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A Viability Result in the Upper Semicontinuous Case*

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We prove the existence of solutions of a differential inclusion $u' \in F(t,u)$ in a separable Banach space X with constraint $u(t) \in D(t)$. F is globally measurable, weakly upper semicontinuous with respect to u and takes convex, weakly compact values. D is upper semicontinuous from the left, and, for every $r > 0$, the sets $D(t) \cap rB$ are compact. F and D fulfil a well-known tangential condition, which is expressed by means of the Bouligand cone.

1. Introduction

In this paper we are going to deal with a differential inclusion of the kind $u' \in F(t,u)$, where $F(t,u)$ is a closed, convex subset of a separable Banach space X. The growth of F with respect to u is at most linear, while u is subject to the constraint

$$u(t) \in D(t), \quad t \in I \subseteq \mathbb{R}.$$

In this field, several results were achieved in the last years, first in the case $D(t) \equiv D$, then in the general case: we recall the books by Aubin-Cellina [3], Aubin [2], Deimling [15], and the wide references therein. In particular, as regards the upper semicontinuous case, we refer to Haddad [19], Deimling [12,13], Tallos [22], Bressan [7], and more recent works by Benabdellah-Castaing-Ibrahim [4], Bothe [5,6], Castaing-Moussaoui-Syam [9], Cavallucci [11], Malaguti [20], Frankowska-Plaskacz [18].

As is known, any viability result needs a tangential condition, in order to keep the trajectory $u(t)$ inside $D(t)$. In some recent papers, the following property is required, at least almost everywhere: whenever $u \in D(t)$, the set $F(t,u)$ must contain a vector v such that $(1,v) \in T_\Gamma(t,u)$, where Γ is the graph of D and $T_\Gamma(z)$ is the Bouligand cone of Γ at the point $z = (t,u)$ (Def. 2.2). This assumption looks quite natural, since the vector $(1,u'(t))$, where it exists, lies in the Bouligand cone of Γ at $z = (t,u(t))$.

If F is globally upper semicontinuous, and the graph of D is closed, the proof can be reduced to the autonomous case, and the given conditions are known to be enough to ensure the existence of global solutions, at least when $X = \mathbb{R}^p$ (see, for instance, [12] for the case $D(t) \equiv D$). If the dimension of X is not finite, some further assumptions must be introduced, about F or D: for instance, if nothing more is required on D, the non-compactness measure of the sets $F(t,u)$ should be controlled in some way, as in [15, §9]. An alternative approach, which is adopted in [20] and the present paper, leads to the

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following assumptions: for every $r > 0$ the sets $D(t) \cap rB$ are compact, but the sets $F(t,u)$ are only supposed to be weakly compact. Moreover, in this context, it looks natural to assume that F is upper semicontinuous with respect to the weak topology on the space of its values ("scalarly" upper semicontinuous) as in [9], [20].

Further difficulties arise when $F(t,u)$ is upper semicontinuous only with respect to u, since the reduction to the autonomous case does not look easy (see [13], [4], [5]). In particular, an example in [5] shows that solutions could miss even if F is measurable with respect to t. Indeed, in [5], F is supposed to be *almost* upper semicontinuous, that is to say, for every $\epsilon > 0$, upper semicontinuous on sets of the kind $\Gamma \cap (I_\epsilon \times X)$, where the measure of $I \backslash I_\epsilon$ does not exceed ϵ. Here we like better to assume that F is upper semicontinuous with respect to u and *globally* measurable, that is measurable with respect to the product of the Lebesgue σ-field on I with the Borel σ-field on X. As is known, the two assumptions are equivalent if X is a euclidean space, and also in more general cases (see, for instance, [23, Thm. 2]). As far as we know, however, the question is left open in our context: indeed, the space where F takes its values is endowed with the weak topology, which does not fulfil the assumptions of [23].

In the case $X = \mathbb{R}^p$, our main result (Theorem 2.3) is equivalent to Theorem 1 of [5] (see Remark 2.10), and strongly related to Theorem 3.1 of [18]. When the dimension of X is not finite, the main difference with respect to [6, §3] lies in the assumptions about compactness and upper semicontinuity, which here are referred to the weak topology in the space where F takes its values. Furthermore, our approximate solutions are built in a different way, since we do not suppose (at least directly) that F is almost upper semicontinuous. Indeed, our approach is rather direct, since it relies on a suitable use of measurable selections and their Lebesgue's points: in particular, it needs neither Scorza-Dragoni's property (as the proof given in [18]), nor the approximation of F through the Aumann integral means, which some authors used in order to get global upper semicontinuity (see, for instance, [9], [20]).

The plan of the work is conceived as follows: in §2 we explain the problem and prove the main result (Theorem 2.3), on the ground of the existence of suitable approximate solutions (Def. 2.4). In §3 we deal with the most important part of the work: the construction of the approximate solutions, which is performed through a suitable use of Zorn's Lemma. In §4 we explain some auxiliary results: in particular, in Prop. 4.1, we exploit a result by Castaing [8], in order to get the compactness of the sequence of approximate solutions. Finally, in Theorem 4.5, we show some useful properties of the function $\phi(t,x) = d(x; D(t))$.

2. Statement of the problem

Let $I = [0,T]$ be an interval of the real line, $(X, \|\cdot\|)$ a separable Banach space, B the closed unit ball of X. We denote respectively by $\mathcal{L}$ and $\mathcal{B}$ the Lebesgue σ-field in I and the Borel σ-field in X. Given $A \in \mathcal{L}$, $|A|$ stands for its Lebesgue measure. If $x \in X$ and $C \subseteq X$, $d(x;C)$ is the distance of x from C, while $\|C\|$ means $\sup\{\|x\|;\, x \in C\}$. We also define in $E = I \times X$ the distance between two points $w = (t,x)$, $w' = (t',x')$ as $d(w,w') = \max(|t-t'|, \|x-x'\|)$. Then, given $w \in E$, $G \subseteq E$, $\delta(w;G)$ will denote the distance of w from G. $AC(I;X)$ will be the space of all absolutely continuous functions $u : I \to X$. Now, for every $t \in I$, let $D(t) \neq \emptyset$ be a closed subset of X, and, for every

$t \in I$, $x \in D(t)$, let $F(t,x) \subseteq X$ be again a closed, non-empty set. Given $x_0 \in D(0)$, we are interested in the following problem: find a function $u \in AC(I;X)$ such that:

$$\begin{aligned} &u(0) = x_0, \quad u(t) \in D(t) \quad \text{on } I \\ &u'(t) \in F(t,u(t)) \quad \text{a.e. on } I \end{aligned} \tag{$\mathcal{P}$}$$

Definition 2.1. If Φ is a multifunction from a set A to X, and $C \subseteq X$, the script $\Phi^{-1}(C)$ denotes the set of those points $\xi \in A$ such that $\Phi(\xi)$ meets C. Then we say that the multifunction $t \mapsto D(t)$ is *upper semicontinuous from the left* if, for every closed subset C of X, the set $D^{-1}(C)$ is closed in (I, τ^-), where τ^- is the "left" topology on I. We say that F is *measurable with respect to* $\mathcal{L} \otimes \mathcal{B}$ if, for every open set $C \subseteq X$, it is $F^{-1}(C) \in \mathcal{L} \otimes \mathcal{B}$. Finally, let $t \in I$ be given: the multifunction $x \mapsto F(t,x)$ is said to be *weakly upper semicontinuous* if, whenever C is weakly closed in X, the set $F(t,\cdot)^{-1}(C)$ is closed in X.

Definition 2.2. Let Γ be the *graph* of the multifunction $D(\cdot)$, i.e. the set of those pairs $(t,x) \in E$ such that $x \in D(t)$. Then the *Bouligand tangent cone* of Γ at a point $z \in \Gamma$ (see, for instance, [15, §4]) is the set of all vectors $w \in E$ such that, for a suitable sequence of positive numbers h_n, it is $h_n \to 0$ and $\delta(z + h_n w; \Gamma)/h_n \to 0$, as $n \to +\infty$. Then we denote by $Q_\Gamma(z)$ the set of all points $y \in X$ such that $(1,y) \in T_\Gamma(z)$.

Roughly speaking, if $z = (a,x)$, $Q_\Gamma(z)$ is the set of "admissible speeds" at the time $t = a$ for a trajectory $u = u(t) \in D(t)$ such that $u(a) = x$, and can be easily characterized as follows: $y \in Q_\Gamma(a,x)$ if and only if there exist positive numbers k_n and points $x_n \in D(a + k_n)$, with $n \in \mathbb{N}$, such that, as $n \to +\infty$,

$$(a)\ k_n \to 0, \quad (b)\ \|\frac{x_n - x}{k_n} - y\| \to 0. \tag{2.1}$$

From now on, we suppose that D and F fulfil the following conditions:

(i) for every $t \in I$, $r > 0$, the set $D(t) \cap rB$ is strongly compact;
(ii) for every $(t,x) \in \Gamma$ the set $F(t,x)$ is convex and weakly compact;
(iii) D is upper semicontinuous from the left;
(iv) F is measurable with respect to $\mathcal{L} \otimes \mathcal{B}$;
(v) for every $t \in I$, $F(t,\cdot)$ is weakly u.s.c. on $D(t)$;
(vi) there exist a function $\theta \in L^1(I)$ and a set $N \in \mathcal{L}$, with $|N| = 0$, such that, for every $(t,x) \in \Gamma$ with $t \notin N$ it is $\|F(t,x)\| \le \theta(t)(1 + \|x\|)$;
(vii) if $(t,x) \in \Gamma$ and $t \notin N$, then $F(t,x) \cap Q_\Gamma(t,x) \neq \emptyset$; otherwise, $Q_\Gamma(t,x) \neq \emptyset$.

Theorem 2.3. *Let X be a separable Banach space, D and F satisfy conditions* (i)–(vii), *$x_0 \in D(0)$. Then problem $(\mathcal{P})$ admits a solution.*

The proof of Theorem 2.3 relies, as usual, on the existence of suitable approximate solutions. In order to define this notion, we recall that a family $\mathcal{V} \subseteq L^1(I;X)$ is *uniformly integrable* if, for every $\epsilon > 0$, we can find $\sigma > 0$ such that, for every $v \in \mathcal{V}$, $A \in \mathcal{L}$, with $|A| \le \sigma$, it is $\int_A \|v(t)\| dt \le \epsilon$. Furthermore, for every $r > 0$, we consider the set Γ^r of those pairs (t,x) such that $t \in I$, $x \in D(t) + rB$, and put

$$F_r(t,x) = \cup\{F(t,y);\ y \in D(t),\ \|y - x\| \le r\}, \quad (t,x) \in \Gamma^r. \tag{2.2}$$

Definition 2.4. We say that $(u_n)_n$ in $AC(I;X)$ is a sequence of approximate solutions for problem $(\mathcal{P})$ if the functions u'_n are uniformly integrable, and there exist numbers $\epsilon_n \in]0,1]$ such that $\sum_n \epsilon_n < +\infty$, and, for every $n \in \mathbb{N}$:

(a) $u_n(0) = x_0$;
(b) for every $t \in I$, $d(u_n(t); D(t)) \leq \epsilon_n$;
(c) there exists a set $E_n \in \mathcal{L}$ such that $|I \backslash E_n| \leq \epsilon_n$, and, for every $t \in E_n$, it is $u'_n(t) \in \Phi_n(t, u_n(t))$, where, for every $(t,x) \in \Gamma^{\epsilon_n}$, we put $\Phi_n(t,x) = F_{\epsilon_n}(t,x) + \epsilon_n B$.

In §3 we shall show the existence of approximate solutions. In this section, on the ground of that result (Theorem 3.4), we are going to prove Theorem 2.3. We put forward some measurability results: first of all, we remark that $\Gamma \in \mathcal{L} \otimes \mathcal{B}$. Indeed, let $A \subseteq X$ be open, and write A as $\cup_n C_n$, where the sets C_n are closed. Then $D^{-1}(A) = \cup_n D^{-1}(C_n) \in \mathcal{L}$, since the sets $D^{-1}(C_n)$ are closed in (I, τ^-), thanks to (iii). Then the multifunction D is measurable, and Theorem III.13 of [10] shows that $\Gamma \in \mathcal{L} \otimes \mathcal{B}$.

Lemma 2.5. *Let $\Lambda \subseteq \Gamma$, $\Lambda \in \mathcal{L} \otimes \mathcal{B}$, and suppose that, for every $t \in I$, the set $C(t) = \{x \in X : (t,x) \in \Lambda\}$ is closed (possibly empty). Let $r > 0$, and Λ^r be the graph of the multifunction $t \mapsto C(t) + rB$: then $\Lambda^r \in \mathcal{L} \otimes \mathcal{B}$.*

Proof. Since $C(t)$ is a closed subset of $D(t)$, and (i) holds, it is easy to see that $y \in C(t) + rB$ if and only if $d(y; C(t)) \leq r$. Now, if $(x_i)_i$ is a dense sequence in X, this happens if and only if, for every $j \in \mathbb{Z}^+$, we can find an index i such that the ball $B_{ij} := x_i + j^{-1}B$ meets $C(t)$, and $y \in U_{ij} := x_i + (r + j^{-1})B$. Now, thanks to Theorem III.23 of [10], the subset E_{ij} of I where $B_{ij} \cap C(t) \neq \emptyset$ is measurable, since $\mathcal{L}$ is a complete σ-field, and E_{ij} is the image of the set $(I \times B_{ij}) \cap \Gamma \in \mathcal{L} \otimes \mathcal{B}$ through the projection $(t,x) \mapsto t$. Now it is enough to point out that $\Lambda^r = \cap_j \cup_i (E_{ij} \times U_{ij})$. □

Proposition 2.6. *Let $r > 0$: then $\Gamma^r \in \mathcal{L} \otimes \mathcal{B}$. Furthermore, the multifunction F_r takes weakly compact values, and is measurable with respect to $\mathcal{L} \otimes \mathcal{B}$.*

Proof. We already checked that $\Gamma \in \mathcal{L} \otimes \mathcal{B}$: thanks to the previous Lemma (in which we put $\Lambda = \Gamma$), we also get $\Gamma^r \in \mathcal{L} \otimes \mathcal{B}$. The set $F_r(t,x)$ is nothing but the image of the compact set $(x + rB) \cap D(t)$ through the multifunction $F(t, \cdot)$: hence, by virtue of (v) and well-known results about upper semicontinuous multifunctions (see [3], Prop. 3, p. 42), $F_r(t,x)$ is weakly compact. Now, let $A \subseteq X$ be open: since X is separable, it is easy to express A as the union of a countable family of closed balls B_n, which are also weakly closed. For every n, the set $\Gamma_n = F^{-1}(B_n)$ can be seen as the graph of the multifunction $t \mapsto \Psi_n(t) = F(t, \cdot)^{-1}(B_n)$: now, from (iv) and (v) we argue respectively that $\Gamma_n \in \mathcal{L} \otimes \mathcal{B}$, and the sets $\Psi_n(t)$ are closed. Then Lemma 2.5 entails that the graph Γ_n^r of the multifunction $t \mapsto \Psi_n(t) + rB$ lies in $\mathcal{L} \otimes \mathcal{B}$ as well. But $\Gamma_n^r = F_r^{-1}(B_n)$, so that $F_r^{-1}(A) = \cup_n \Gamma_n^r \in \mathcal{L} \otimes \mathcal{B}$. □

Definition 2.7. Let $C \subseteq X$, and X^* be the dual space of X: then the *support function* of C is defined, for every $p \in X^*$, by $\delta^*(p; C) = \sup\{\langle p, v \rangle; v \in C\}$. If $t \in I$ is given, the multifunction $x \mapsto F(t,x)$ is said to be *scalarly* upper semicontinuous if, for every $p \in X^*$, the function $x \mapsto \delta^*(p; F(t,x))$ is upper semicontinuous.

In our context, the condition given in Def. 2.7 is actually equivalent to (v) [10, Theorem II.20]. We also point out that, through simple arguments, (2.2) allows to express the upper semicontinuity of the function $x \mapsto \delta^*(p; F(t,x))$ as follows:

$$\lim_{r \to 0^+} \delta^*(p; F_r(t,x)) \leq \delta^*(p; F(t,x)). \tag{2.3}$$

Proof of Theorem 2.3. Let us consider the approximate solutions u_n of problem $(\mathcal{P})$, whose existence will be shown in Theorem 3.4 of the next section. Since the sequence $(u_n')_n$ is uniformly integrable on the bounded interval I, it is easy to see that it is bounded in $L^1(I;X)$ by some constant $M > 0$. Now, let $R = M+1$, recall (2.2) and, for every $t \in I$, put $W(t) = F_R(t, x_0)$. Then, for every $t \in I$, $\|u_n(t) - x_0\| \leq M$, so that $u_n(t) + \epsilon_n B \subseteq x_0 + RB$. Hence $\Phi_n(t, u_n(t)) \subseteq W(t) + \epsilon_n B$, and condition (c) implies that, for every $t \in E_n$, $d(u_n'(t); W(t)) \leq \epsilon_n$. Now we can apply Proposition 4.2 to $v_n = u_n'$, and infer that the sequence $(v_n)_n$ admits a limit function v in the weak topology of $L^1(I;X)$. More precisely $v_{n_k} \to v$ weakly in $L^1(I;X)$ as $k \to +\infty$, where $(n_k)_k$ is an increasing sequence of positive integers. Now let $u \in AC(I;X)$ be the function such that $u(0) = x_0$ and $u' = v$ almost everywhere, and put, for every $k \in \mathbb{N}$, $x_k = u_{n_k}$: we are going to show that u solves problem $(\mathcal{P})$. First of all, we notice that, as $k \to +\infty$,

$$x_k(t) = x_0 + \int_0^t v_{n_k}(\tau) d\tau \to x_0 + \int_0^t v(\tau) d\tau = u(t) \quad \text{weakly in } X, \quad t \in I. \tag{2.4}$$

Now it is easy to check that actually $x_k \to u$ uniformly on I as $k \to +\infty$. Indeed, the functions x_k are equicontinuous, because their derivatives are uniformly integrable. Furthermore, for every $t \in I$, $k \in \mathbb{N}$, $x_k(t) \in x_0 + RB$. On the other hand, $\epsilon_{n_k} \leq 1$, so that condition (b) of Def. 2.4 entails that the distance of $x_k(t)$ from the compact set $D(t) \cap (x_0 + (R+1)B)$ does not exceed the number $\sigma_k := \epsilon_{n_k}$, which tends to 0 as $k \to +\infty$. Then, obviously, for every $t \in I$ the sequence $(x_k(t))_k$ lies in a strongly compact set, and Prop. 7.3b of [14] ensures that, in the space $C(I;X)$ of continuous functions $x : I \to X$, the sequence $(x_k)_k$ is relatively compact with respect to $\|\cdot\|_\infty$. Now, let us argue by contradiction and suppose that there exist $\eta > 0$ and an infinite set $H \subseteq \mathbb{N}$ such that, for every $i \in H$, $\|x_i - u\|_\infty \geq \eta$. Thanks to the previous compactness arguments, we should find an infinite subset L of H and a continuous function x such that $\|x_j - x\|_\infty \to 0$ as $j \to +\infty$, with $j \in L$. But (2.4) entails $x = u$, then it should be $\|x_i - u\|_\infty < \eta$ for infinitely many $i \in H$, in contrast with the previous inequality. Hence $x_k \to u$ uniformly as $k \to +\infty$. In particular, from condition (b) of Def. 2.4 (with $n = n_k$), we get, as $k \to +\infty$, $d(u(t); D(t)) = 0$: since $D(t)$ is closed, we get actually that $u(t) \in D(t)$. Finally, let us put $I_k = E_{n_k}$, and $n = n_k$ in condition (c) of Def. (2.4), so as to get

$$x_k'(t) \in \Phi_{n_k}(t, x_k(t)), \quad t \in I_k. \tag{2.5}$$

Let us also put $\rho_k = \|x_k - u\|_\infty$, $r_k = \rho_k + \sigma_k$: then $x_k(t) \in u(t) + \rho_k B$, so that $x_k(t) + \sigma_k B \subseteq u(t) + r_k B$, and

$$\Phi_{n_k}(t, x_k(t)) \subseteq F_{r_k}(t, u(t)) + \sigma_k B. \tag{2.6}$$

Now we are going to apply a result of Section 4, Prop. 4.2. To this end, let p be a function in the space $L_s^\infty(I;X^*)$, which is defined before Prop. 4.1: then, from (2.5) and (2.6),

$$\langle p(t), x_k'(t) \rangle \leq \delta^*(p(t); F_{r_k}(t, u(t))) + \sigma_k \|p(t)\|^*, \quad t \in I_k, \tag{2.7}$$

where $\|\cdot\|^*$ is the norm of X^*. Now, let $J = \liminf_k I_k$ be the set of those points $t \in I$ which lie definitively in I_k: since Def. 2.4 entails that $\sum_k |I \backslash I_k| < +\infty$, we get $|I \backslash J| = 0$. Furthermore, for every $t \in J$, we can take the upper limit in (2.7) as $k \to +\infty$, so as to get, thanks to (2.3),

$$\limsup_{k \to +\infty} \langle p(t), x_k'(t) \rangle \leq \delta^*(p(t); F(t, u(t))) \quad \text{a. e. on } I. \tag{2.8}$$

Now, let us put $\Phi(t) = F(t, u(t))$ and recall condition (vi) of Theorem 2.3, so as to get $\|\Phi(t)\| \leq C\theta(t)$, with $C = 1 + \|x_0\| + R$: then we can apply Prop. 4.2 with $v_k = x_k'$, $v = u'$, and conclude that $u'(t) \in F(t, u(t))$ for almost every $t \in I$. □

Remark 2.8. As we can easily check, condition (iii) implies that, for every $x \in X$, the function $t \to d(x; D(t))$ is lower semicontinuous from the left. Furthermore, we recall that an upper semicontinuous multifunction with closed values has a closed graph [15, Prop. 1.2b]: then condition (iii) also entails that Γ is closed from the left, that is to say, closed with respect to the product of τ^- with the strong topology of X. When $X = \mathbb{R}^p$, this property (which is used also in [6, §3]) could replace (iii): indeed, as we are going to see in the next sections, what we actually need in the proof of Theorem 2.3 is the lower semicontinuity from the left of the functions $t \to d(x; D(t))$, $x \in X$, which in this case holds as soon as Γ is closed from the left.

Remark 2.9. If $N = \emptyset$, condition (vii) is a standard assumption: this particular, slightened formulation is also given in [5]. We point out that in [18] a weaker condition is considered, which replaces $T_\Gamma(z)$ with its closed, convex hull. Thanks to the characterization of Q_Γ given in (2.1), the condition $F(t, x) \cap Q_\Gamma(t, x) \neq \emptyset$ entails the following property: there exist numbers $k_n > 0$ and points $x_n \in D(t + k_n)$, with $n \in \mathbb{N}$, such that k_n and the distance of $(x_n - x)/k_n$ from $F(t, x)$ tend to 0 as $n \to +\infty$. In the case $N = \emptyset$, this condition looks less restrictive than (vii) (see [4] for a comparison between the two conditions).

Remark 2.10. As we point out in the introduction, when the dimension of X is finite, Theorem 2.3 is actually equivalent to Theorem 1 of [5]. Indeed, in this case, (v) simply says that $F(t, \cdot)$ is upper semicontinuous. Then (iv) and (v), thanks to Theorem 2 of [23], entail that F is *almost upper semicontinuous,* as in [5].

3. The approximate solutions

Throughout this section we fix a number $\epsilon \in]0, 1]$ and build an approximate solution of problem $(\mathcal{P})$ in correspondence with it (Lemma 3.3). Then Theorem 3.4 will follow easily. Let $(x_i)_i$ be a dense sequence in X and, for every $i \in \mathbb{N}$, $t \in I$, put $\phi_i(t) = d(x_i; D(t))$, $B_i(t) = x_i + \phi_i(t)B$, $D_i(t) = D(t) \cap B_i(t)$. Thanks to conditions (i), (iii) of Theorem 2.3 the multifunctions D_i have non-empty values and are measurable: hence they admit measurable selections u_i [10, Thm. III-6]. Now let us put, for every $i \in \mathbb{N}$, $t \in I$, $\Phi_i^\epsilon(t) = F_{\epsilon/2}(t, u_i(t))$. Since we already saw that $F_{\epsilon/2}$ is measurable with respect to $\mathcal{L} \otimes \mathcal{B}$, and takes weakly compact values (Prop. 2.6), we argue that Φ_i^ϵ is measurable on I, and its values are strongly closed: then it admits a Castaing representation [10, Thm. III-7], that is a countable family $\mathcal{V}_i^\epsilon$ of measurable selections such that, for every $t \in I$, the set $\{v(t);\, v \in \mathcal{V}_i^\epsilon\}$ is dense in $\Phi_i^\epsilon(t)$. Let L^ϵ be the set of those points $t \in I$ which are

Lebesgue points for all the functions v of the family $\mathcal{V}^\epsilon = \cup_i \mathcal{V}_i^\epsilon$: since $\mathcal{V}^\epsilon$ is countable as well, $|I \backslash L^\epsilon| = 0$. Furthermore, for every $i \in \mathbb{N}$, $v \in \mathcal{V}_i^\epsilon$, $a \in L^\epsilon \cap [0, T[$ it is

$$\lim_{h \to 0^+} \frac{1}{h} \int_a^{a+h} v(t) dt = v(a). \tag{3.1}$$

Given an initial datum $(a, x) \in \Gamma$, we are going to build, on a "small" interval $I^* = [a, a^*]$, an approximate solution w of problem $(\mathcal{P})$. To this end, we need a control *a priori* on the moving set $D(t)$: so we exploit a result of §4 (Theorem 4.5), and take the constant $c > 0$ which appears in (4.6). Then we put $\beta = \max(c\theta, 2\theta + 1)$, consider the function $r \in AC(I; \mathbb{R})$ such that $r(0) = \|x_0\|$, $r' = \beta(1 + r)$ a. e., and put $\psi = (1 + r(T))\beta$. Now we should like to get on I^* the following conditions:

$$\begin{array}{ll} \text{(a) } w(a) = x,\ w(a^*) \in D(a^*), & \text{(b) } d(w(t); D(t)) \leq \epsilon, \\ \text{(c) } w'(t) \in F_\epsilon(t, w(t)) + \epsilon B \quad \text{a. e.}, & \text{(d) } \|w(a^*)\| \leq r(a^*), \\ \text{(e) } \|w'(t)\| \leq \psi(t) \quad \text{a. e.} & \end{array} \tag{3.2}$$

Unfortunately, we are not able to satisfy all previous conditions for any initial datum, but only when a lies outside the set $N_\epsilon = (I \backslash L^\epsilon) \cup N$, where N is the set which appears in conditions (vi), (vii) of Theorem 2.3. Then we proceed in two different ways, according to whether $a \notin N_\epsilon$ (Lemma 3.1) or $a \in N_\epsilon$ (Lemma 3.2). We put forward the following inequality, which follows easily from the definition of r:

$$r(t) - r(a) \geq (1 + r(a)) \int_a^t \beta(\tau) d\tau, \quad 0 \leq a \leq t \leq T. \tag{3.3}$$

Lemma 3.1. *Let $\epsilon \in]0, 1]$, $z = (a, x) \in \Gamma$, with $a < T$, $a \notin N_\epsilon$, $\|x\| \leq r(a)$. Then there exist $a^* \in]a, T]$ and a function $w \in AC(I^*)$ (where $I^* = [a, a^*]$) such that conditions* (3.2) *hold on I^*.*

Proof. Let $\sigma \in]0, \epsilon/8]$. Then we can find $i \in \mathbb{N}$ such that

$$\text{(a)} \quad \|x_i - x\| \leq \sigma, \quad \text{(b)} \quad \phi_i(a) = d(x_i; D(a)) \leq \sigma, \tag{3.4}$$

so that $\|u_i(a) - x\| \leq \|u_i(a) - x_i\| + \|x_i - x\| \leq \phi_i(a) + \sigma \leq 2\sigma \leq \epsilon/2$. Since $a \notin N$, thanks to condition (vii) of Theorem 2.3 we can find $y \in F(a, x) \cap Q_\Gamma(a, x)$, and from the previous inequality we get $y \in \Phi_i^\epsilon(a)$. Since $\mathcal{V}_i^\epsilon$ is a Castaing representation of Φ_i^ϵ, there must be a function $v_i \in \mathcal{V}_i^\epsilon$ such that

$$\|y - v_i(a)\| \leq \sigma. \tag{3.5}$$

We recall that the function r of (3.3) is continuous, and that (3.1) holds, with $v = v_i$. Furthermore, thanks to Theorem 4.5, ϕ_i is upper semicontinuous from the right. Then we can find $\hat{h} \in]0, \sigma]$ such that, for every $k \in]0, \hat{h}]$ the following inequalities hold:

$$\begin{array}{l} \text{(a) } r(a + k) - r(a) \leq \sigma, \quad \text{(b) } \|v_i(a) - \frac{1}{k} \int_a^{a+k} v_i(\tau) d\tau\| \leq \sigma, \\ \text{(c) } \phi_i(t) \leq \phi_i(a) + \sigma, \quad t \in [a, a + k]. \end{array} \tag{3.6}$$

In particular we get, whenever $a \le t \le a+k$,

$$\text{(a)} \quad \|u_i(t) - x\| \le 3\sigma, \quad \text{(b)} \quad \|v_i(t)\| \le \theta(t)(2 + \|x\|). \tag{3.7}$$

Indeed, since $\|u_i(t) - x_i\| \le \phi_i(t)$, (3.7a) follows from (3.4a), (3.6c) and (3.4b). On the other hand, since $v_i(t) \in F_{\epsilon/2}(t, u_i(t))$, from (3.7a) and the inequality $\epsilon/2 + 3\sigma \le 1$, we get $v_i(t) \in F_1(t,x)$: then condition (vi) of Theorem 2.3 implies (3.7b). Now, let us recall Remark 2.1, and take n so large as to ensure that k_n in (2.1a) and the norm which appears in (2.1b) do not exceed respectively $\hat{h}$ and σ. Then let us put $k = k_n$, $x^* = x_n$, $a^* = a+k$, $I^* = [a, a^*]$, $q = (x^* - x)/k$. In particular, $k = a^* - a$ fulfils conditions (3.6). On the other hand, if we put $y_i = \frac{1}{k}\int_a^{a+k} v_i(\tau)d\tau$, $q_i = q - y_i$, from (3.5) and (3.6b) we get $\|y - y_i\| \le 2\sigma$, so that

$$\|q_i\| \le \|q - y\| + \|y - y_i\| \le \sigma + 2\sigma \le \epsilon. \tag{3.8}$$

We claim that the function $w : I^* \to X$ defined by $w(t) = x + \int_a^t v_i(\tau)d\tau + (t-a)q_i$ fulfils conditions (3.2). First of all, (3.2a) holds obviously, since $w(a) = x$ and $w(a^*) = x^*$. In order to prove (3.2d), we recall (3.7b), the inequalities $\|x\| \le r(a)$, $2\theta + 1 \le \beta$ and (3.3) so as to get:

$$\begin{aligned}\|w(t) - x\| &\le \int_a^{a+k} \|v_i(\tau)\|d\tau + k\|q_i\| \le (2 + \|x\|)\int_a^{a+k} \theta(t)dt + k \le \\ &\le (1 + \|x\|)\int_a^{a+k} (2\theta(t) + 1)dt \le (1 + r(a))\int_a^{a+k} \beta(t)dt \le r(a+k) - r(a).\end{aligned}$$

In particular, for $t = a^* = a + k$, (3.2d) follows. Furthermore, thanks to (3.6a), we get

$$\|w(t) - x\| \le \sigma. \tag{3.9}$$

In order to prove (3.2b) we deduce from (3.9) and (3.4a) that $\|w(t) - x_i\| \le 2\sigma$. Then $d(w(t); D(t)) \le 2\sigma + \phi_i(t)$ and, thanks to (3.6c), (3.4b) $d(w(t); D(t)) \le 4\sigma \le \epsilon$. We also argue easily (3.2e) on I^*, since it is, almost everywhere on I^*, $w'(t) = v_i(t) + q_i$, so that, from (3.8)

$$\|w'(t) - v_i(t)\| \le \epsilon. \tag{3.10}$$

Then $\|w'(t)\| \le \|v_i(t)\| + \epsilon$, and from (3.7b) and the inequalities $\epsilon \le 1$, $2\theta + 1 \le \beta$ we get $\|w'(t)\| \le \beta(t)(1 + \|x\|)$. Since $\|x\| \le r(a) \le r(T)$, (3.2e) holds as well. Now let us prove (3.2c) on I^*: from (3.9) and (3.7a) we get $\|w(t) - u_i(t)\| \le 4\sigma \le \epsilon/2$, so that

$$u_i(t) + \frac{\epsilon}{2}B \subseteq w(t) + \epsilon B. \tag{3.11}$$

On the other hand $v_i(t) \in \Phi_i^\epsilon(t) \subseteq F(t, (u_i(t) + \frac{1}{2}\epsilon B) \cap D(t))$, and we get actually from (3.11), almost everywhere on I^*, $v_i(t) \in F_\epsilon(t, w(t))$. Then (3.2c) follows from (3.10). □

Now we are going to deal with the case $a \in N_\epsilon$. Since $|N_\epsilon| = 0$, and the function ψ of (3.2e) is integrable, we can find an open subset $\hat{N}_\epsilon$ of I such that $\hat{N}_\epsilon \supseteq N_\epsilon$, $|\hat{N}_\epsilon| \le \epsilon$, $\int_{\hat{N}_\epsilon} \psi(t) \le \epsilon$.

Lemma 3.2. *Let $\epsilon \in]0,1]$, $z = (a,x) \in \Gamma$, with $a < T$, $a \in N_\epsilon$, $\|x\| \le r(a)$. Then there exist $a^* \in]a,T]$ and a function $w \in AC(I^*)$ (where $I^* = [a,a^*]$) such that conditions* (3.2a,b,d) *hold on I^*. Furthermore $I^* \subseteq \hat{N}_\epsilon$, and $\int_{I^*} \|w'(t)\|dt \le \int_{I^*} \psi(t)dt$.*

Proof. Let $0 < \sigma \le \epsilon/2$. We recall that the set $\hat{N}_\epsilon$ is open, and r is continuous. Furthermore, thanks to Theorem 4.5 (which will be proved in the next section), the function $\phi(t,x) = d(x; D(t))$ is upper semicontinuous from the right with respect to t, and $\phi(a,x) = 0$: then we can find $k \in]0,\sigma]$ such that $[a, a+k] \subseteq \hat{N}_\epsilon$, (3.6a) holds and, for every $t \in [a, a+k]$, $\phi(t,x) \le \sigma$. Let us put $a^* = a+k$, $I^* = [a,a^*]$ and take $x^* \in D(a^*)$ such that $\|x^* - x\| = \phi(a^*, x)$. Then, from (4.6) and the inequalities $c\theta \le \beta$, $\|x\| \le r(a)$, we get $\|x^* - x\| \le (1 + r(a)) \int_a^{a+k} \beta(t)dt$. Then, from (3.3) and the definition of ψ,

$$\text{(a)} \quad \|x^* - x\| \le r(a+k) - r(a), \quad \text{(b)} \quad \|x^* - x\| \le \int_a^{a+k} \psi(t)dt. \tag{3.12}$$

Now let us put $q = (x^* - x)/k$, and define $w : I^* \to X$ by $w(t) = x + (t-a)q$. Then $w(a) = x$ and $w(a^*) = x^*$, so that (3.2a) holds obviously. In order to prove (3.2b), we point out that $\|w(t) - x\| \le \|x^* - x\|$. Then, from (3.12a) and (3.6a), $\|w(t) - x\| \le \sigma$, so that $d(w(t); D(t)) \le \phi(t,x) + \sigma$. On the other hand, $\phi(t,x) \le \sigma$, and $2\sigma \le \epsilon$: then (3.2b) holds as well, and, again from (3.12a), we easily get (3.2d). Finally, $I^* \subseteq \hat{N}_\epsilon$, and, from (3.12b), $\int_{I^*} \|w'(t)\|dt = \|x^* - x\| \le \int_{I^*} \psi(t)dt$. □

Now we are going to deduce, from the two previous lemmas, the existence of approximate solutions of problem $(\mathcal{P})$ on the whole interval I, through a suitable use of Zorn's Lemma.

Lemma 3.3. *Let $\epsilon > 0$, $x_0 \in D(0)$. Then there exist a function $w \in AC(I)$ and a set $\Lambda_\epsilon \in \mathcal{L}$ such that $w(0) = x_0$,* (3.2b) *holds on I, $|\Lambda_\epsilon| \le \epsilon$ and*

$$\begin{aligned} &\text{(a)} \quad \int_A \|w'(t)\|dt \le \int_A \psi(t)dt + \epsilon, \quad A \in \mathcal{L}, \\ &\text{(b)} \quad w'(t) \in F_\epsilon(t, w(t)) + \epsilon B, \quad t \in I \backslash \Lambda_\epsilon. \end{aligned} \tag{3.13}$$

Proof. For every $a \in I$, let $\lambda_\epsilon(a)$ be the integral of ψ on the set $\hat{N}_\epsilon \cap [0,a]$, and define $\mathcal{W}_a$ as the space of those functions $w \in AC([0,a])$ such that $w(0) = x_0$, $w(a) \in D(a)$, $\|w(a)\| \le r(a)$, (3.2b) holds on $[0,a]$, (3.2c) holds a. e. on $[0,a]\backslash\hat{N}_\epsilon$, and, for every $A \in \mathcal{L}$, with $A \subseteq [0,a]$,

$$\int_A \|w'(t)\|dt \le \int_A \psi(t)dt + \lambda_\epsilon(a). \tag{3.14}$$

Let $\mathcal{Q}$ be the family of all pairs (a,w) such that $0 < a \le T$ and $w \in \mathcal{W}_a$: we define on $\mathcal{Q}$ the usual partial order $\preceq$ according to which $(a_1, w_1) \preceq (a_2, w_2)$ if and only if $a_1 \le a_2$ and w_2 agrees with w_1 on $[0, a_1]$. Thanks to the previous Lemmas, in which we put $a = 0$, $\mathcal{Q}$ is non-empty. Furthermore, we can easily check that every non-empty family $\mathcal{F} \subseteq \mathcal{Q}$ which is totally ordered with respect to $\preceq$ admits an upper bound in $\mathcal{Q}$: indeed, let J be the set of those points $\alpha \in I$ such that $\mathcal{W}_\alpha \ne \emptyset$, and, for some $w \in \mathcal{W}_\alpha$, $(\alpha, w) \in \mathcal{F}$. If $a = \sup J$, we can certainly find in $\mathcal{F}$ a monotone sequence $(a_i, w_i)_i$ such that $a = \sup_i a_i$: then let us define w on $[0,a[$ as the only function which agrees with w_i on each interval

$[0, a_i]$. Of course, w is absolutely continuous on these intervals, and admits a. e. on $[0, a[$ a derivative which agrees a. e. with w_i' on each interval $[0, a_i]$. Then, for every $t \in [0, a[$, it is $w(t) = x_0 + \int_0^t w'(t)dt$. On the other hand, let us put $A = [0, a_i]$ in (3.14), and take the limit as $i \to +\infty$: then the function $\|w'\|$ turns out to be integrable on $[0, a[$. Hence $w(t)$ converges in X as $t \to a^-$, and w can be extended to $t = a$ in such a way that $w \in AC([0, a]; X)$. Now we are going to check that $w \in \mathcal{W}_a$: of course, $w(0) = x_0$; furthermore, the points $(a_i, w(a_i))$ lie in Γ, and Γ is closed from the left, according to Remark 2.8; then the point $(a, w(a)) = \lim_i (a_i, w(a_i))$ lies in Γ as well, that is to say, $w(a) \in D(a)$. Since w and r are continuous, it is also $\|w(a)\| \leq r(a)$. Furthermore, (3.2b) holds on $[0, a]$, while (3.2c) holds a.e. on $[0, a]\backslash\hat{N}_\epsilon$. In order to get (3.14) in $[0, a]$, we notice that, for every measurable subset A of $[0, a]$, that inequality holds for $A \cap [0, a_i]$: then it is enough to let $i \to +\infty$. Now we can conclude that $(a, w) \in \mathcal{Q}$, so that (a, w) is an upper bound for the pairs (a_i, w_i), then, actually, for the whole family $\mathcal{F}$.

Now we can apply Zorn's Lemma, and deduce that $\mathcal{Q}$ admits a maximal element (a, w): we claim that $a = T$. Indeed, let us suppose, by contradiction, that $a < T$. Then we can put $z = (a, w(a))$ and apply Lemma 3.1 or Lemma 3.2 according to whether a lies outside N_ϵ or not. In both cases we can extend w to an interval $[0, a^*]$ in such a way that $w(a^*) \in D(a^*)$, $\|w(a^*)\| \leq r(a^*)$ and (3.2b) holds on $[0, a^*]$, while (3.2c) holds a.e. on $[0, a^*]\backslash\hat{N}_\epsilon$. As regards (3.14), let us take $A \in \mathcal{L}$, $A \subseteq [0, a^*]$, and call λ^* the integral of $\|w'\|$ on $A \cap I^*$: then, according to whether $a \notin N_\epsilon$ or $a \in N_\epsilon$, from (3.2e) or the last statement of Lemma 3.2 we argue that

$$\text{(a)} \quad \lambda^* \leq \int_{A\cap I^*} \psi(t)dt \quad \text{or} \quad \text{(b)} \quad \lambda^* \leq \int_{I^*} \psi(t) = \lambda_\epsilon(a^*) - \lambda_\epsilon(a). \tag{3.15}$$

On the other hand, since $A\backslash I^* \subseteq [0, a]$ and (3.14) holds in $[0, a]$,

$$\int_A \|w'(t)\|dt = \int_{A\backslash I^*} \|w'(t)\|dt + \int_{A\cap I^*} \|w'(t)\|dt \leq \int_{A\backslash I^*} \psi(t)dt + \lambda_\epsilon(a) + \lambda^*.$$

Now is is easy to check that (3.15a) or (3.15b), together with the last inequality, implies (3.14) (with a^* instead of a) whenever $A \in \mathcal{L}$, $A \subseteq [0, a^*]$. Then $w \in \mathcal{W}_{a^*}$ and $(a, w) \prec (a^*, w)$, in contrast with our assumption. Then actually $a = T$, and (3.2b) holds on I, while (3.2c) holds a. e. on $I\backslash\hat{N}_\epsilon$. In particular, if Z is the subset of $I\backslash\hat{N}_\epsilon$ where (3.2c) does not hold, let us put $\Lambda_\epsilon = \hat{N}_\epsilon \cup Z$ and recall that $|\hat{N}_\epsilon| \leq \epsilon$. Since $|Z| = 0$, we get $|\Lambda_\epsilon| \leq \epsilon$, and (3.13b) holds. Finally, since $\lambda_\epsilon(T) = \int_{\hat{N}_\epsilon} \psi(t)dt \leq \epsilon$, we get also (3.13a). Hence w is the function we looked for. □

Theorem 3.4. *Let X be a separable Banach space, D and F satisfy conditions* (i)–(vii), $x_0 \in D(0)$. *Then problem* $(\mathcal{P})$ *admits a sequence of approximate solutions, according to* Def. 2.4.

Proof. For every $n \in \mathbb{N}$, let us apply the previous lemma with $\epsilon = \epsilon_n$ and put $u_n = w$, $E_n = I\backslash\Lambda_{\epsilon_n}$. Then conditions (a), (b) and (c) follow at once. Finally, if we put in (3.13a) $w = u_n$ and $\epsilon = \epsilon_n$, we also argue easily that $(u_n')_n$ is uniformly integrable. □

4. Some auxiliary results

In the first part of this section, we give two results (Prop. 4.1 and 4.2) which are useful in the proof of Theorem 2.3: the first one exploits a compactness result by Castaing [8], while the second one can be related to [10, Thm. VI-4]. First of all we recall the definition of uniformly integrable family in $L^1(I;X)$, given after Theorem 2.3, and other well-known notions: a multifunction $t \to C(t)$ from I to X is said to be measurable if, for every open subset Q of X, $C^{-1}(Q) \in \mathcal{L}$; X^* is the dual space of X, with norm $\|\cdot\|^*$ and closed unit ball B^*, while $\mathcal{B}^*$ denotes its Borel σ-field with respect to the strong topology. If $C \subseteq X$, $\|C\|$ was defined at the beginning of §2, while the *support function* $p \mapsto \delta^*(p;C)$ was introduced in Def. 2.7. We also denote by $L^\infty_s(I;X^*)$ the space of all essentially bounded functions $p: I \to X^*$ which are *weakly* or *scalarly measurable* [16, p. 41]. As is known, $L^\infty_s(I;X^*)$ is the dual space of $L^1(I;X)$ [21, p.301].

Proposition 4.1. *For every $t \in I$, let $W(t)$ be a weakly compact subset of X, and suppose that the multifunction $W(\cdot)$ is measurable. For every $n \in \mathbb{N}$, let $v_n \in L^1(I;X)$, $\epsilon_n > 0$, $E_n \in \mathcal{L}$ be such that*

$$d(v_n(t); W(t)) \leq \epsilon_n \quad \textit{for every} \;\; t \in E_n, \tag{4.1}$$

and, as $n \to +\infty$, $\epsilon_n \to 0$, $|I \backslash E_n| \to 0$. Suppose that the sequence $(v_n)_n$ is uniformly integrable: then $(v_n)_n$ is relatively weakly compact in $L^1(I;X)$.

Proof. For every $n \in \mathbb{N}$, $t \in E_n$, let us put $\hat{W}_n(t) = W(t) \cap (v_n(t) + \epsilon_n B)$. Since $W(t)$ is weakly compact and (4.1) holds, $\hat{W}_n(t)$ is non-empty on E_n. Let $\hat{v}_n : I \to X$ agree with v_n on $I \backslash E_n$, and be a measurable selection of $\hat{W}_n$ on E_n; it is easy to see that, for every $A \in \mathcal{L}$, the integral on A of $\|\hat{v}_n\| - \|v_n\|$ cannot exceed $\epsilon_n |A|$. Then the functions $\hat{v}_n$ are uniformly integrable as well. On the other hand, it is easy to check that the sequence $(\hat{v}_n)_n$ is $\mathcal{R}_w$*-tight*, according to the definition given in [1, p.38]: then it is relatively weakly compact in $L^1(I;X)$, thanks to a result given in [8, Thm.1, p. 2-14] and also reported in [1, Thm.4], so that it admits a limit function $v \in L^1(I;X)$. More precisely: $\hat{v}_{n_k} \to v$ weakly in $L^1(I;X)$ as $k \to +\infty$, where $(n_k)_k$ is an increasing sequence of integers. Since, obviously, $\lim_k \|\hat{v}_{n_k} - v_{n_k}\|_1 = 0$, we actually proved that $v_{n_k} \to v$ weakly in $L^1(I;X)$ as $k \to +\infty$. □

Proposition 4.2. *For every $t \in I$, let $\Phi(t)$ be a closed, convex subset of X, suppose that the multifunction $\Phi(\cdot)$ is measurable and that the function $t \to \|\Phi(t)\|$ is integrable. Let $(v_k)_k$ be a sequence in $L^1(I;X)$ which converges weakly to a function v, and assume that, for every $p \in L^\infty_s(I;X^*)$,*

$$\limsup_{k \to +\infty} \langle p(t), v_k(t) \rangle \leq \delta^*(p(t); \Phi(t)) \quad \textit{a.e. on } I. \tag{4.2}$$

Then $v(t) \in \Phi(t)$ a. e. on I.

Proof. If we take a Castaing representation of the multifunction $\Phi(\cdot)$ [10, Thm. III-7], we can easily check that the mapping $(t,p) \mapsto \delta^*(p;\Phi(t))$ is measurable with respect to $\mathcal{L} \otimes \mathcal{B}^*$: in particular, the right-hand side of (4.2), say $\gamma(t)$, is measurable with respect to t. Furthermore, $\gamma(t) \leq \eta(t) = \|p(t)\|^* \|\Phi(t)\|$, where η is obviously integrable. Now, let $p \in L^\infty_s(I;X^*)$, and, for every $k \in \mathbb{Z}^+$, $t \in I$, put $\phi_k(t) = \langle p(t), v_k(t) \rangle$. Let $J_k \subseteq I$ be the set where $\phi_k \leq 1 + \gamma$, and put $\psi_k = \phi_k \wedge (1+\gamma)$: from (4.2) we easily argue that, a. e. on I, the equality $\psi_k = \phi_k$ holds definitively, so that $|I \setminus J_k| \to 0$ as $k \to +\infty$. On the other hand, since the sequence $(v_k)_k$ is weakly convergent, it is also uniformly integrable: then, as $k \to +\infty$, the integrals of $\|v_k\|$ and $1+\gamma$ on $I \setminus J_k$ tend to 0, so that

$$\begin{aligned}\int_I \langle p(t), v(t)\rangle dt &= \lim_{k\to+\infty} \int_{J_k} \phi_k(t)dt = \lim_{k\to+\infty} \int_I \psi_k(t)dt \leq \\ &\leq \int_I \limsup_{k\to+\infty} \psi_k(t)dt = \int_I \limsup_{k\to+\infty} \phi_k(t)dt \leq \int_I \delta^*(p(t), \Phi(t))dt.\end{aligned} \tag{4.3}$$

We remark that in the first inequality of (4.3) it is right to apply Fatou's lemma, since $\psi_k \leq 1+\gamma$, and γ is integrable. Hence (4.3) holds for every $p \in L^\infty_s(I; X^*)$. Now, let us argue by contradiction, and suppose that there exist $r > 0$, $E_r \in \mathcal{L}$ such that $|E_r| > 0$ and, for every $t \in E_r$, $d(v(t); \Phi(t)) \geq r$. In particular, by virtue of the Hahn-Banach theorem, for every $t \in E_r$ the sets $\Phi(t)$ and $v(t) + rB^\circ$ can be separated by a functional $p \in X^*$, with $\|p\|^* \leq 1$: then, for every $z \in \Phi(t)$, $u \in B^\circ$, it is $\langle p, z\rangle \leq \langle p, v(t) + ru\rangle$, that is to say, $\delta^*(p; \Phi(t)) \leq \langle p, v(t)\rangle - r$. Let $Q_r(t) \neq \emptyset$ be the set of such p's: thanks to the previous remarks about the mapping $(t,p) \mapsto \delta^*(p; \Phi(t))$, the multifunction $t \mapsto Q_r(t)$ from E_r to $(B^*, \|\cdot\|^*)$ is measurable, since its graph lies in $\mathcal{L} \times \mathcal{B}^*$. Now, let us consider the space (B^*, d), where d is a distance which gives rise to the $\sigma(X^*, X)$ topology on B^*: since (B^*, d) is compact, it is complete and separable. Furthermore, the open sets of (B^*, d) are open in $(B^*, \|\cdot\|^*)$, and the values of Q_r are certainly closed with respect to $\sigma(X^*, X)$. Hence, if we regard Q_r as a multifunction from E_r to (B^*, d), we can conclude as well that Q_r is measurable and takes non-empty, closed values, so that Theorem III.6 in [10] ensures the existence of a selection $\bar{p}$ of Q_r which is the pointwise limit, with respect to the $\sigma(X^*, X)$ topology, of simple, measurable functions on E_r. Now, let p be the extension of $\bar{p}$ which vanishes outside E_r, and put $\psi(t) = \langle p(t), v(t)\rangle - \delta^*(p(t); \Phi(t))$: then we get $\psi \geq r$ on E_r and $\psi \equiv 0$ outside E_r, so that $\int_I \psi(t)dt \geq r|E_r| > 0$. On the other hand, $p \in L^\infty_s(I; X^*)$, so that (4.3) entails $\int_I \psi(t)dt \leq 0$, in contrast with the previous inequality. Then $v(t) \in \Phi(t)$ a. e. on I, as claimed. □

Now we are going to state two results about Dini's derivatives of a real function. To this end, given a function $\rho : I \to \mathbb{R}$ and a point $t \in [0, T[$, we denote by $D_+\rho(t)$, as usual, the lower limit, as $\tau \to t^+$, of $(\rho(\tau) - \rho(t))/(\tau - t)$. In Lemma 4.3 we give a particular version of the "chain rule", which involves $D_+\rho$; in Lemma 4.4 we give a generalized version of Gronwall's Lemma: both results are exploited in Theorem 4.5, which provides with useful properties of the functions $t \mapsto d(x; D(t))$.

Lemma 4.3. *Let $\rho : I \to \mathbb{R}$ be bounded from below, $J \supseteq \rho(I)$ an interval, $a = \inf \rho(I) \in J$, $f \in C^1(J; \mathbb{R})$ such that $f' > 0$ on J, $\mu = f \circ \rho$. Then, for every $t \in [0, T[$, $D_+\mu(t) = f'(\rho(t))D_+\rho(t)$, where, for every $\alpha > 0$, $\alpha \cdot (\pm\infty)$ is to be understood as $\pm\infty$.*

Proof. Given $t \in [0, T[$, let us prove that $D_+\mu(t) \leq f'(\rho(t))D_+\rho(t)$: if $D_+\rho(t) = +\infty$ the inequality is obvious, since $f' > 0$. If $D_+\rho(t) = l < +\infty$, let us consider, for every $\tau \in]t, T]$, the ratio between $\rho(\tau) - \rho(t)$ and $\tau - t$, and call it $\Delta(t, \tau)$: then there exist points $t_n \in]t, T]$ such that $t_n \to t^+$ and $\Delta(t, t_n) \to l$ as $n \to +\infty$. Furthermore, the mean value theorem ensures that, for every n, the segment S_n with endpoints $\rho(t)$ and $\rho(t_n)$ contains a point c_n such that $f(\rho(t_n)) - f(\rho(t)) = f'(c_n)(\rho(t_n) - \rho(t))$. Now, let γ be the lower limit of the numbers c_n as $n \to +\infty$: then $\gamma = \lim_k \gamma_k$, where $\gamma_k = c_{n_k}$ and the indexes n_k strictly increase with k. Furthermore, since $l < +\infty$ and $\gamma_k \geq a \in J$, we easily argue that $\gamma \in J$. Now, for every $k \in \mathbb{N}$, let us put $\tau_k = t_{n_k}$, $r_k = \rho(\tau_k)$, $r = \rho(t)$, and let Δ_k be the ratio between $f(r_k) - f(r)$ and $\tau_k - t$: then $\Delta_k = f'(\gamma_k)\Delta(t, \tau_k)$. Now we consider two cases: if $l = -\infty$, we get $\lim_k \Delta_k = -\infty$, because $f'(\gamma) > 0$; if $l \in \mathbb{R}$, it must be $r_k \to r$, so that $\gamma = r$: then $\lim_k \Delta_k = f'(r)l$. In both cases, we conclude that $\Delta_k \to f'(\rho(t))D_+\rho(t)$ as $k \to +\infty$. Since $D_+\mu(t) \leq \lim_k \Delta_k$ we actually get the required inequality. In order to get the reverse inequality, it is enough to consider $g = f^{-1}$ in place of f and exchange the rôles of ρ and μ. □

Lemma 4.4. *Let $\rho : I \to [0, +\infty[$ be lower semicontinuous from the left, $M > 0$, $\theta \in L^1(I)$, and suppose that, for every $t \in [0, T[$, $D_+\rho(t) \leq \theta(t)(M + \rho(t))$. Then:*

$$\rho(t) \leq (M + \rho(a))\, exp\,(\int_a^t \theta(\tau)d\tau) - M, \quad 0 \leq a < t \leq T. \tag{4.4}$$

Proof. Let us apply Lemma 4.3 with $J = [0, +\infty[$, and, for every $r \in J$, $f(r) = \log(M + r)$: if $\mu = f \circ \rho$, then $D_+\mu(t) = D_+\rho(t)/(M + \rho(t))$, so that $D_+\mu(t) \leq \theta(t)$. Furthermore, μ is lower semicontinuous from the left, like ρ. Now, for every $\epsilon > 0$ let $\theta_\epsilon > \theta$ be a lower semicontinuous function such that $\int_I \theta_\epsilon(t)dt \leq \int_I \theta(t)dt + \epsilon$ (see, for instance, [17]). Let $0 \leq a < t < T$: then we can find a sequence of points $t_n \in]t, T]$ such that $t_n \to t$ as $n \to +\infty$, and

$$\mu(t_n) - \mu(t) \leq \int_t^{t_n} \theta_\epsilon(\tau)d\tau + \epsilon(t_n - t). \tag{4.5}$$

Indeed, let $\sigma > 0$ be such that, for every $s \in]0, \sigma]$, $\theta_\epsilon(t) \leq \theta_\epsilon(t + s) + \epsilon$, and, consequently, $\theta_\epsilon(t)s \leq \int_t^{t+s} \theta_\epsilon(\tau)d\tau + \epsilon s$. Thanks to the strict inequality $D_+\mu(t) < \theta_\epsilon(t)$, we can find numbers $h_n \in]0, \sigma]$ such that $h_n \to 0$ as $n \to +\infty$ and, for every $n \in \mathbb{N}$, $\mu(t + h_n) - \mu(t) \leq \theta_\epsilon(t)h_n$. By virtue of the previous inequality, the points $t_n = t + h_n$ fulfil (4.5). Now, let $I_\epsilon(a)$ be the set of all points $t \in [a, T]$ such that

$$\mu(t) - \mu(a) \leq \int_a^t \theta_\epsilon(\tau)d\tau + \epsilon(t - a). \tag{4.6}$$

Since μ is lower semicontinuous from the left, $I_\epsilon(a)$ is closed from the left. Furthermore, every point $t \in I_\epsilon(a) \cap [0, T[$ is a cluster point for $I_\epsilon(a) \cap]t, T]$: in order to check this property, it is enough to choose points t_n as in (4.5) and take the sum of (4.6) and (4.5) so as to prove that, for every $n \in \mathbb{N}$, $t_n \in I_\epsilon(a)$. Now we argue easily that actually $I_\epsilon(a) = [a, T]$, so that (4.6) holds on $[a, T]$: so, if $\epsilon \to 0$, we get $\mu(t) \leq \mu(a) + \int_a^t \theta(\tau)d\tau$, and, in order to prove (4.4), it is enough to take the exponential of both sides in the last inequality. □

Theorem 4.5. *Under the assumptions of* Theorem 2.3, *for every $\xi \in X$ the function $t \mapsto d(\xi; D(t))$ is upper semicontinuous from the right. Furthermore, there exists a constant $c > 0$ such that, whenever $\xi \in D(a)$ and $t > a$,*

$$d(\xi; D(t)) \leq c(1 + \|\xi\|) \int_a^t \theta(\tau)d\tau. \tag{4.7}$$

Proof. Let $\xi \in X$, put $\rho = d(\xi; D(\cdot))$ and take $t \in [0, T[$. Let $x \in D(t)$ such that $\|\xi - x\| = \rho(t)$ and, according to condition (vii) of Theorem 2.3, take $y = y_t \in Q_\Gamma(t, x)$.

Let us choose k_n and x_n as in (2.1): then

$$\frac{\rho(t+k_n)-\rho(t)}{k_n} \leq \frac{\|x_n-\xi\|-\|x-\xi\|}{k_n} \leq \frac{\|x_n-x\|}{k_n}. \tag{4.8}$$

Now, let l be the lower limit of the left hand side of (4.8): from (2.1b) we get easily $D_+\rho(\tau) \leq l \leq \|y_t\|$. Now let us recall the set N of conditions (vi), (vii). Since $|N| = 0$, it is right to change θ on N: then we put, if $t \in N$, $\theta(t) = \|y_t\|/(1+\|x\|)$. If $t \notin N$, we can suppose that $y_t \in F(t,x)$ as well, so as to get $D_+\rho(t) \leq \|F(t,x)\|$. Then, thanks to (vi), we get, in both cases, $D_+\rho(t) \leq \theta(t)(1+\|x\|)$. On the other hand, $\|x\| \leq \|\xi\| + \rho(t)$: hence $D_+\rho(t) \leq \theta(t)(M+\rho(t))$, where $M = 1+\|\xi\|$. Thanks to the previous Lemma, (4.4) holds, so that ρ, that is the function $t \mapsto d(\xi; D(t))$, is actually upper semicontinuous from the right at the point a. Finally, let $\xi \in D(a)$, so as to get $\rho(a) = 0$ in (4.4). Let $c > 0$ be such that $e^\lambda \leq 1 + c\lambda$ whenever $0 \leq \lambda \leq \int_I \theta(\tau)d\tau$, let us consider that inequality for $\lambda = \int_a^t \theta(\tau)d\tau$ and combine it with (4.4): then we get $\rho(t) \leq Mc\int_a^t \theta(\tau)d\tau$. Since $M = 1 + \|\xi\|$, the proof is complete. ☐

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