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A GRÜSS TYPE INEQUALITY FOR SEQUENCES OF VECTORS IN INNER PRODUCT SPACES AND APPLICATIONS

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ABSTRACT. A Grüss type inequality for sequences of vectors in inner product spaces which complement a recent result from [6] and applications for differentiable convex functions defined on inner product spaces and applications for Fourier and Mellin transforms, are given.

Key words and phrases: Grüss' Inequality, Inner Product Spaces.

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1. INTRODUCTION

In 1935, G. Grüss proved the following integral inequality (see [11] or [12])

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq \frac{1}{4}(\Phi-\phi)(\Gamma-\gamma), \tag{1.1}$$

provided that f and g are two integrable functions on $[a,b]$ and satisfy the condition

$$\phi\leq f(x)\leq\Phi \text{ and } \gamma\leq g(x)\leq\Gamma \text{ for all } x\in[a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is the *best possible* and is achieved for

$$f(x)=g(x)=sgn\left(x-\frac{a+b}{2}\right).$$

The discrete version of (1.1) states that:

If $a\leq a_i\leq A$, $b\leq b_i\leq B$ $(i=1,...,n)$ where $a,\ A,\ a_i,\ b,\ B,\ b_i$ are real numbers, then

$$\left|\frac{1}{n}\sum_{i=1}^n a_ib_i-\frac{1}{n}\sum_{i=1}^n a_i\,\frac{1}{n}\sum_{i=1}^n b_i\right|\leq\frac{1}{4}(A-a)(B-b) \tag{1.3}$$

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and the constant $\frac{1}{4}$ is the best possible.

In the recent paper [2], the author proved the following generalisation in inner product spaces.

Theorem 1.1. *Let* $(X;\langle\cdot,\cdot\rangle)$ *be an inner product space over* $\mathbb{K}, \mathbb{K}=\mathbb{C},\mathbb{R}$, *and* $e\in X$, $\|e\|=1$. *If* $\phi,\Phi,\gamma,\Gamma\in\mathbb{K}$ *and* $x,y\in X$ *such that*

$$\operatorname{Re}\langle\Phi e-x,x-\phi e\rangle\geq 0 \text{ and } \operatorname{Re}\langle\Gamma e-y,y-\gamma e\rangle\geq 0 \tag{1.4}$$

holds, then we have the inequality

$$|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|\leq\frac{1}{4}|\Phi-\phi||\Gamma-\gamma|. \tag{1.5}$$

The constant $\frac{1}{4}$ *is the best possible.*

It has been shown in [1] that the above theorem, for the real case, contains the usual integral and discrete Grüss inequality and also some Grüss type inequalities for mappings defined on infinite intervals.

Namely, if $\rho:(-\infty,+\infty)\to(-\infty,+\infty)$ is a probability density function, i.e., $\int_{-\infty}^{\infty}\rho(t)\,dt=1$, then $\rho^{\frac{1}{2}}\in L^2(-\infty,\infty)$ and obviously $\|\rho^{\frac{1}{2}}\|_2=1$. Consequently, if we assume that $f,g\in L^2(-\infty,\infty)$ and

$$\alpha\rho^{\frac{1}{2}}\leq f\leq\psi\rho^{\frac{1}{2}},\ \beta\rho^{\frac{1}{2}}\leq g\leq\theta\rho^{\frac{1}{2}} \text{ a.e. on } (0,\infty), \tag{1.6}$$

then we have the inequality

$$\left|\int_{-\infty}^{\infty}f(t)\,g(t)\,dt-\int_{-\infty}^{\infty}f(t)\,\rho^{\frac{1}{2}}(t)\,dt\int_{-\infty}^{\infty}g(t)\,\rho^{\frac{1}{2}}(t)\,dt\right| \leq \frac{1}{4}(\psi-\alpha)(\theta-\beta). \tag{1.7}$$

In a similar way, if $e=(e_i)_{i\in\mathbb{N}}\in l^2(\mathbb{R})$ with $\sum_{i\in\mathbb{N}}|e_i|^2=1$ and $x=(x_i)_{i\in\mathbb{N}}$, $y=(y_i)_{i\in\mathbb{N}}\in l^2(\mathbb{R})$ are such that

$$\alpha e_i\leq x_i\leq\psi e_i,\quad \beta e_i\leq y_i\leq\theta e_i \tag{1.8}$$

for all $i\in\mathbb{N}$, then we have

$$\left|\sum_{i\in\mathbb{N}}x_iy_i-\sum_{i\in\mathbb{N}}x_ie_i\sum_{i\in\mathbb{N}}y_ie_i\right|\leq\frac{1}{4}(\psi-\alpha)(\theta-\beta). \tag{1.9}$$

In the recent paper [6], the author also proved the following discrete inequality in inner product spaces:

$$\left\|\sum_{i=1}^{n}p_ia_ix_i-\sum_{i=1}^{n}p_ia_i\sum_{i=1}^{n}p_ix_i\right\|\leq\frac{1}{4}|A-a|\,\|X-x\| \tag{1.10}$$

provided $x_i\in H$, $a_i\in\mathbb{K}$ $(\mathbb{K}=\mathbb{C},\mathbb{R})$ and $a,A\in\mathbb{K}$, $x,X\in H$ are such that

$$\operatorname{Re}[(A-a_i)(\bar{a}_i-\bar{a})]\geq 0 \text{ and } \operatorname{Re}\langle X-x_i,x_i-x\rangle\geq 0 \text{ for all } i\in\{1,...,n\}. \tag{1.11}$$

The constant $\frac{1}{4}$ is sharp.

For other recent developments of the Grüss inequality, see the papers [1]-[6], [10] and the website `http://rgmia.vu.edu.au/Gruss.html`

In this paper we point out some other Grüss type inequalities in inner product spaces which will complement the above result (1.10).

2. PRELIMINARY RESULTS

The following lemma is of interest in itself (see also [6]).

Lemma 2.1. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be an inner product space over the real or complex number field* $\mathbb{K}$, $x_i \in H$ *and* $p_i \geq 0$ *(*$i=1,...,n$*) such that* $\sum_{i=1}^{n} p_i = 1$ *(*$n \geq 2$*).*
If $x, X \in H$ *are such that*

$$\operatorname{Re}\langle X - x_i, x_i - x\rangle \geq 0 \text{ for all } i \in \{1,...,n\}, \tag{2.1}$$

then we have the inequality

$$0 \leq \sum_{i=1}^{n} p_i \|x_i\|^2 - \left\|\sum_{i=1}^{n} p_i x_i\right\|^2 \leq \frac{1}{4}\|X - x\|^2. \tag{2.2}$$

The constant $\frac{1}{4}$ *is sharp.*

Proof. Define

$$I_1 := \left\langle X - \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i x_i - x\right\rangle$$

and

$$I_2 := \sum_{i=1}^{n} p_i \langle X - x_i, x_i - x\rangle.$$

Then

$$I_1 = \sum_{i=1}^{n} p_i \langle X, x_i\rangle - \langle X, x\rangle - \left\|\sum_{i=1}^{n} p_i x_i\right\|^2 + \sum_{i=1}^{n} p_i \langle x_i, x\rangle$$

and

$$I_2 = \sum_{i=1}^{n} p_i \langle X, x_i\rangle - \langle X, x\rangle - \sum_{i=1}^{n} p_i \|x_i\|^2 + \sum_{i=1}^{n} p_i \langle x_i, x\rangle.$$

Consequently

$$I_1 - I_2 = \sum_{i=1}^{n} p_i \|x_i\|^2 - \left\|\sum_{i=1}^{n} p_i x_i\right\|^2. \tag{2.3}$$

Taking the real value in (2.3) we can state

$$\sum_{i=1}^{n} p_i \|x_i\|^2 - \left\|\sum_{i=1}^{n} p_i x_i\right\|^2 = \operatorname{Re}\left\langle X - \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i x_i - x\right\rangle - \sum_{i=1}^{n} p_i \operatorname{Re}\langle X - x_i, x_i - x\rangle, \tag{2.4}$$

which is an identity of interest in itself.

Using the assumption (2.1), we can conclude, by (2.4), that

$$\sum_{i=1}^{n} p_i \|x_i\|^2 - \left\|\sum_{i=1}^{n} p_i x_i\right\|^2 \leq \operatorname{Re}\left\langle X - \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i x_i - x\right\rangle. \tag{2.5}$$

It is known that if $y, z \in H$, then

$$4\operatorname{Re}\langle z, y\rangle \leq \|z + y\|^2, \tag{2.6}$$

with equality iff $z = y$.

Now, by (2.6), we can state that

$$\operatorname{Re}\left\langle X - \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i x_i - x \right\rangle \leq \frac{1}{4}\left\| X - \sum_{i=1}^{n} p_i x_i + \sum_{i=1}^{n} p_i x_i - x \right\|^2 = \frac{1}{4}\left\| X - x \right\|^2.$$

Using (2.5), we can easily deduce (2.2).

To prove the sharpness of the constant $\frac{1}{4}$, let us assume that the inequality (2.2) holds with a constant $c > 0$, i.e.,

$$0 \leq \sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 \leq c \|X - x\|^2 \tag{2.7}$$

for all p_i, x_i and x, X as in the hypothesis of Lemma 2.1.

Assume that $n = 2,\ p_1 = p_2 = \frac{1}{2},\ x_1 = x$ and $x_2 = X$ with $x, X \in H$ and $x \neq X$. Then, obviously,

$$\langle X - x_1, x_1 - x \rangle = \langle X - x_2, x_2 - x \rangle = 0,$$

which shows that the condition (2.1) holds.

If we replace n, p_1, p_2, x_1, x_2 in (2.7), we obtain

$$\sum_{i=1}^{2} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{2} p_i x_i \right\|^2 = \frac{1}{2}\left(\|x\|^2 + \|X\|^2 - \left\| \frac{x + X}{2} \right\|^2 \right) = \frac{\|X - x\|^2}{4} \leq c \|X - x\|^2,$$

from where we deduce $c \geq \frac{1}{4}$, which proves the sharpness of the constant factor $\frac{1}{4}$. □

Remark 2.2. The assumption (2.1) can be replaced by the more general condition

$$\sum_{i=1}^{n} p_i \operatorname{Re}\langle X - x_i, x_i - x \rangle \geq 0 \tag{2.8}$$

and the conclusion (2.2) will still remain valid.

The following corollary is natural.

Corollary 2.3. *Let* $a_i \in \mathbb{K}$, $p_i \geq 0$ $(i = 1, ..., n)$ $(n \geq 2)$ *with* $\sum_{i=1}^{n} p_i = 1$. *If* $a, A \in \mathbb{K}$ *are such that*

$$\operatorname{Re}\left[(A - a_i)(\bar{a}_i - \bar{a}) \right] \geq 0 \text{ for all } i \in \{1, ..., n\}, \tag{2.9}$$

then we have the inequality

$$0 \leq \sum_{i=1}^{n} p_i |a_i|^2 - \left| \sum_{i=1}^{n} p_i a_i \right|^2 \leq \frac{1}{4} |A - a|^2. \tag{2.10}$$

The constant $\frac{1}{4}$ *is sharp.*

The proof follows by the above Lemma 2.1 by choosing $H = \mathbb{K}$, $\langle x, y \rangle := x\bar{y}$, $x_i = a_i$, $x = a$, $X = A$. We omit the details.

Remark 2.4. The condition (2.9) can be replaced by the more general assumption

$$\sum_{i=1}^{n} p_i \operatorname{Re}\left[(A - a_i)(\bar{a}_i - \bar{a}) \right] \geq 0. \tag{2.11}$$

Remark 2.5. If we assume that $\mathbb{K} = \mathbb{R}$, then (2.8) is equivalent with

$$a \leq a_i \leq A \text{ for all } i \in \{1, ..., n\} \tag{2.12}$$

and then, with the assumption (2.12), we get the discrete Grüss type inequality

$$0 \leq \sum_{i=1}^{n} p_i a_i^2 - \left(\sum_{i=1}^{n} p_i a_i \right)^2 \leq \frac{1}{4} (A - a)^2 \tag{2.13}$$

and the constant $\frac{1}{4}$ is sharp.

3. A DISCRETE INEQUALITY OF GRÜSS TYPE

The following Grüss type inequality holds.

Theorem 3.1. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{K}$; $\mathbb{K} = \mathbb{C}, \mathbb{R}$, $x_i, y_i \in H$, $p_i \geq 0$ $(i = 0, ..., n)$ $(n \geq 2)$ with $\sum_{i=1}^{n} p_i = 1$. If $x, X, y, Y \in H$ are such that*

$$\operatorname{Re} \langle X - x_i, x_i - x \rangle \geq 0 \textit{ and } \operatorname{Re} \langle Y - y_i, y_i - y \rangle \geq 0 \textit{ for all } i \in \{1, ..., n\}, \tag{3.1}$$

then we have the inequality

$$\left| \sum_{i=1}^{n} p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i \right\rangle \right| \leq \frac{1}{4} \|X - x\| \|Y - y\| . \tag{3.2}$$

The constant $\frac{1}{4}$ is sharp.

Proof. A simple calculation shows that

$$\sum_{i=1}^{n} p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i \right\rangle = \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \langle x_i - x_j, y_i - y_j \rangle . \tag{3.3}$$

Taking the modulus in both parts of (3.3), and using the generalized triangle inequality, we obtain

$$\left| \sum_{i=1}^{n} p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i \right\rangle \right| \leq \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j |\langle x_i - x_j, y_i - y_j \rangle| . \tag{3.4}$$

By Schwartz's inequality in inner product spaces we have

$$|\langle x_i - x_j, y_i - y_j \rangle| \leq \|x_i - x_j\| \|y_i - y_j\| \tag{3.5}$$

for all $i, j \in \{1, ..., n\}$, and therefore

$$\left| \sum_{i=1}^{n} p_i \langle x_i, y_i \rangle - \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i \right\rangle \right| \leq \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\| \|y_i - y_j\| . \tag{3.6}$$

Using the Cauchy-Buniakowsky-Schwartz inequality for double sums, we can state that

$$\begin{aligned} &\frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\| \|y_i - y_j\| \\ &\qquad \leq \left(\frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \|x_i - x_j\|^2 \right)^{\frac{1}{2}} \times \left(\frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \|y_i - y_j\|^2 \right)^{\frac{1}{2}} \end{aligned} \tag{3.7}$$

and, a simple calculation shows that,

$$\frac{1}{2}\sum_{i,j=1}^{n} p_i p_j \left\|x_i - x_j\right\|^2 = \sum_{i=1}^{n} p_i \left\|x_i\right\|^2 - \left\|\sum_{i=1}^{n} p_i x_i\right\|^2$$

and

$$\frac{1}{2}\sum_{i,j=1}^{n} p_i p_j \left\|y_i - y_j\right\|^2 = \sum_{i=1}^{n} p_i \left\|y_i\right\|^2 - \left\|\sum_{i=1}^{n} p_i y_i\right\|^2 .$$

We obtain

$$\left|\sum_{i=1}^{n} p_i \left\langle x_i, y_i\right\rangle - \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i\right\rangle\right| \leq \left(\sum_{i=1}^{n} p_i \left\|x_i\right\|^2 - \left\|\sum_{i=1}^{n} p_i x_i\right\|^2\right)^{\frac{1}{2}} \times \left(\sum_{i=1}^{n} p_i \left\|y_i\right\|^2 - \left\|\sum_{i=1}^{n} p_i y_i\right\|^2\right)^{\frac{1}{2}} . \tag{3.8}$$

Using Lemma 2.1, we know that

$$\left(\sum_{i=1}^{n} p_i \left\|x_i\right\|^2 - \left\|\sum_{i=1}^{n} p_i x_i\right\|^2\right)^{\frac{1}{2}} \leq \frac{1}{2}\left\|X - x\right\|$$

and

$$\left(\sum_{i=1}^{n} p_i \left\|y_i\right\|^2 - \left\|\sum_{i=1}^{n} p_i y_i\right\|^2\right)^{\frac{1}{2}} \leq \frac{1}{2}\left\|Y - y\right\| .$$

Therefore, by (3.8) we may deduce the desired inequality (3.3).

To prove the sharpness of the constant $\frac{1}{4}$, let us assume that (3.2) holds with a constant $c > 0$, i.e.,

$$\left|\sum_{i=1}^{n} p_i \left\langle x_i, y_i\right\rangle - \left\langle \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i\right\rangle\right| \leq c\left\|X - x\right\|\left\|Y - y\right\| \tag{3.9}$$

under the above assumptions for $p_i,\ x_i,\ y_i,\ x,\ X,\ y,\ Y$ and $n \geq 2$.

If we choose $n = 2$, $x_1 = x,\ x_2 = X$, $y_1 = y,\ y_2 = Y\ \ (x \neq X,\ y \neq Y)$ and $p_1 = p_2 = \frac{1}{2}$, then

$$\begin{aligned}\sum_{i=1}^{2} p_i \left\langle x_i, y_i\right\rangle - \left\langle \sum_{i=1}^{2} p_i x_i, \sum_{i=1}^{2} p_i y_i\right\rangle &= \frac{1}{2}\sum_{i,j=1}^{2} p_i p_j \left\langle x_i - x_j, y_i - y_j\right\rangle \\ &= \sum_{1\leq i<j\leq 2} p_i p_j \left\langle x_i - x_j, y_i - y_j\right\rangle \\ &= \frac{1}{4}\left\langle x - X, y - Y\right\rangle\end{aligned}$$

and then

$$\left|\sum_{i=1}^{2} p_i \left\langle x_i, y_i\right\rangle - \left\langle \sum_{i=1}^{2} p_i x_i, \sum_{i=1}^{2} p_i y_i\right\rangle\right| = \frac{1}{4}\left|\left\langle x - X, y - Y\right\rangle\right| .$$

Choose $X - x = z$, $Y - y = z$, $z \neq 0$. Then using (3.9), we derive

$$\frac{1}{4}\left\|z\right\|^2 \leq c\left\|z\right\|^2 ,\ \ z \neq 0$$

which implies that $c \geq \frac{1}{4}$, and the theorem is proved. □

Remark 3.2. The condition (3.1) can be replaced by the more general assumption

$$\sum_{i=1}^{n} p_i \operatorname{Re}\langle X - x_i, x_i - x\rangle \geq 0, \quad \sum_{i=1}^{n} p_i \operatorname{Re}\langle Y - y_i, y_i - y\rangle \geq 0 \tag{3.10}$$

and the conclusion (3.2) still remains valid.

The following corollary for real or complex numbers holds.

Corollary 3.3. *Let* $a_i, b_i \in \mathbb{K}\,(\mathbb{K} = \mathbb{C}, \mathbb{R})\,, p_i \geq 0\ \ (i = 1, ..., n)$ *with* $\sum_{i=1}^{n} p_i = 1$. *If* $a, A, b, B \in \mathbb{K}$ *are such that*

$$\operatorname{Re}\left[(A - a_i)\left(\bar{a}_i - \bar{a}\right)\right] \geq 0 \ \text{ and } \quad \operatorname{Re}\left[(B - b_i)\left(\bar{b}_i - \bar{b}\right)\right] \geq 0, \tag{3.11}$$

then we have the inequality

$$\left|\sum_{i=1}^{n} p_i a_i \bar{b}_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i \bar{b}_i\right| \leq \frac{1}{4}\left|A - a\right|\left|B - b\right| \tag{3.12}$$

and the constant $\frac{1}{4}$ *is sharp.*

The proof is obvious by Theorem 3.1 applied for the inner product space $(\mathbb{C}, \langle\cdot,\cdot\rangle)$ where $\langle x, y\rangle = x \cdot \bar{y}$. We omit the details.

Remark 3.4. The condition (3.11) can be replaced by the more general condition

$$\sum_{i=1}^{n} p_i \operatorname{Re}\left[(A - a_i)\left(\bar{a}_i - \bar{a}\right)\right] \geq 0, \quad \sum_{i=1}^{n} p_i \operatorname{Re}\left[(B - b_i)\left(\bar{b}_i - \bar{b}\right)\right] \geq 0 \tag{3.13}$$

and the conclusion of the above corollary will still remain valid.

Remark 3.5. If we assume that a_i, b_i, a, b, A, B are real numbers, then (3.11) is equivalent to

$$a \leq a_i \leq A, \quad b \leq b_i \leq B \ \text{ for all } i \in \{1, ..., n\} \tag{3.14}$$

and (3.12) becomes

$$0 \leq \left|\sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i\right| \leq \frac{1}{4}\left(A - a\right)\left(B - b\right), \tag{3.15}$$

which is the classical Grüss inequality for sequences of real numbers.

4. APPLICATIONS FOR CONVEX FUNCTIONS

Let $(H; \langle\cdot,\cdot\rangle)$ be a real inner product space and $F : H \to \mathbb{R}$ a Fréchet differentiable convex mapping on H. Then we have the "gradient inequality"

$$F(x) - F(y) \geq \langle\nabla F(y), x - y\rangle \tag{4.1}$$

for all $x, y \in H$, where $\nabla F : H \to H$ is the gradient operator associated to the differentiable convex function F.

The following theorem holds.

Theorem 4.1. *Let* $F : H \to \mathbb{R}$ *be as above and* $x_i \in H\ (i = 1, ..., n)$. *Suppose that there exists the vectors* $\gamma, \phi \in H$ *such that* $\langle x_i - \gamma, \phi - x_i\rangle \geq 0$ *for all* $i \in \{1, ..., m\}$ *and* $m, M \in H$

such that $\langle \nabla F(x_i) - m,\ M - \nabla F(x_i) \rangle \geq 0$ *for all* $i \in \{1, ..., m\}$. *Then for all* $p_i \geq 0$ $(i = 1, ..., m)$ *with* $P_m := \sum_{i=1}^{m} p_i > 0$, *we have the inequality*

$$0 \leq \frac{1}{P_m}\sum_{i=1}^{m} p_i F(x_i) - F\left(\frac{1}{P_m}\sum_{i=1}^{m} p_i x_i\right) \leq \frac{1}{4}\|\phi - \gamma\| \|M - m\| . \tag{4.2}$$

Proof. Choose in (4.1), $x = \frac{1}{P_M}\sum_{i=1}^{m} p_i x_i$ and $y = x_j$ to obtain

$$F\left(\frac{1}{P_m}\sum_{i=1}^{m} p_i x_i\right) - F(x_j) \geq \left\langle \nabla F(x_j), \frac{1}{P_m}\sum_{i=1}^{m} p_i x_i - x_j \right\rangle \tag{4.3}$$

for all $j \in \{1, ..., n\}$.

If we multiply (4.3) by $p_j \geq 0$ and sum over j from 1 to m, we have

$$P_m F\left(\frac{1}{P_m}\sum_{i=1}^{m} p_i x_i\right) - \sum_{j=1}^{m} p_j F(x_j) \geq \frac{1}{P_m}\left\langle \sum_{j=1}^{m} p_j \nabla F(x_j), \sum_{i=1}^{m} p_i x_i \right\rangle - \sum_{i=1}^{m} \langle \nabla F(x_i), x_i \rangle .$$

Dividing by $P_m > 0$, we obtain the inequality

$$\begin{aligned} 0 &\leq \frac{1}{P_m}\sum_{i=1}^{m} p_i F(x_i) - F\left(\frac{1}{P_m}\sum_{i=1}^{m} p_i x_i\right) \\ &\leq \frac{1}{P_m}\sum_{i=1}^{m} p_i \langle \nabla F(x_i), x_i \rangle - \left\langle \frac{1}{P_m}\sum_{i=1}^{m} p_i \nabla F(x_i), \frac{1}{P_m}\sum_{i=1}^{m} p_i x_i \right\rangle \end{aligned} \tag{4.4}$$

which is a generalisation for the case of inner product spaces of the result by Dragomir and Goh established in 1996 for the case of differentiable mappings defined on $\mathbb{R}^n$ [9].

Applying Theorem 3.1 for real inner product spaces, $X = \phi$, $x = \gamma$, $y_i = \nabla F(x_i)$, $y = m$, $Y = M$ and $n = m$, we easily deduce

$$\frac{1}{P_m}\sum_{i=1}^{m} p_i \langle x_i, \nabla F(x_i) \rangle - \left\langle \frac{1}{P_m}\sum_{i=1}^{m} p_i x_i, \frac{1}{P_m}\sum_{i=1}^{m} p_i \nabla F(x_i) \right\rangle \leq \frac{1}{4}\|\Phi - \phi\| \|M - m\| \tag{4.5}$$

and then, by (4.4) and (4.5) we can conclude that the desired inequality (4.2) holds. □

Remark 4.2. The conditions

$$\langle x_i - \gamma, \phi - x_i \rangle \geq 0, \quad \langle \nabla F(x_i) - m, M - \nabla F(x_i) \rangle \geq 0, \tag{4.6}$$

for all $i \in \{1, ..., m\}$ can be replaced by the more general conditions

$$\sum_{i=1}^{m} p_i \langle x_i - \gamma, \phi - x_i \rangle \geq 0 \ \text{ and } \ \sum_{i=1}^{m} p_i \langle \nabla F(x_i) - m, M - \nabla F(x_i) \rangle \geq 0 \tag{4.7}$$

and the conclusion (4.2) will still be valid.

Remark 4.3. Even if the inequality (4.2) is not as sharp as (4.4), it can be more useful in practice when only some bounds of the gradient operator ∇F and of the vectors x_i $(i = 1, ..., n)$ are known. In other words, it provides the opportunity to estimate the difference

$$\Delta(F, x, p) := \frac{1}{P_m}\sum_{i=1}^{m} p_i F(x_i) - F\left(\frac{1}{P_m}\sum_{i=1}^{m} p_i x_i\right),$$

where the differences $\|\phi - \gamma\|$ and $\|M - m\|$ are known.

Remark 4.4. For example, if we know that $\langle \nabla F(x_i) - m,\ M - \nabla F(x_i)\rangle \geq 0$ for all $i \in \{1, ..., m\}$ and the vectors x_i $(i = 1, ..., n)$ are not too far from each other in the sense that $\langle x_i - \gamma, \phi - x_i\rangle \geq 0$ for all $i \in \{1, ..., m\}$ and $\|\phi - \gamma\| \leq \frac{4\varepsilon}{\|M-m\|}$ $(\varepsilon > 0)$, then by (4.2), we can conclude that

$$0 \leq \Delta(F, x, p) \leq \varepsilon.$$

5. APPLICATIONS FOR SOME DISCRETE TRANSFORMS

Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over $\mathbb{K}$, $\mathbb{K} = \mathbb{C}, \mathbb{R}$ and $\bar{x} = (x_1, ..., x_n)$ be a sequence of vectors in H.

For a given $m \in \mathbb{K}$, define the *discrete Fourier transform*

$$\mathcal{F}_w(\bar{x})(m) = \sum_{k=1}^{n} \exp(2wimk) \times x_k, \quad m = 1, ..., n. \tag{5.1}$$

The complex number $\sum_{k=1}^{n} \exp(2wimk)\langle x_k, y_k\rangle$ is actually the usual Fourier transform of the vector $(\langle x_1, y_1\rangle, ..., \langle x_n, y_n\rangle) \in \mathbb{K}^n$ and will be denoted by

$$\mathcal{F}_w(\bar{x}\cdot\bar{y})(m) = \sum_{k=1}^{n} \exp(2wimk)\langle x_k, y_k\rangle, \quad m = 1, ..., n. \tag{5.2}$$

The following result holds.

Theorem 5.1. *Let $\bar{x}, \bar{y} \in H^n$ be sequences of vectors such that there exists the vectors $c, C, y, Y \in H$ with the properties*

$$\operatorname{Re}\langle C - \exp(2wimk)x_k, \exp(2wimk)x_k - c\rangle \geq 0, \quad k, m = 1, ..., n \tag{5.3}$$

and

$$\operatorname{Re}\langle Y - y_k, y_k - y\rangle \geq 0, \quad k = 1, ..., n. \tag{5.4}$$

Then we have the inequality

$$\left|\mathcal{F}_w(\bar{x}\cdot\bar{y})(m) - \left\langle \mathcal{F}_w(\bar{x})(m), \frac{1}{n}\sum_{k=1}^{n} y_k \right\rangle\right| \leq \frac{n}{4}\|C - c\|\|Y - y\|, \tag{5.5}$$

for all $m \in \{1, ..., n\}$.

The proof follows by Theorem 3.1 applied for $p_k = \frac{1}{n}$ and for the sequences $x_k \to c_k = \exp(2wimk)x_k$ and y_k $(k = 1, ..., n)$. We omit the details.

We can also consider the *Mellin transform*

$$\mathcal{M}(\bar{x})(m) := \sum_{k=1}^{n} k^{m-1}x_k, \quad m = 1, ..., n, \tag{5.6}$$

of the sequence $\bar{x} = (x_1, ..., x_n) \in H^n$.

We remark that the complex number $\sum_{k=1}^{n} k^{m-1}\langle x_k, y_k\rangle$ is actually the Mellin transform of the vector $(\langle x_1, y_1\rangle, ..., \langle x_n, y_n\rangle) \in \mathbb{K}^n$ and will be denoted by

$$\mathcal{M}(\bar{x}\cdot\bar{y})(m) := \sum_{k=1}^{n} k^{m-1}\langle x_k, y_k\rangle. \tag{5.7}$$

The following theorem holds.

Theorem 5.2. *Let $\bar{x},\bar{y}\in H^n$ be sequences of vectors such that there exist the vectors $d,D,y,Y\in H$ with the properties*

$$\operatorname{Re}\left\langle D-k^{m-1}x_k,k^{m-1}x_k-d\right\rangle\geq 0 \tag{5.8}$$

for all $k,m\in\{1,...,n\}$, and (5.4) is fulfilled.
Then we have the inequality

$$\left|\mathcal{M}\left(\bar{x}\cdot\bar{y}\right)(m)-\left\langle\mathcal{M}\left(\bar{x}\right)(m),\frac{1}{n}\sum_{k=1}^{n}y_k\right\rangle\right|\leq\frac{n}{4}\left\|D-d\right\|\left\|Y-y\right\| \tag{5.9}$$

for all $m\in\{1,...,n\}$.

The proof follows by Theorem 3.1 applied for $p_k=\frac{1}{n}$ and for the sequences $x_k\rightarrow d_k=kx_k$ and y_k $(k=1,...,n)$. We omit the details.

Another result which connects the Fourier transforms for different parameters w also holds.

Theorem 5.3. *Let $\bar{x},\bar{y}\in H^n$ and $w,z\in\mathbb{K}$. If there exists the vectors $e,E,f,F\in H$ such that*

$$\operatorname{Re}\left\langle E-\exp\left(2wimk\right)x_k,\exp\left(2wimk\right)x_k-e\right\rangle\geq 0,\quad k,m=1,...,n$$

and

$$\operatorname{Re}\left\langle F-\exp\left(2zimk\right)y_k,\exp\left(2zimk\right)y_k-f\right\rangle\geq 0,\quad k,m=1,...,n$$

then we have the inequality:

$$\left|\frac{1}{n}\mathcal{F}_{w+z}\left(\bar{x}\cdot\bar{y}\right)(m)-\left\langle\frac{1}{n}\mathcal{F}_{w}\left(\bar{x}\right)(m),\frac{1}{n}\mathcal{F}_{z}\left(\bar{y}\right)(m)\right\rangle\right|\leq\frac{1}{4}\left\|E-e\right\|\left\|F-f\right\|,$$

for all $m\in\{1,...,n\}$.

The proof follows by Theorem 3.1 for the sequences $\exp\left(2wimk\right)x_k$, $\exp\left(2zimk\right)y_k$ $(k=1,...,n)$. We omit the details.

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