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Minimax Equalities by Reconstruction of Polytopes

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Given a quasi-concave-convex function $f : X \times Y \to \overline{\mathbb{R}}$ defined on the product of two convex sets we would like to know if $\inf_Y \sup_X f = \sup_X \inf_Y f$. In [4] we showed that that question is very closely linked to the following "reconstruction" problem: given a polytope (i.e. the convex hull of a finite set of points) X and a family $\mathbb{F}$ of subpolytopes of X, we would like to know if $X \in \mathbb{F}$, knowing that any polytope which is obtained by cutting an element of $\mathbb{F}$ with a hyperplane or by pasting two elements of $\mathbb{F}$ along a common facet is also in $\mathbb{F}$. Here, we consider a similar "reconstruction" problem for arbitrary convex sets. Our main geometric result, Theorem 1.1, gives necessary and sufficient conditions for a subset-stable family $\mathbb{F}$ of subsets of a convex set X to verify $X \in \mathbb{F}$. Theorem 1.1 leads to some nontrivial minimax equalities, some of which are presented here: Theorems 1.3, 1.5, 3.4, 3.5, 4.1 and their corollaries. Further applications of our method to minimax equalities will be carried out in a forthcoming paper [5].

1. Introduction

From a sharp analysis of Flåm-Greco's paper [2] we derive a new flexible and powerful geometric method of reconstruction of convex sets (see Theorem 1.1, below) which we apply to problems concerning *nonempty intersection properties* for multifunctions and, consequently, *minimax equalities*. Such a method which was silently initiated by Greco in [3], was announced by Greco-Horvath in [4] as a consequence of their "reconstruction theorem by convex pastings".

Let A, B, and C be subsets of a convex set of a real vector space. Following Flåm-Greco [2], we say that C *separates* A and B (or A from B) if, for every $a \in A$ and $b \in B$, one has $C \cap [a, b] \neq \emptyset$.

Let us say that a family of sets $\mathbb{F}$ *covers* a set X if $\cup\mathbb{F} \supset X$; $\mathbb{F}$ is called *subset-stable* if any subset of a set belonging to $\mathbb{F}$ is itself an element of $\mathbb{F}$. A net of sets $\{A_j\}_j$ is said to be *increasing*, if $A_j \subset A_{j'}$ for $j \leq j'$.

Theorem 1.1 (reconstruction of convex sets). *Let X be a convex set of a real vector space and $\mathbb{F}$ be a subset-stable family of sets such that $\cup\mathbb{F} = X$. Then $X \in \mathbb{F}$, if the following properties hold:*

(1) **[increasing-net-stable]** $\cup_j A_j \in \mathbb{F}$, *if $\{A_j\}_j \subset \mathbb{F}$ is an increasing net,*

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(2) **[separating set property]** *either* $A \in \mathbb{F}$ *or* $B \in \mathbb{F}$, *if* A *and* B *are subsets of* X *which are separated by some* $C \in \mathbb{F}$,

(3) **[pasting property]** $A \cup B \in \mathbb{F}$, *if* A *and* B *belong to* $\mathbb{F}$ *and are separated by their intersection,*

(4) **[simplex property]** $S \in \mathbb{F}$, *if* $S \subset X$ *is a simplex with* $\dim(S) \geq 1$ *and there is a vertex* x *of* S *such that* $S \setminus \{x\} \in \mathbb{F}$.

At first, properties (1)–(4) seem awkward. But, it is worthwhile to notice that they furnish a minimal set of (necessary and) sufficient conditions to have $X \in \mathbb{F}$, as we can see from the following examples.

Example 1.2. In the following, each item (Ex i) gives an example of a family $\mathbb{F}$ on the unit interval $[0, 1]$ showing that property (i) cannot be dropped in Theorem 1.1.

(Ex 1) $\mathbb{F} := \big\{\emptyset, \{0\}\big\} \cup \big\{A \subset [0, 1] : \inf A > 0\big\}$.
(Ex 2) $\mathbb{F} := \{\emptyset\} \cup \big\{\{x\} : x \in [0, 1]\big\}$.
(Ex 3) $\mathbb{F} := \big\{A : A \subset [0, \frac{1}{2}]\big\} \cup \big\{A : A \subset [\frac{1}{2}, 1]\big\}$.
(Ex 4) $\mathbb{F} := \big\{A : A \subset [0, \frac{1}{2}[\big\} \cup \big\{A : A \subset [\frac{1}{2}, 1]\big\}$.

Theorem 1.1 will be proved in Section 2 where we will also introduce the basic notion of "reconstruction of polytopes and convex sets" and two criteria on "reconstruction basis". Then, Theorem 1.1 will be used in Section 4 to "solve minimax equalities".

In Section 3, following Greco [3], a family of sets $\mathbb{F}(\Omega)$ is associated to a multifunction Ω, as described in (5) of the proof of Theorem 1.3 below. Properties (1)–(4) for $\mathbb{F}(\Omega)$ are investigated.

To offer some motivation to the reader, and also to illustrate our method, we conclude this section with the proofs of two minimax equalities using Theorem 1.1, one of which, Flåm-Greco's minimax Theorem [2], was the motivation for condition (4): the *simplex property.*

Other intriguing applications of Theorem 1.1 concerning the finite intersection property for multifunctions will be investigated in subsequent papers.

Theorem 1.3 (Flåm-Greco [2]). *Let* X, Y *be convex sets of real vector spaces and* Y *equipped with a linear topology. Assume* $f : X \times Y \to \overline{\mathbb{R}}$ *is a quasi-concave-convex function* (1) *which is lower semicontinuous on* Y *and inf-compact on* Y *at every point of* X (2). *If*

(1)Let X and Y be convex sets of real vector space. A function $f : X \times Y \to \overline{\mathbb{R}}$ is *quasi-concave-convex* if it is both *quasi-concave on* X (i.e. for every $\bar{y} \in Y$ and $\alpha \in \mathbb{R}$, the set $\{x \in X : f(x, \bar{y}) \geq \alpha\}$ is a convex subset of X) and *quasi-convex on* Y (i.e. for every $\bar{x} \in X$ and $\alpha \in \mathbb{R}$, the set $\{y \in Y : f(\bar{x}, y) \leq \alpha\}$ is a convex subset of Y).

(2)Let X and Y be topological spaces. A function $f : X \times Y \to \overline{\mathbb{R}}$ is *lower semicontinuous on* Y, if for every $\bar{x} \in X$ and $\alpha \in \mathbb{R}$, the set $\{y \in Y : f(\bar{x}, y) \leq \alpha\}$ is closed in Y. Moreover, we say that f is *inf-compact on* Y *at every point* (resp. *at some point*) of X, if, for every $\alpha \in \mathbb{R}$, the set $\{y \in Y : f(x, y) \leq \alpha\}$ is relatively compact in Y for every (resp. at least one) point $x \in X$. Generally, when we say that "f has a given property on Y" we mean that for every $x \in X$ the function $y \mapsto f(x, y)$ has that property.

$$(\mathbf{S}) \qquad \inf_Y \sup_S f = \inf_Y \sup_{S\setminus\{x\}} f \quad {}^{(3)}$$

whenever $S \subset X$ *is a simplex with* $\dim(S) \geq 1$ *and* x *is a vertex of* S*, then* $\inf_Y \sup_X f = \sup_X \inf_Y f$.

Proof. It is enough to prove the inequality $\inf_Y \sup_X f \leq \sup_X \inf_Y f$, which means showing that for any real number $\lambda > \sup_X \inf_Y f$ the intersection over X of the sets

$$\Omega x := \{y \in Y : f(x, y) \leq \lambda\}$$

is not empty. Now, consider the family $\mathbb{F}$ defined by

$$\mathbb{F} := \{A \subset X : \cap_{x\in A}\Omega x \neq \emptyset\}. \tag{5}$$

We have to see that $X \in \mathbb{F}$ (i.e. the intersection of the values of Ω is nonempty). The inequality $\lambda > \sup_X \inf_Y f$ implies that the values of Ω are nonempty; hence $\mathbb{F}$ *covers* X. Obviously $\mathbb{F}$ is *subset-stable*. Lower semicontinuity and inf-compactness of f entail that the values of Ω are both closed and compact. Hence, $\mathbb{F}$ contains the union of any increasing net of sets belonging to $\mathbb{F}$; therefore $\mathbb{F}$ is *increasing-net-stable*. By the definition of quasi-concavity, we have

$$\Omega x \subset \Omega x_0 \cup \Omega x_1 \quad \forall\, x \in [x_0, x_1] \subset X. \tag{6}$$

As observed in Flåm-Greco [2], it is easy to check that (6) amounts to

$$\cap_{x\in C}\Omega x \subset (\cap_{x\in A}\Omega x) \cup (\cap_{x\in B}\Omega x) \tag{7}$$

whenever C separates A and B. Hence, $\mathbb{F}$ verifies the *separating set property*. If A and B are separated by their intersection $A \cap B$, from (7) it follows that

$$\cap_{x\in A\cap B}\Omega x = (\cap_{x\in A}\Omega x) \cup (\cap_{x\in B}\Omega x)\,. \tag{8}$$

On the other hand, quasi-convexity and lower semicontinuity of f on Y imply that the values of Ω are both convex and closed. Furthermore a convex subset of Y is connected; these facts together with (8) entail that $\cap_{x\in A\cup B}\Omega x = (\cap_{x\in A}\Omega x)\cap(\cap_{x\in B}\Omega x) \neq \emptyset$, whenever $A, B \in \mathbb{F}$. In other words, the *pasting property* holds for $\mathbb{F}$. Finally, taking into account lower semicontinuity of f and its inf-compactness on Y, the inf's of the equality (S) are attained. Using this, it is easy to check that condition (S) implies that $\mathbb{F}$ verifies the *simplex property*. Now, that all the required properties on $\mathbb{F}$ have been checked, Theorem 1.1 yields the desired result: $X \in \mathbb{F}$. $\square$

Remark 1.4. One can notice that (6) involves only the linear structure on X. The proof of (7), and consequently of (8) also, uses only the linear structure on X. The linear structure on Y enters only in the proof through the fact that convex sets are connected, and therefore

$$\cap_{x\in C}\Omega x \text{ is connected, for every subset } C \text{ of } X. \tag{9}$$

(3) An expression of the type $\inf_B \sup_A f$ (resp. $\sup_C \inf_D f$) has to be seen as short hand for the following $\inf_{y\in B} \sup_{x\in A} f(x, y)$ (resp. $\sup_{x\in C} \inf_{y\in D} f(x, y)$). Therefore, the variable of maximization will be in X, the minimization variable in Y.

Theorem 1.1 is also a useful tool to investigate minimax equalities involving linear structure only on one variable (see Horvath [8], Simons [10] and references therein). As an example, in the next theorem we will remove the linear structure on Y which is required by the previous minimax theorem. The quasi-convexity on Y will be replaced by condition (10) below which is less demanding than other similar known conditions (for example, see König [9] where (9) is used).

Theorem 1.5 (Flåm-Greco's minimax theorem without linear structure on Y). *Let X be a convex set of a real vector space, Y a topological space and $f : X \times Y \to \overline{\mathbb{R}}$ a function which is quasi-concave on X, lower semicontinuous on Y and inf-compact on Y at every point of X. Then* $\inf_Y \sup_X f = \sup_X \inf_Y f$, *if* (S) *holds and the following property is satisfied:*

$$\cap_{x \in C} \{y \in Y : f(x,y) \leq \alpha\} \text{ is connected} \tag{10}$$

whenever α is a real number and $C \subset X$ is a polytope with $\dim C < \dim X$.

Proof. First, let us notice that one can proceed exactly as in the proof of Theorem 1.3, if Property (9) (which is obviously stronger than (10)) holds, whenever $\alpha \in \mathbb{R}$ and Ω is defined by $\Omega x := \{y \in Y : f(x,y) \leq \alpha\}$.

Otherwise, let us say that two polytopes are *interfaced*, if their union is a convex set and their intersection is a common facet. Then, keeping the notation of the proof of Theorem 1.3, by (8), (10) and Remark 1.4 it follows immediately that

$$A \cup B \in \mathbb{F}, \text{ whenever } A \text{ and } B \text{ are interfaced polytopes belonging to } \mathbb{F}. \tag{11}$$

Finally, Theorem 2.7 below implies that Theorem 1.1 still holds when (3) is replaced by (11). From here, to complete the proof one can argue as in the proof of Theorem 1.3. □

Now, let us agree on the notation. "Convex set" always means "convex subset of a real vector space". If X is a convex set, the expression "simplex of X" and "polytope of X" stand for "simplex included in X" and "polytope included in X", respectively.

If topological terminology is used in relation to convex sets, we will, implicitly, assume that the convex set is equipped with an arbitrary topology coarser that the *finite topology.* Recall that a subset A of a convex set Z is said to be closed with respect to the finite topology, if its intersection with any polytope P of Z is closed in P with respect to the Euclidean topology (on P). We say that a family $\mathbb{B}$ of subsets of a convex set X *openly covers* X, if the topological interiors of the sets B cover X, where B runs in $\mathbb{B}$.

For a subset A of a convex set the *affine dimension* is denoted by $\dim A$ and the *affine span* by $\mathrm{aff}(A)$. Basic notions on convex sets and (convex) polytopes can be found in Grünbaum [6] and in Brøndsted [1].

2. Reconstructing convex sets by small simplices

Let A and B be subsets of a convex set. We will denote by $[A, B]$ the *segment-join* of A and B, that is:

$$[A, B] := \cup\{[a, b] : a \in A, b \in B\}. \tag{12}$$

If A is a singleton, say $\{x\}$, the symbol $[x, B]$ (resp. $[B, x]$) stands for $[\{x\}, B]$ (resp. $[B, \{x\}]$).

A *pointed simplex* is a pair (S, x) where x is a vertex of the simplex S and $\dim S \geq 1$. We will also say that the simplex S is *pointed at* x. If B stands for the facet of S which does not contain the vertex x, let us define, for every $\lambda \in]0, 1[$, the sets $S_\lambda^- x$ and $S_\lambda^+ x$ by

$$S_\lambda^- x := [x, (1-\lambda)x + \lambda B] \quad \text{and} \quad S_\lambda^+ x := [(1-\lambda)x + \lambda B, B]. \tag{13}$$

Notice that $S_\lambda^- x \cap S_\lambda^+ x = (1-\lambda)x + \lambda B$.

We say that two subsets A and B of a convex subset of a vector space are *interfaced* (by their intersection), if $A \cup B$ is convex and there exists an hyperplane H in the affine span of $A \cup B$ such that $A \cap B = (A \cup B) \cap H$; in the case where neither A nor B is included in the hyperplane H, we said that A and B are *properly interfaced*(4). Clearly, two interfaced convex sets are separated by their intersection(5).

Also, notice that two polytopes P_1 and P_2 are properly interfaced if and only if $P_1 \cup P_2$ is convex and $P_1 \cap P_2$ is a common facet.

Definition 2.1. A family $\mathbb{P}$ of polytopes of a vector space E is called a *polytope ideal* on E, if $\mathbb{P}$ contains every polytope of E which is a subset of some element of $\mathbb{P}$ and the union of every properly interfaced pair of polytopes belonging to $\mathbb{P}$.

Notice that for any family $\mathbb{F}$ satisfying the assumptions of Theorem 1.1, the family of all polytopes belonging to $\mathbb{F}$ is a polytope ideal. Let us recall the following useful result.

Theorem 2.2 (see Theorem A in Greco-Horvath [4])**.** *Let $\mathbb{P}$ be a polytope ideal on a vector space E. Then $\mathbb{P}$ contains every polytope of E which is covered by finitely many polytopes belonging to $\mathbb{P}$.*

Given a family $\mathbb{B}$ of subsets of a vector space E, we define the infinite sequence of families of polytopes $\mathbb{B}_0$, $\mathbb{B}_1$, $\mathbb{B}_2$, …, (by recursion) as follows:

(14) a polytope P of E is in $\mathbb{B}_0$, if there exists a set $S \in \mathbb{B}$ such that $P \subset S$ (in the sequel, the elements of $\mathbb{B}_0$ will be called $\mathbb{B}$-*basic polytopes*);

(15) a polytope P of E is in $\mathbb{B}_{n+1}$ if either $P \in \mathbb{B}_n$ or there exist two properly interfaced polytopes $P_i \in \mathbb{B}_n$, $i = 1, 2$, such that $P = P_1 \cup P_2$.

Polytopes of E belonging to

$$\mathcal{R}ec(\mathbb{B}) := \cup_{n \geq 0} \mathbb{B}_n. \tag{16}$$

will be said to be *reconstructible from* $\mathbb{B}$.

(4) All interfaced pairs of convex sets can be obtained by cutting arbitrary convex sets along hyperplanes which they meet. All the pairs of properly interfaced convex sets can be obtained by imposing that the cutting hyperplanes strictly separate at least two distinct points of the convex set.

(5) Observe that two convex sets A and B are *properly interfaced*, if (a) they are separated by $A \cap B$, (b) $\mathrm{aff}(A \cap B)$ is an hyperplane in $\mathrm{aff}(A \cup B)$ and (c) neither A nor B are subsets of $\mathrm{aff}(A \cap B)$. Moreover, notice that two finite dimensional convex sets A and B are properly interfaced if and only if their intersection $A \cap B$ separates A and B and $\dim(A \cap B) < \min\{\dim A, \dim B\}$.

Proposition 2.3. *The following properties hold:*

(17) $\mathcal{R}ec(\mathbb{B})$ *is a polytope ideal on* E*;*

(18) *every polytope ideal on* E *containing the family of* $\mathbb{B}$*-basic polytopes, includes* $\mathcal{R}ec(\mathbb{B})$*;*

(19) $\mathcal{R}ec(\mathbb{B})$ *is the family of all polytopes of* E *which are covered by finitely many* $\mathbb{B}$*-basic polytopes.*

Proof. By (15) the family $\mathcal{R}ec(\mathbb{B})$ contains the union of every properly interfaced pair of polytopes belonging to $\mathcal{R}ec(\mathbb{B})$. Moreover, every polytope of E which is a subset of an element of $\mathcal{R}ec(\mathbb{B})$ belongs to $\mathcal{R}ec(\mathbb{B})$, because each of the families $\mathbb{B}_n$ satisfies that same property. Hence (17) holds.

Clearly, any polytope ideal $\mathbb{P}$ containing the $\mathbb{B}$-basic polytopes must include each of the families $\mathbb{B}_n$; hence $\mathcal{R}ec(\mathbb{B}) \subset \mathbb{P}$. Therefore (18) holds.

Let $\mathbb{P}$ be the family of all polytopes of E which are covered by finitely many $\mathbb{B}$-basic polytopes. Clearly, $\mathbb{P}$ is a polytope ideal containing the $\mathbb{B}$-basic polytopes; hence, from (18) it follows that $\mathcal{R}ec(\mathbb{B}) \subset \mathbb{P}$. On the other hand, from Theorem 2.2 it follows that $\mathcal{R}ec(\mathbb{B}) \supset \mathbb{P}$, because the polytope ideal $\mathcal{R}ec(\mathbb{B})$ contains the $\mathbb{B}$-basic polytopes. Hence (19) holds. □

Let X be a convex subset E. Then we will say that $\mathbb{B}$ is a *reconstruction base* for X, if every polytope of X is reconstructible from $\mathbb{B}$. This extends the definition of reconstruction base given in [4].

Our first reconstruction criterion is a direct consequence of property (19) of Proposition 2.3.

Theorem 2.4 (first reconstruction base criterion). $\mathbb{B}$ *is a reconstruction base for* X *if and only if every polytope of* X *is covered by finitely many* $\mathbb{B}$*-basic polytopes.*

As shown in Greco-Horvath [4], there are plenty of reconstruction basis for a given convex set.

Example 2.5. Here are some examples of reconstruction basis for a convex set X:

(R_1) any family which contains all simplices of X,

(R_2) any family $\mathbb{B}$ which covers X and for which there exists a point $x \in X$ such that all simplices of X which are pointed at x are in $\mathbb{B}$,

(R_3) any family of subsets of X which openly covers X, [6]

(R_4) any finite family of closed convex subsets of X which covers X, [7]

(R_5) any family $\mathbb{B}$ which covers X such that, for every $x \in X$ and for every pointed simplex $S \subset X$ at x, there exists a real number $\lambda \in]0,1[$ and $B \in \mathbb{B}$ such that $S_\lambda^- x \subset B$. [8]

[6] Using Theorem 2.4 (*first reconstruction base criterion*) (R_1)-(R_3) are easily checked.

[7] This can be easily proved, using Theorem 2.6, (*second reconstruction base criterion*). Alternatively, take a polytope P of X and assume that the convex sets $C_1,\dots, C_n$ cover X and are closed in X. Therefore, $P = \cup_{i=1}^n (P \cap C_i)$ and the convex sets $P \cap C_i$ are closed in P. Hence, by Theorem 1 of Greco-Horvath [4] or by Hoffman [7], there exist polytopes $P_1,\dots,P_n$ such that $P = \cup_{i=1}^n P_i$ and, for any i, $P_i \subset P \cap C_i$. By the *first reconstruction base criterion* (see Theorem 2.4), this implies that the families described in (R_4) are reconstruction bases.

[8] It is an immediate consequence of the *second reconstruction base criterion.* Alternatively, (R_5) can be proved using Proposition 3 of Greco-Horvath [4] jointly with the *first reconstruction base criterion.*

The reconstructibility of a polytope depends on basic polytopes of the family; consequently, it depends on basic simplices of the family (in virtue of Carathéodory's Lemma). Now, we will show that reconstructibility depends on "small" basic simplices. As above, let $\mathbb{B}$ be a family of subsets of a vector space E and $X \subset E$ a convex set.

Theorem 2.6 (second reconstruction base criterion). *$\mathbb{B}$ is a reconstruction base for X if and only if it covers X and for every $\bar{x} \in X$ and for every pointed simplex $S \subset X$ at $\bar{x}$*

(20) *there exist $\lambda \in]0,1[$ and a finite family $\{S_i\}_{i=1}^n$ of pointed simplices at $\bar{x}$ which are $\mathbb{B}$-basic such that $S_\lambda^- \bar{x} = \cup_{i=1}^n S_i$.*

Proof. First, we prove the *if part* from the first reconstruction base criterion. Let P be a polytope of X; we have to show that P is covered by a finite family of $\mathbb{B}$-basic polytopes. In the case where the dimension of P is zero, there is nothing to prove, because $\mathbb{B}$ covers X. Now suppose the dimension of P is not zero. By compactness of P, it is enough to verify that any point of P has a neighborhood which is covered by a finite family of $\mathbb{B}$-basic polytopes. Then fix $\bar{x} \in P$. List the facets of P which do not contain $\bar{x}$: $F_1, \ldots, F_n$. Clearly $P = \cup_{i=1}^n [\bar{x}, F_i]$. A facet F_i being a polytope it is also, by Carathéodory's Lemma, the union of finitely many simplices. Hence, there are finitely many simplices $G_1, \ldots, G_k$ such that $\bar{x}$ is not in $\operatorname{aff}(G_i)$ for any $i = 1, \cdots k$, and $P = \cup_{i=1}^k [\bar{x}, G_i]$. Applying (20) to any of the pointed simplex $[\bar{x}, G_i]$, shows that there exists $\lambda \in]0,1[$ such that any $[\bar{x}, (1-\lambda)\bar{x} + \lambda G_i]$ is covered by finitely many $\mathbb{B}$-basic simplices. Finally, since $(1-\lambda)\bar{x} + \lambda P = \cup_{i=1}^k [\bar{x}, (1-\lambda)\bar{x} + \lambda G_i]$, we also have that $(1-\lambda)\bar{x} + \lambda P$ is covered by finitely many $\mathbb{B}$-basic polytopes. Since $(1-\lambda)\bar{x} + \lambda P$ is a neighborhood of $\bar{x}$ in P, we have reached the desired conclusion.

To prove the *only if part* assume that $\mathbb{B}$ is a reconstruction base which covers X and that $S \subset X$ is a pointed simplex at $\bar{x}$. From the *first reconstruction base criterion* (see Theorem 2.4) it follows that there exist finitely many $\mathbb{B}$-basic polytopes, $P_1, \ldots, P_n$ such that $S = \cup_{i=1}^n P_i$; consequently, by Carathéodory's Lemma, there exist finitely many $\mathbb{B}$-basic simplices, $S_1, \ldots, S_k$, such that $S = \cup_{i=1}^k S_i$. Now, take $\lambda \in]0,1[$ such that the simplex $S_\lambda^- \bar{x}$ does not contain any of the vertices, other than $\bar{x}$, of any of the simplices S_i. Define $J := \{i : 1 \leq i \leq k\}$ such that $S_\lambda^- \bar{x} \cap S_i \neq \emptyset$ and $S_i \neq \{\bar{x}\}$. Clearly, $J \neq \emptyset$ and, for every $j \in J$, the simplex $S_\lambda^- \bar{x} \cap S_j$ is pointed at $\bar{x}$. On the other hand it is easy to check that $S_\lambda^- \bar{x} = \cup_{j \in J} S_\lambda^- \bar{x} \cap S_j$. This completes the proof. □

Finally, we come to the proof of Theorem 1.1. But first, recall that if a family $\mathbb{F}$ fulfils all the conditions of Theorem 1.1, then the family of all $\mathbb{F}$-basic polytopes is a polytope ideal (i.e. $\mathcal{R}ec(\mathbb{F}) \subset \mathbb{F}$). Notice also that for such a family conditions (21), (22) and (23) of Theorem 2.7 below hold. Indeed, (21) follows from (13), (12) clearly implies (22), and (23) is the same as (15). Therefore a proof of Theorem 2.7 proves also Theorem 1.1.

Theorem 2.7 (reconstruction of polytopes by small simplices). *Let $\mathbb{F}$ be a family of subsets of a vector space E and $X \subset E$ a convex set. Assume that $\mathbb{F}$ covers X and that the family of all $\mathbb{F}$-Basic polytopes is a polytope ideal. Then every polytope of X belongs to $\mathbb{F}$ if, for every $x \in X$ and for every simplex S of X pointed at x, the following three properties hold:*

(21) *either $S_\lambda^- x$ or $S_\lambda^+ x$ belongs to $\mathbb{F}$, whenever $\lambda \in]0,1[$ and $S_\lambda^- x \cap S_\lambda^+ x$ belongs to $\mathbb{F}$,*

(22) *$S \setminus \{x\}$ belongs to $\mathbb{F}$, whenever, for any $\lambda \in \,]0,1[$, the set $S^+_\lambda x$ belongs to $\mathbb{F}$,*
(23) *S belongs to $\mathbb{F}$, whenever $S \setminus \{x\}$ belongs to $\mathbb{F}$.*

Proof. It is clear that $\mathbb{F}$ contains all polytopes of X of dimension zero, because $\mathbb{F}$ covers X. Now, arguing by induction, let $n \geq 1$ and suppose that

(24) $\mathbb{F}$ contains all polytopes of X of dimension $< n$.

We want to prove that any polytope of X having dimension n is contained in $\mathbb{F}$. For simplicity, we can suppose that X is a polytope of dimension n. Then, it is enough to show that $\mathbb{F}$ is a reconstruction base of the type described in (R_5). Let $S \subset X$ be an arbitrary simplex of X pointed at x. By the induction hypothesis ($*1$), for every $\lambda \in \,]0,1[$, the simplex $S^-_\lambda x \cap S^+_\lambda x$ belongs to $\mathbb{F}$, because its dimension is less than n. Hence, from (21) one of the following two cases holds:

(25) there exists $\lambda \in \,]0,1[$ such that $S^-_\lambda x \in \mathbb{F}$
(26) $S^+_\lambda x \in \mathbb{F}$, for every $\lambda \in \,]0,1[$

In the second case, from (22) and (23) it follows that S belongs to $\mathbb{F}$. Hence, all the conditions required by (R_5) hold. Therefore $X \in \mathbb{F}$, as desired. □

3. Families of sets associated to concave multifunctions

Let $\Omega : X \twoheadrightarrow Y$ be a multifunction. The family of sets $\mathbb{F}(\Omega)$ is defined by

$$\mathbb{F}(\Omega) := \{A \subset X : \cap_{x \in A} \Omega x \neq \emptyset\}.$$

To prove Theorems 1.3 and 1.5 we applied Theorem 1.1 on reconstruction of convex sets to families of type $\mathbb{F}(\Omega)$. We therefore had to verify the following properties:

(P^+_1) $\mathbb{F}(\Omega)$ covers X,
(P^+_2) $\mathbb{F}(\Omega)$ is subset-stable,
(P^+_3) $\mathbb{F}(\Omega)$ is increasing-net-stable,
(P^+_4) $\mathbb{F}(\Omega)$ has the pasting property,
(P^+_5) $\mathbb{F}(\Omega)$ has the separating set property,
(P^+_6) $\mathbb{F}(\Omega)$ has the simplex property.

Clearly (P^+_1) says that Ω has nonempty values; while (P^+_2) is always true. Now let us investigate how the other properties of $\mathbb{F}(\Omega)$ depend on Ω.

All the multifunctions which will be considered in this paper are *concave*, that is: X is a convex set and

$$\Omega x \subset \Omega x_0 \cup \Omega x_1 \text{ whenever } x \in [x_0, x_1] \subset X.$$

A concave multifunction Ω is said to be *concave-convex*, if Y is convex and the values of Ω are convex subsets of Y. Let us start by giving a straightforward and useful key characterization, (27) below, of concavity which is due to Flåm-Greco, see [2] where it was called *separating set property.*

Lemma 3.1 (separating set property). *Let $\Omega : X \twoheadrightarrow Y$ be a concave multifunction where Y is an arbitrary set. If A, B and C are subsets of X, then*

$$\cap_{x \in C} \Omega x \subset (\cap_{x \in A} \Omega x) \cup (\cap_{x \in B} \Omega x) \tag{27}$$

whenever C separates A and B. Hence $\mathbb{F}(\Omega)$ verifies the separating set property. Moreover, if $A \cap B$ separates A and B, then

$$\cap_{x \in A \cap B} \Omega x = (\cap_{x \in A} \Omega x) \cup (\cap_{x \in B} \Omega x) . \tag{28}$$

Proof. For every $a \in A$ and $b \in B$ choose $c_{a,b} \in [a, b] \cap C$. If $y \in \cap_{x \in C} \Omega x$ then $y \in \Omega c_{a,b}$ for every $(a, b) \in A \times B$. By hypothesis we have $\Omega c_{a,b} \subset \Omega a \cup \Omega b$, and therefore $y \in \Omega a \cup \Omega b$ for every $(a, b) \in A \times B$. This implies that $y \in (\cap_{x \in A} \Omega x) \cup (\cap_{x \in B} \Omega x)$.

The second part is a consequence of the first and the fact that the other inclusion always holds. □

The term "pasting property" was introduced in Greco [3], where it was shown that the union of two closed convex sets A and B belongs to $\mathbb{F}(\Omega)$, whenever the following conditions are fulfilled: (a) A and B belong to $\mathbb{F}(\Omega)$, (b) the union $A \cup B$ is convex and (c) the multifunction Ω is concave-convex with closed values.

Lemma 3.2 (about the pasting property). *Let $\Omega : X \twoheadrightarrow Y$ be a concave multifunction and Y a topological space. If the following properties hold*

(29) *A, B are two subsets of X which are separated by their intersection,*
(30) *$\cap_{x \in A \cap B} \Omega x$ is connected,*
(31) *both $\cap_{x \in A} \Omega x$ and $\cap_{x \in B} \Omega x$ are nonempty and closed,*

then $\cap_{x \in A \cup B} \Omega x \neq \emptyset$.

Proof. From (28)–(31) it follows that the connected set $\cap_{x \in A \cap B} \Omega x$ is covered by the two nonempty closed sets $\cap_{x \in A} \Omega x$ and $\cap_{x \in B} \Omega x$. Therefore $(\cap_{x \in A} \Omega x) \cap (\cap_{x \in B} \Omega x) \neq \emptyset$. Hence, the conclusion follows from $(\cap_{x \in A} \Omega x) \cap (\cap_{x \in B} \Omega x) = \cap_{x \in A \cup B} \Omega x$. □

Lemma 3.3 (about the increasing-net stability). *Let Y be a topological space and $\Omega : X \twoheadrightarrow Y$ a multifunction with closed and compact values, then $\mathbb{F}(\Omega)$ is increasing-net-stable.*

Proof. Let $\{A_j\}_j$ be an increasing net in $\mathbb{F}(\Omega)$. Then $\{\cap_{x \in A_j} \Omega x\}_j$ is a decreasing net of nonempty, closed and compact subsets of Y. By compactness we can conclude that $\cap_j (\cap_{x \in A_j} \Omega x) = \cap_{x \in \cup_j A_j} \Omega x \neq \emptyset$, which means that $\cup_j A_j \in \mathbb{F}(\Omega)$. □

Let us say that a multifunction $\Omega : X \twoheadrightarrow Y$ has *marginally connected* values, if Y is a topological space and

(32) for every polytope $P \subset X$ with $\dim(P) < \dim(X)$, the set $\cap_{x \in P} \Omega x$ is connected.

Observe that, if Y is convex and a multifunction $\Omega : X \twoheadrightarrow Y$ has convex values, then Ω has marginally connected values, whenever Y is equipped with any topology whose segments are connected (for example: any topology coarser than the finite topology on Y).

An immediate consequence of the previous lemmas is the following theorem.

Theorem 3.4. *Let Y be a topological space and $\Omega : X \twoheadrightarrow Y$ a concave multifunction with closed and marginally connected values. Then $\mathbb{F}(\Omega)$ is subset-stable and has both the separating set property and the pasting property. Moreover, $\mathbb{F}(\Omega)$ is increasing-net-stable, if the values of Ω are compact.*

Let us say that a multifunction Ω has the *simplex property*, if the family $\mathbb{F}(\Omega)$ has the simplex property. From Theorem 3.4 and Theorem 1.1 we can derive the following intersection theorem which was proved by Flåm-Greco in [2] for concave-convex multifunctions.

Theorem 3.5 (nonempty intersection property criterion). *Let Y be a topological space and $\Omega : X \twoheadrightarrow Y$ a concave multifunction with nonempty closed, compact and marginally connected values. Then $\cap_{x \in X} \Omega x \neq \emptyset$ if and only if Ω has the simplex property.*

From Theorem 3.4 and the reconstruction base criteria of the previous section, we have

Proposition 3.6. *Let Y be a topological space and $\Omega : X \twoheadrightarrow Y$ a concave multifunction with closed and marginally connected values. Then $\mathbb{F}(\Omega)$ contains all polytopes of X which are reconstructible from $\mathbb{F}(\Omega)$. In particular, $\mathbb{F}(\Omega)$ contains all polytopes of X which are union of finitely many simplices belonging to $\mathbb{F}(\Omega)$.*

The six conditions (P_1^+)–(P_6^+) are necessary and sufficient to have that $X \in \mathbb{F}(\Omega)$ (that is, the intersection of the values of Ω is not empty). As observed, the subset-stable property is always true for $\mathbb{F}(\Omega)$, while the five other properties can be violated. In each of the following examples of multifunctions, all except one of the these five properties are satisfied by $\mathbb{F}(\Omega)$.

Example 3.7. The family of sets associated to the multifunction $\Omega : [0,1] \twoheadrightarrow [0,1]$ defined by

$$\Omega x := \emptyset \tag{Ex1}$$

does not cover X, but verifies all other five properties.

Example 3.8 (about the increasing-net stability of $\mathbb{F}(\Omega)$). The family of sets associated to the following multifunction $\Omega : [0,1] \twoheadrightarrow \mathbb{R}$

$$\Omega x := \begin{cases} \{0\} & x = 0 \\ [\frac{1}{x}, +\infty[& \text{otherwise} \end{cases} \tag{Ex3}$$

is not increasing net stable; all other five properties are satisfied. Moreover, observe that Ω satisfies all the assumptions of Lemma 3.3 except the compactness of the values and that the conclusion of the Lemma fails for Ω.

Example 3.9 (about the separating set property for $\mathbb{F}(\Omega)$). The family of sets associated to the following multifunction $\Omega : [0,1] \twoheadrightarrow [0,1]$

$$\Omega x := \{x\} \tag{Ex4}$$

does not verify the separating set property, but all other five properties are satisfied. This multifunction Ω shows that the concavity of Ω cannot be dropped in Lemma 3.1.

Example 3.10 (about the pasting property for $\mathbb{F}(\Omega)$). The families of sets associated to the multifunctions $\Gamma_1, \Gamma_2, \Gamma_3 : [0,1] \twoheadrightarrow [0,1]$ defined by

(Ex5)
$$\Gamma_1 x := \begin{cases} \{0\} & \text{if } x \in [0, \frac{1}{2}[\\ \{0,1\} & \text{if } x = \frac{1}{2} \\ \{1\} & \text{if } x \in]\frac{1}{2}, 1] \end{cases} \qquad \Gamma_2 x := \begin{cases} [0, \frac{1}{2}[& \text{if } x \in [0, \frac{1}{2}[\\ [0,1] & \text{if } x = \frac{1}{2} \\ [\frac{1}{2}, 1] & \text{if } x \in]\frac{1}{2}, 1] \end{cases}$$
$$\Gamma_3 x := \begin{cases} \{0\} & \text{if } x \in [0, \frac{1}{2}[\\ [0,1] & \text{if } x = \frac{1}{2} \\ \{1\} & \text{if } x \in]\frac{1}{2}, 1] \end{cases}$$

do not have the pasting property but all the other five properties are verified. The first Γ_1 is concave and has closed values, but some value is not convex; the second Γ_2 is concave and its values are convex but some value is not closed; the third Γ_3 is not concave but its values are both closed and convex. Hence concavity, convexity and closedness of the values are essential to have the pasting property in Lemma 3.2.

Example 3.11 (about the simplex property for $\mathbb{F}(\Omega)$). Let $n \geq 1$ and consider $n+1$ affinely independent points: $\{x_i\}_{i=0}^n$. They generate a simplex. We let both X and Y be that simplex. Denote by F_i the facet opposite to vertex x_i. Let $S_{-1} := \emptyset$ and, for $0 \leq i \leq n$, let S_i be the simplex generated by $\{x_s\}_{s=0}^i$. Now, consider the multifunction $\Omega : X \twoheadrightarrow Y$ defined by

(Ex6)
$$\Omega x := F_i \text{ whenever } 0 \leq i \leq n \text{ and } x \in S_i \setminus S_{i-1}.$$

Notice that $\mathbb{F}(\Omega)$ does not verify the simplex property, but all other five properties are satisfied. Furthermore,

(33) if $S \subset X$ is a simplex with $1 \leq \dim S < \dim X$ and x is a vertex of S such that $S \setminus \{x\} \in \mathbb{F}(\Omega)$, then $S \in \mathbb{F}(\Omega)$.

This shows that, in the simplex property, one cannot generally impose an upper bound constraint on the dimension of the pointed simplex S.

4. Solving minimax equalities

A *minimax equality* is an equality of the following type

(34)
$$\sup_X \inf_Y f = \inf_Y \sup_X f$$

where X and Y are nonempty sets and f is a function defined on the product $X \times Y$ with values in the extended real line $\overline{\mathbb{R}}$. A function f which verifies (34) is called a *minimax function.*

Given a *scenario* (that is: a list of initial conditions on X, Y, f and the values of f[9]), *solving a minimax equality* means *describing the whole class or part of the class* of functions f enjoying (34) among those which belong to the scenario. A theorem (or a condition) which characterizes *all* minimax functions belonging to a scenario will be called

[9] We suggest that the reader sees a scenario as a class of quadruples (X, Y, f, L) where L is a subset of $\overline{\mathbb{R}}$ and f is a function from $X \times Y \to \overline{\mathbb{R}}$ such that $f(X \times Y) \subset L$. Accordingly, *subscenario* will stand for *subclass.* Roughly speaking, quadruples (X, Y, f, L) are identified to f

minimax criterion (with respect to such a scenario). A minimax criterion for the class consisting of all quadruples $(X, Y, f, \overline{\mathbb{R}})$ where X and Y are arbitrary nonempty sets and $f : X \times Y \to \overline{\mathbb{R}}$ is any function, is called an *abstract minimax criterion.*

Here is an obvious example of abstract minimax criterion:

(35) a function f is a minimax function if and only if, for all $\alpha > \sup_X \inf_Y f, \cap_{x\in X}\{f \leq \alpha\}x \neq \emptyset$.

Here and in the sequel the symbol $\{f \leq \alpha\}$ denotes the multifunction from X to Y defined by $\{f \leq \alpha\}x := \{y \in Y : f(x, y) \leq \alpha\}$.

By considering the indicator function of the graph of a multifunction, we see that a minimax criterion for a class of two valued functions becomes a necessary and sufficient condition for multifunctions of a given class to have the nonempty intersection property.

Our primary aim in this section is to exploit Theorem 1.1 on reconstruction of convex sets and reconstruction base criteria to give minimax criteria for scenarios where X and Y are convex sets and f is a quasi-concave-convex function.

Going back to Flåm-Greco's minimax Theorem 1.3, we recognize therein a scenario given by all quadruples $(X, Y, f, \overline{\mathbb{R}})$ satisfying the following list of conditions:

(FG-S) (a) X and Y are convex sets and $f : X \times Y \to \overline{\mathbb{R}}$ is quasi-concave-convex
(b) Y is equipped with a topology coarser than the finite topology on Y and f is lower semicontinuous on Y and inf-compact on Y at any point of X.

In the proof of Flåm-Greco's minimax Theorem 1.3, the abstract minimax criterion (35) was used to reduce the minimax equality problem to the nonempty intersection problem for

(36) the multifunctions $\Omega : X \twoheadrightarrow Y$ which are concave-convex and have nonempty closed and compact values.

In such a subscenario (4.3), the necessary condition:

(37) Ω has the simplex property

becomes sufficient to have that $\cap_{x\in X}\Omega x \neq \emptyset$ (see Theorem 3.5). In other words, condition (37) gives us a minimax criterion for the subscenario (36).

Theorem 4.1 (A minimax criterion in the scenario (FG-S)). *Under the assumptions* (FG-S)*, f is a minimax function if and only if,*

(38) *for every $\bar{x} \in X$ and pointed simplex $S \subset X$ at $\bar{x}$, one has*

$$\max\left\{\sup_X \inf_Y f, \inf_Y \sup_{S\setminus\{\bar{x}\}} f\right\} \geq \inf_Y \sup_S f.$$

Clearly, condition (38) holds, if the function f is lower semicontinuous on X [(10)]. This theorem follows from Theorem 3.5 and minimax criterion (35), because, in term of sublevel sets of the function f, property (38) amounts to

[(10)]Observe that an arbitrary function $g : X \to \overline{\mathbb{R}}$ is lower semicontinuous on X if and only if, for every subset A of X one has $\sup_A g = \sup_{\overline{A}} g$.

(39) for every $\alpha > \sup_X \inf_Y f$, the multifunction $\Omega := \{f \leq \alpha\}$ has the simplex property.

Observe that property (S) of Flåm-Greco's minimax Theorem 1.3 says that:

(**S′**) for every real number α the multifunction $\Omega := \{f \leq \alpha\}$ has the simplex property.

Now, we consider the scenario given by all quadruple $(X, Y, f, \overline{\mathbb{R}})$ satisfying the following list of conditions:

(G-S) (a) X and Y are convex sets and $f : X \times Y \to \overline{\mathbb{R}}$ is quasi-concave-convex
(b) Y is equipped with a topology coarser than the finite topology on Y and f is lower semicontinuous on Y
(c) X is equipped with a topology coarser than the finite topology on X
(d) f is sup-compact on X at some point of Y[11] or f is inf-compact on Y at some point of X.

Denote by f^+ the *upper regularization* of f on X, that is, for every $(x, y) \in X \times Y$,

(40)
$$f^+(x, y) := \limsup_{x' \to x} f(x', y).$$

Theorem 4.2 (A minimax criterion in the scenario (G-S)). *Under the assumptions* (G-S)*, a function f is a minimax function if and only if*

(41)
$$\sup_X \inf_Y f = \sup_X \inf_Y f^+.$$

Proof. We carry out the proof under the sup-compactness condition. The other case is left to the reader. Let $\alpha > \sup_X \inf_Y f$. By sup-compactness there exists $y_0 \in Y$ and a compact closed set K such that $\{x \in X : f(x, y_0) \geq \alpha\} \subset K$; hence

(42)
$$y_0 \in \cap_{x \in X \setminus K} \{f \leq \alpha\} x \neq \emptyset$$

By (41) we have that $\alpha > \sup_X \inf_Y f^+$; hence, from the definition of upper regularization it follows that

(43)
$$\text{for every } \bar{x} \in X \text{ there exists a open neighborhood } V \text{ of } \bar{x} \text{ in } X \text{ such that } \cap_{x \in V} \{f \leq \alpha\} x \neq \emptyset.$$

Hence, by compactness of K there exist finitely many open sets in X (say: $V_1, \ldots, V_n$) such that

(44)
$$K \subset \cup_{i=1}^n V_i \quad \text{and, for any } i, \cap_{x \in V_i} \{f \leq \alpha\} x \neq \emptyset.$$

Now, for every i, choose $y_i \in \cap_{x \in V_i} \{f \leq \alpha\} x$. Denote by Q the polytope of Y generated by the points: y_0, $y_1 \ldots$, y_n. And observe that the multifunction $\Omega : X \twoheadrightarrow Y$ defined by

(45)
$$\Omega x := \{f \leq \alpha\} x \cap Q$$

is concave-convex with closed compact values and the open sets $X \setminus K$, $V_1, \ldots, V_n$ cover X and belong to the family $\mathbb{F}(\Omega)$. Therefore, by the *first reconstruction base criterion* (see

[11] for every real number α, there exists $y_0 \in Y$ such that the set $\{x \in X : f(x, y_0) \geq \alpha\}$ is relatively compact in X.

Theorem 2.4) the family $\mathbb{F}(\Omega)$ is a reconstruction base for X. But, $\mathbb{F}(\Omega)$ is a polytope ideal; hence, every polytope of X belongs to $\mathbb{F}(\{f \leq \alpha\})$. Finally, the values of Ω being both closed and compact, we have that X belongs to $\mathbb{F}(\Omega)$, that is, $\cap_{x\in X}\Omega x \neq \emptyset$. *A fortiori*, it yields that $\cap_{x\in X}\{f \leq \alpha\}x \neq \emptyset$. In conclusion, by the abstract minimax criterion (35) we have that f is a minimax function. $\square$

Let f^- denote the *lower regularization* of f on Y; that is, for every $(x, y) \in X \times Y$, f^- is defined by

$$f^-(x, y) := \liminf_{y'\to y} f(x, y) \tag{46}$$

Now, let us iterate the "regularizations" to have the functions

$$f^{+-} := (f^+)^- \ , \ f^{-+} := (f^-)^+ \quad \text{and} \quad f^{+-+} := (f^{+-})^+ \ , \ f^{-+-} := (f^{-+})^-. \tag{47}$$

Iteration cannot yield other functions, because

$$(f^{+-})^{+-} = f^{+-} \quad \text{ and } (f^{-+})^{-+} = f^{-+}. \tag{48}$$

In the following corollary we will assume the following additional condition on the topology of a convex set: "*the closure of a convex set is a convex set*".

Corollary 4.3 (Greco [3]). *Let $f : X \times Y \to \overline{\mathbb{R}}$ an arbitrary quasi-convex-concave function which is either sup-compact on X at some point of Y or inf-compact on Y at some point of X. Then the functions f^{+-}, f^{-+-}, f^{-+} and f^{+-+} are minimax functions.*

Proof. Assume that f is quasi-concave-convex and sup-compact on X at some point of Y. It is easy to check the following elementary facts:

(49) f^-, f^+ and f^{+-} are quasi-concave-convex and sup-compact on X at some point of Y,

(50) $\inf_{y\in Y} g^-(y) = \inf_{y\in Y} g(y)$ and $\sup_{x\in X} h^+(x) = \sup_{x\in X} h(x)$

where $g : Y \to \overline{\mathbb{R}}$ and $h : X \to \overline{\mathbb{R}}$ are arbitrary functions. 1^{st} *claim: f^{+-} is a minimax function.* Using (50) and (48), we have:

(51) $\sup_X \inf_Y f^{+-} = \sup_X \inf_Y f^{+-+-} = \sup_X \inf_Y f^{+-+} = \sup_X \inf_Y (f^{+-})^+$

Therefore, the function f^{+-} which is quasi-concave-convex, lower semicontinuous on Y and sup-compact on X at some point of Y, verifies (41). Thus, from Theorem 4.2 it follows that f^{+-} is a minimax function.

2^{nd} *claim: f^{-+-} is a minimax function.* Observe that f^- is quasi-concave-convex and sup-compact on X at some point of Y. Apply the first claim to the function f^- to have that f^{-+-} is a minimax function.

3^{rd} *claim: f^{-+} is a minimax function.* From (43) and (48) deduce the following equalities

(52) $\sup_X \inf_Y f^{-+} = \sup_X \inf_Y f^{-+-}$ and
$\inf_Y \sup_X f^{-+} = \inf_Y \sup_X f^{-+-+} = \inf_Y \sup_X f^{-+-}$.

By the 2^{nd} *claim*, f^{-+-} is a minimax function; from (45) it follows that f^{-+} is a minimax function.

4^{th} *claim: f^{+-+} is a minimax function.* Apply the 3^{rd} *claim* to the function f^+.

In the case where f is inf-compact on Y at some point of X, apply what we have just shown to the function $-f$. □

Finally, we consider the scenario given by all quadruples $(X, Y, f, \overline{\mathbb{R}})$ satisfying the following list of conditions:

(GG-S) (a) X and Y are convex sets and $f : X \times Y \to \overline{\mathbb{R}}$ is quasi-concave-convex,
(b) Y is a subset of a locally convex vector space E and is equipped with the induced topology,
(c) f is both lower semicontinuous on Y and inf-compact on Y at some point of X.

Denote by $f^{(+,-)}$ the function from $X \times Y \to \overline{\mathbb{R}}$ defined by

$$f^{(+,-)}(x,y) := \sup_{A\in\mathcal{N}(y)} \limsup_{x'\to x} \inf_{y'\in A} f(x',y'). \tag{53}$$

where $\mathcal{N}(y)$ is the neighborhood filter of y in Y. Moreover, denote by $\mathcal{V}$ the family of all convex closed neighborhoods of 0 in E and, for every $V \in \mathcal{V}$ define the function $f_V : X \times Y \to \overline{\mathbb{R}}$ by

$$f_V(x,y) := \inf_{y'\in(y+V)\cap Y} f(x,y'). \tag{54}$$

Theorem 4.4 (A minimax criterion in the scenario (GG-S)). *Under the assumptions* (GG-S)*, the following properties are equivalent:*

(55) *f is a minimax function,*
(56) $\sup_X \inf_Y f = \sup_X \inf_Y f^{(+,-)}$.

Proof. First, we show that (55)⟹(56). Observe the following inequalities

$$\begin{aligned}\inf_Y \sup_X f &= \inf_Y \sup_X f^+ \geq \sup_X \inf_Y f^+ \geq \sup_X \inf_Y f^{(+,-)} \geq \sup_X \inf_Y f^- \\ &= \sup_X \inf_Y f.\end{aligned} \tag{57}$$

The first and the last equality are obvious, and so is the first inequality; the remaining two inequalities follow from "$f^+ \geq f^{(+,-)} \geq f^-$". Hence, from (57) and (55) follows (56).

Now, we will show that (56)⟹(55). Let P denote an arbitrary polytope contained in X. First, we notice that

$$\sup_{V\in\mathcal{V}} \inf_Y \sup_P (f_V)^- = \inf_Y \sup_P f. \tag{58}$$

The sequences of inequalities (59) and (60) below are easily checked.

$$\sup_X \inf_Y f = \sup_X \inf_Y f^{(+,-)} \geq \sup_P \inf_Y f^{(+,-)} \geq \sup_P \inf_Y (f_V)^+ = \sup_P \inf_Y (f_V)^{+-}. \tag{59}$$

$$\inf_Y \sup_P (f_V)^{+-} \geq \inf_Y \sup_P (f_V)^-. \tag{60}$$

Corollary 4.3 applied to the function $(f_V)^+$ restricted to $P \times Y$ yields[12]

(61) $$\sup_P \inf_Y (f_V)^{+-} = \inf_Y \sup_P (f_V)^{+-}.$$

From (59)–(61) we get

(62) $$\sup_X \inf_Y f \geq \inf_Y \sup_P (f_V)^- \quad \text{for any polytope} \quad P \subset X.$$

Finally, from (58) and (62) it follows that

(63) $$\sup_X \inf_Y f \geq \inf_Y \sup_P f \quad \text{for any polytope} \quad P \subset X.$$

Now, taking into account the inf-compactness condition, we get from (63)

$$\sup_X \inf_Y f \geq \inf_Y \sup_X f$$

and this concludes the proof. □

One more *saddling transformation* can now be added to those listed in Corollary 4.3.

Corollary 4.5 (Greco [3]). *Let X and Y be convex sets, with Y a subspace of a locally convex vector space E, and $f : X \times Y \to \overline{\mathbb{R}}$ a quasi-concave-convex function which is inf-compact on Y at some point of X. Then $f^{(+,-)}$ is a minimax function.*

Proof. We carry out the proof in two steps.

1^{st} *step.* Assume here that f is lower semicontinuous on Y. Let $\alpha = \sup_X \inf_Y f^{(+,-)}$. Since $f^{(+,-)} \geq f^- = f$ we have $\alpha \geq \sup_X \inf_Y f$, and therefore

(64) $$\begin{aligned}\sup_X \inf_Y \max\{f, \alpha\}^{(+,-)} &= \sup_X \inf_Y f^{(+,-)} = \max\{\sup_X \inf_Y f, \alpha\} \\ &= \sup_X \inf_Y \max\{f, \alpha\}.\end{aligned}$$

Theorem 4.4 tells us that $\max\{f, \alpha\}$ is a minimax function, therefore

(65) $$\sup_X \inf_Y \max\{f, \alpha\} = \inf_Y \sup_X \max\{f, \alpha\}.$$

From (57) and (58) we get

(66) $$\sup_X \inf_Y f^{(+,-)} = \inf_Y \sup_X \max\{f, \alpha\} \geq \inf_Y \sup_X f = \inf_Y \sup_X f^+.$$

Finally, the inequality $f^+ \geq f^{(+,-)}$ implies

(67) $$\inf_Y \sup_X f^+ \geq \inf_Y \sup_X f^{(+,-)}.$$

Taken together, (66) and (67) yield the desired conclusion, namely

$$\sup_X \inf_Y f^{(+,-)} \geq \inf_Y \sup_X f^{(+,-)}.$$

2^{nd} *step.* Now, to remove the lower semicontinuity assumption on f one just has to notice that $(f^-)^{(+,-)} = f^{(+,-)}$. □

[12] Notice that the regularizations appearing in (58)–(62) are taken with respect to $X \times Y$.

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