is a constant for any vertices x, y, z with $\partial(x,y)=1$, $\partial(y,z)=i-1$, $\partial(x,z)=i$.

Definition 3.2 Let s_i denote the expression in (3.1). Note that $s_i = 0$ if and only if Γ contains no parallelograms of length i.

Lemma 3.3 *Let Γ denote a distance-regular graph with classical parameters (d, b, α, β). Suppose that intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Then $\alpha < 0$ and $b < -1$.*

Proof Since $a_1 = 0$ and $a_2 \neq 0$, from (2.19) and (2.20) we have

$$-\alpha(b+1)^2 = a_2 - (b+1)a_1 = a_2 > 0. \tag{3.6}$$

Hence

$$\alpha < 0. \tag{3.7}$$

By direct calculation from (2.17) we get

$$(c_2 - b)\big(b^2 + b + 1\big) = c_3 > 0. \tag{3.8}$$

Since

$$b^2 + b + 1 > 0, \tag{3.9}$$

(3.8) implies that

$$c_2 > b. \tag{3.10}$$

Using (2.17) and (3.10), we get

$$\alpha(1 + b) = c_2 - b - 1 \geq 0. \tag{3.11}$$

Hence, $b < -1$ by (3.7), since $b \neq -1$. □

Proof of Theorem 1.1 (ii) ⇒ (i) This is clear.

(iii) ⇒ (ii) Suppose that Γ has classical parameters. Then Γ is Q-polynomial with associated dual eigenvalues $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ satisfying

$$\theta_i^* - \theta_0^* = (\theta_1^* - \theta_0^*)\begin{bmatrix} i \\ 1 \end{bmatrix} b^{1-i} \quad \text{for } 1 \leq i \leq d. \tag{3.12}$$

We need to prove that $s_i = 0$ for $3 \leq i \leq d$. To compute s_i in (3.2), observe from (3.12) that

$$\theta_{i-1}^* - \theta_i^* = \big(\theta_0^* - \theta_1^*\big)b^{1-i} \quad \text{for } 1 \leq i \leq d. \tag{3.13}$$

Summing (3.13) for consecutive i, we find

$$\big(\theta_1^* - \theta_i^*\big) = \big(\theta_0^* - \theta_1^*\big)\big(b^{-1} + b^{-2} + \cdots + b^{1-i}\big), \tag{3.14}$$

$$\big(\theta_1^* - \theta_{i-1}^*\big) = \big(\theta_0^* - \theta_1^*\big)\big(b^{-1} + b^{-2} + \cdots + b^{2-i}\big), \tag{3.15}$$

$$\big(\theta_2^* - \theta_i^*\big) = \big(\theta_0^* - \theta_1^*\big)\big(b^{-2} + b^{-3} + \cdots + b^{1-i}\big), \tag{3.16}$$

$$\left(\theta_0^* - \theta_{i-1}^*\right) = \left(\theta_0^* - \theta_1^*\right)\left(b^0 + b^{-1} + \cdots + b^{2-i}\right) \tag{3.17}$$

for $3 \leq i \leq d$. Evaluating (3.2) by using (3.13–3.17), we find that $s_i = 0$ for $3 \leq i \leq d$.

(i) $\Rightarrow$ (iii) Note that $s_3 = 0$. Then by setting $i = 3$ in (3.2) and using the assumption $a_2 \neq 0$, we find

$$\left(\theta_0^* - \theta_2^*\right)\left(\theta_2^* - \theta_3^*\right) - \left(\theta_1^* - \theta_2^*\right)\left(\theta_1^* - \theta_3^*\right) = 0. \tag{3.18}$$

Set

$$b := \frac{\theta_1^* - \theta_0^*}{\theta_2^* - \theta_1^*}. \tag{3.19}$$

Then

$$\theta_2^* = \theta_0^* + \frac{(\theta_1^* - \theta_0^*)(b+1)}{b}. \tag{3.20}$$

Eliminating θ_2^*, θ_3^* in (3.18) by using (3.20) and (2.16), we have

$$\frac{-(\theta_1^* - \theta_0^*)^2(\sigma b^2 + \sigma b + \sigma - b)}{\sigma b^2} = 0 \tag{3.21}$$

for an appropriate $\sigma \in \mathbb{R} \setminus \{0\}$. Since $\theta_1^* \neq \theta_0^*$, we have

$$\sigma b^2 + \sigma b + \sigma - b = 0$$

and hence

$$\sigma^{-1} = \frac{b^2 + b + 1}{b}. \tag{3.22}$$

By Theorem 2.2, to prove that Γ has classical parameter, it suffices to prove that

$$\theta_i^* - \theta_0^* = (\theta_1^* - \theta_0^*)\begin{bmatrix} i \\ 1 \end{bmatrix} b^{1-i} \quad \text{for } 1 \leq i \leq d. \tag{3.23}$$

We prove (3.23) by induction on i. The case $i = 1$ is trivial, and the case $i = 2$ is from (3.20). Now suppose that $i \geq 3$. Then (2.16) implies

$$\theta_i^* = \sigma^{-1}\left(\theta_{i-1}^* - \theta_{i-2}^*\right) + \theta_{i-3}^* \quad \text{for } 3 \leq i \leq d. \tag{3.24}$$

Evaluating (3.24), using (3.22) and the induction hypothesis, we find that $\theta_i^* - \theta_0^*$ is as in (3.23). Therefore, Γ has classical parameters (d, b, α, β) for some scalars α, β. Note that $b < -1$ by Lemma 3.3. $\square$

4 Proof of Theorem 1.2

Recall that a sequence x, y, z of vertices of Γ is *geodetic* whenever

$$\partial(x, y) + \partial(y, z) = \partial(x, z).$$

Recall that a sequence x, y, z of vertices of Γ is *weak-geodetic* whenever

$$\partial(x, y) + \partial(y, z) \leq \partial(x, z) + 1.$$

Definition 4.1 A subset $\Omega \subseteq X$ is *weak-geodetically closed* if, for any weak-geodetic sequence x, y, z of Γ,

$$x, z \in \Omega \Longrightarrow y \in \Omega.$$

Theorem 4.2 [12, Proposition 6.7, Theorem 4.6] *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$. Assume that the intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Suppose that Γ contains no parallelograms of length 3. Then, for each pair of vertices $v, w \in X$ at distance $\partial(v, w) = 2$, there exists a weak-geodetically closed subgraph Ω of diameter 2 in Γ containing v, w. Furthermore, Ω is strongly regular with intersection numbers*

$$a_i(\Omega) = a_i(\Gamma), \tag{4.1}$$

$$c_i(\Omega) = c_i(\Gamma), \tag{4.2}$$

$$b_i(\Omega) = a_2(\Gamma) + c_2(\Gamma) - a_i(\Omega) - c_i(\Omega) \tag{4.3}$$

for $0 \leq i \leq 2$.

Corollary 4.3 *Let Γ denote a distance-regular graph with classical parameters (d, b, α, β), where $d \geq 3$. Assume that Γ has intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Then there exists a weak-geodetically closed subgraph Ω of diameter 2. Furthermore, the intersection numbers of Ω satisfy*

$$b_0(\Omega) = (1 + b)(1 - \alpha b), \tag{4.4}$$

$$b_1(\Omega) = b(1 - \alpha - \alpha b), \tag{4.5}$$

$$c_2(\Omega) = (1 + b)(1 + \alpha), \tag{4.6}$$

$$a_2(\Omega) = -(1 + b)^2\alpha, \tag{4.7}$$

$$|\Omega| = \frac{(1 + b)(b\alpha - 2)(b\alpha - 1 - \alpha)}{(1 + \alpha)}. \tag{4.8}$$

Proof Note that $b < -1$ by Lemma 3.3 and Γ contains no parallelograms of length 3 by Theorem 1.1. Hence there exists a weak-geodetically closed subgraph Ω of diameter 2 by Theorem 4.2. By applying (2.17), (2.18), and (2.20) to (4.1–4.3), we immediately have (4.4–4.7). Note that $|\Omega| = 1 + k(\Omega) + k(\Omega)b_1(\Omega)/c_2(\Omega)$. Equation (4.8) follows from this and from (4.4–4.6). □

Proposition 4.4 ([12, Proposition 3.2]) *Let Γ denote a distance-regular graph with diameter $d \geq 3$. Suppose that there exists a weak-geodetically closed subgraph Ω of Γ with diameter 2. Then the intersection numbers of Γ satisfy the following inequality*

$$a_3 \geq a_2(c_2 - 1) + a_1. \tag{4.9}$$

Corollary 4.5 *Let Γ denote a distance-regular graph with classical parameters (d, b, α, β), where $d \geq 3$. Suppose that the intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Then*

$$c_2 \leq b^2 + b + 2. \tag{4.10}$$

Proof Applying $a_1 = 0$ in (2.20), we have that $a_3 = -\alpha(b^2+b+1)(b+1)^2$. Then by applying (4.9), using Lemma 3.3, (4.1), and (4.7), the result immediately follows. □

We will decrease the upper bound of c_2 in (4.10). We need the following lemma.

Lemma 4.6 *Let Γ denote a distance-regular graph with classical parameters (d, b, α, β), where $d \geq 3$. Assume that the intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Let Ω be a weak-geodetically closed subgraph of diameter 2 in Γ. Let $r > s$ denote the nontrivial eigenvalues of the strongly regular graph Ω. Then the following* (i)–(ii) *hold*:

(i) *The multiplicity of r is*

$$f = \frac{(b\alpha - 1)(b\alpha - 1 - \alpha)(b\alpha - 1 + \alpha)}{(\alpha - 1)(\alpha + 1)}. \tag{4.11}$$

(ii) *The multiplicity of s is*

$$g = \frac{-b(b\alpha - 1)(b\alpha - 2)}{(\alpha - 1)(\alpha + 1)}. \tag{4.12}$$

Proof From [9, Theorem 21.1] we have

$$f = \frac{1}{2}\left\{v - 1 + \frac{(v-1)(c_2 - a_1) - 2k}{\sqrt{(c_2 - a_1)^2 + 4(k - c_2)}}\right\}, \tag{4.13}$$

$$g = \frac{1}{2}\left\{v - 1 - \frac{(v-1)(c_2 - a_1) - 2k}{\sqrt{(c_2 - a_1)^2 + 4(k - c_2)}}\right\}, \tag{4.14}$$

where $v = |\Omega|$, and k is the valency of Ω. Note that $c_2(\Omega) = (1+b)(1+\alpha)$ by (2.17), $k(\Omega) = (1+b)(1-\alpha b)$ by (4.4), and $v = (1+b)(b\alpha - 2)(b\alpha - 1 - \alpha)/(1+\alpha)$ by (4.8). Now (4.11) and (4.12) follow from (4.13) and (4.14). □

Corollary 4.7 *Let Γ denote a distance-regular graph with classical parameters (d, b, α, β), where $d \geq 3$. Assume that Γ has intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Then*

$$\frac{b(b+1)^2(b+2)}{c_2}, \tag{4.15}$$

$$\frac{(b-2)(b-1)b(b+1)}{2 + 2b - c_2} \tag{4.16}$$

are both integers.

Proof Let f and g be as in (4.11–4.12). Set $\rho = \alpha(1+b)$. Note that ρ is an integer, since $\rho = c_2 - 1 - b$. Then both

$$f + g - \left(1 - 3b^2 - b\rho + b^2\rho - b^3\right) = \frac{2b + 5b^2 + 4b^3 + b^4}{1 + b + \rho} = \frac{b(b+1)^2(b+2)}{c_2}$$

and

$$f - g - \left(1 - 3b^2 - b\rho + b^2\rho + b^3\right) = \frac{2b - b^2 - 2b^3 + b^4}{-1 - b + \rho} = \frac{(b-2)(b-1)b(b+1)}{c_2 - 2 - 2b}$$

are integers, since f, g, b, and ρ are integers. □

Proposition 4.8 *Let Γ denote a distance-regular graph with classical parameters (d, b, α, β), where $d \geq 3$. Assume that Γ has intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Then $c_2 \leq b(b+1)$.*

Proof Recall that $c_2 \leq b^2 + b + 2$ by (4.10). First, suppose that

$$c_2 = b^2 + b + 2. \tag{4.17}$$

Then the integral condition (4.15) becomes

$$b^2 + 3b + \frac{-4b}{b^2 + b + 2}. \tag{4.18}$$

Since $0 < -4b < b^2 + b + 2$ for $b \leq -5$, we have $-4 \leq b \leq -2$. For $b = -4$ or -3, expression (4.18) is not an integer. The remaining case $b = -2$ implies $\alpha = -5$ by (4.6), $v = 28$ by (4.8), and $g = 6$ by (4.12). This contradicts to $v \leq \frac{1}{2}g(g+3)$ [9, Theorem 21.4]. Hence $c_2 \neq b^2 + b + 2$. Next, suppose that $c_2 = b^2 + b + 1$. Then (4.16) becomes

$$-b^2 + b + 1 + \frac{1}{b^2 - b - 1}. \tag{4.19}$$

It fails to be an integer, since $b < -1$. □

Proof of Theorem 1.2 The results come from Corollary 4.7 and Proposition 4.8. □

Example 4.9 [3] Hermitian forms graph $\mathrm{Her}_2(d)$ is a distance-regular graph with classical parameters (d, b, α, β) with $b = -2$, $\alpha = -3$, and $\beta = -((-2)^d + 1)$, which satisfies $a_1 = 0$, $a_2 \neq 0$, and $c_2 = b(b+1)$.

Example 4.10 [9, p. 237] Gewirtz graph is a distance-regular graph with diameter 2 and intersection numbers $a_1 = 0$, $c_2 = 2$, $k = 10$, which can be written as classical parameters (d, b, α, β) with $d = 2$, $b = -3$, $\alpha = -2$, $\beta = -5$, so we have $c_2 = \frac{(b+1)^2}{2}$.

Conjecture 4.11 (Gewirtz graph does not grow) *There is no distance-regular graph with classical parameters $(d, -3, -2, -\frac{1+(-3)^d}{2})$, where $d \geq 3$.*

There is a conjecture similar to Conjecture 4.11 for the complement part in $a_1 \neq 0$. See [13, Theorem 10.3] for details.

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