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OSSTROWSKI TYPE INEQUALITIES FROM A LINEAR FUNCTIONAL POINT OF VIEW

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Abstract

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Abstract

Inequalities are obtained using P_0 -simple functionals. Applications to Lipschitzian mappings are given.

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1. Introduction

Let I be a bounded interval of the real axis. We denote by $B(I)$ the set of all functions which are bounded on $[a, b]$.

Let A be a positive linear functional $A : B(I) \rightarrow \mathbb{R}$, such that $A(e_0) = 1$, where $e_i : I \rightarrow \mathbb{R}, e_i(x) = x^i, \forall x \in I, i \in \mathbb{N}$.

The following inequality is known in literature as the Grüss inequality for the functional A .

Theorem 1.1. *Let $f, g : I \rightarrow \mathbb{R}$ be two bounded functions such that $m_1 \leq f(x) \leq M_1$ and $m_2 \leq g(x) \leq M_2$ for all $x \in I$, m_1, M_1, m_2 and M_2 are constants. Then the inequality:*

$$(1.1) \quad |A(fg) - A(f)A(g)| \leq \frac{1}{4}(M_1 - m_1)(M_2 - m_2)$$

holds.

In 1938 Ostrowski (cf. for example [7, p. 468]) proved the following result:

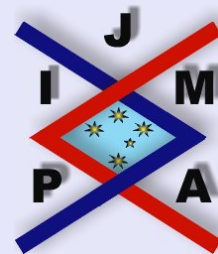
Theorem 1.2. *Let $f : I \rightarrow \mathbb{R}$ be continuous on (a, b) whose derivative $f' : (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b) , i.e.*

$$\|f'\|_\infty := \sup_{t \in (a, b)} |f'(t)| < \infty.$$

Then

$$(1.2) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ is best.



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In the recent paper [4] S.S. Dragomir and S. Wang proved the following version of Ostrowski's inequality.

Theorem 1.3. *Let $f : I \rightarrow \mathbb{R}$ be a differentiable mapping in the interior of I and $a, b \in \text{int}(I)$ with $a < b$. If $f' \in L_1[a, b]$ and $\gamma \leq f'(x) \leq \Gamma$ for all $x \in [a, b]$ then we have the following inequality:*

$$(1.3) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt - \frac{f(b) - f(a)}{b-a} \left(x - \frac{a+b}{2} \right) \right| \leq \frac{1}{4}(b-a)(\Gamma - \gamma)$$

for all $x \in [a, b]$.

The following inequality for mappings with bounded variation can be found in [1]:

Theorem 1.4. *Let $f : I \rightarrow \mathbb{R}$ be a mapping of bounded variation. Then for all $x \in [a, b]$ we have the inequality*

$$(1.4) \quad \left| \int_a^b f(t) dt - f(x)(b-a) \right| \leq \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b f,$$

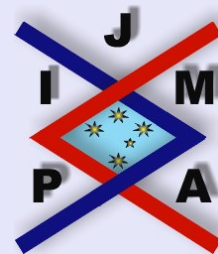
where $\bigvee_a^b f$ denotes the total variation of f .

The constant $\frac{1}{2}$ is the best possible one.

In [2] S.S. Dragomir gave the following result for Lipschitzian mappings:

Theorem 1.5. *Let $f : [a, b] \rightarrow \mathbb{R}$ be an L -Lipschitzian mapping on $[a, b]$, i.e.*

$$|f(x) - f(y)| \leq L|x - y|, \text{ for all } x, y \in [a, b].$$



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Then we have the inequality

$$(1.5) \quad \left| \int_a^b f(t)dt - f(x)(b-a) \right| \leq L(b-a)^2 \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right]$$

for all $x \in [a, b]$.

The constant $\frac{1}{4}$ is the best possible one.

S.S. Dragomir, P. Cerone, J. Roumeliotis and S. Wang in [3] proved the following theorem:

Theorem 1.6. Let $f, w : (a, b) \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be so that $w(s) \geq 0$ on (a, b) , w is integrable on (a, b) and $\int_a^b w(s)ds > 0$, f is of r -Hölder type, i.e.

$$(1.6) \quad |f(x) - f(y)| \leq H|x - y|^r, \text{ for all } x, y \in (a, b)$$

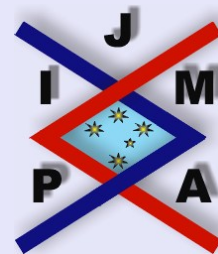
where $H > 0$ and $r \in (0, 1]$ are given. If $w, f \in L_1(a, b)$, then we have the inequality:

$$(1.7) \quad \left| f(x) - \frac{1}{\int_a^b w(s)ds} \int_a^b w(s)f(s)ds \right| \leq H \frac{1}{\int_a^b w(s)ds} \int_a^b |x - s|^r w(s)ds$$

for all $x \in (a, b)$.

The constant factor 1 in the right hand side cannot be replaced by a smaller one.

The aim of this paper is to improve the results from Theorems 1.1 – 1.6 using a unitary method.



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2. Auxiliary results

Let $X = (X, d)$ be a compact metric space and $C(X)$ the Banach lattice of real-valued continuous functions on the compact metric space $X = (X, d)$, endowed with the max norm $\|\cdot\|_X$.

For a function $f \in C(X)$, the modulus of continuity (with respect to the metric d) is defined by:

$$\omega(f; t) = \omega_d(f; t) = \sup_{d(x,y) \leq t} |f(x) - f(y)|, \quad t \geq 0.$$

The least concave majorant of this modulus with respect to the variable t is given by

$$\tilde{\omega}(f; t) = \begin{cases} \sup_{\substack{0 \leq x \leq t \leq y \\ x \neq y}} \frac{(t-x)\omega(f; y) + (y-t)\omega(f; x)}{y-x} & \text{for } 0 \leq t \leq d(X); \\ \omega(f; d(X)) & \text{for } t > d(X), \end{cases}$$

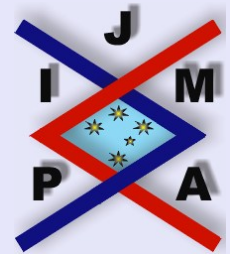
where $d(X) < \infty$ is the diameter of the compact space X .

We denote by $Lip_M \alpha = Lip_M(\alpha; X)$ the set of all Lipschitzian functions of order α , $\alpha \in [0, 1]$ having the same Lipschitz constant M . That is $f \in Lip_M \alpha$ iff for all $x, y \in X$

$$|f(x) - f(y)| \leq M d^\alpha(x, y).$$

We see that

$$Lip_M(\alpha; X) = \{g \in C(X) : \omega(g; t) \leq M t^\alpha\}.$$



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Let $I = [a, b]$ be a compact interval of the real axis, S a subspace of $C(I)$, and A a linear functional defined on S . The following definition was given by T. Popoviciu in [8].

Definition 2.1. The linear functional A defined on the subspace S which contains all polynomials is P_n -simple ($n \geq -1$) if

- (i) $A(e_{n+1}) \neq 0$
- (ii) for every $f \in S$ there are the distinct points t_1, t_2, \dots, t_{n+2} in $[a, b]$ such that

$$A(f) = A(e_{n+1})[t_1, t_2, \dots, t_{n+2}; f],$$

where $[t_1, t_2, \dots, t_{n+2}; f]$ is the divided difference of the function f on the points t_1, t_2, \dots, t_{n+2} .

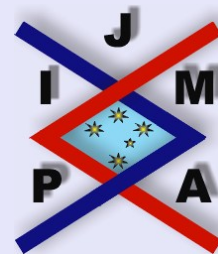
In [5] the following result is proved. The proof is reproduced here for completeness.

Theorem 2.1. Let A be a bounded linear functional, $A : C(I) \rightarrow \mathbb{R}$. If A is P_0 -simple then for all $f \in C(I)$ we have

$$(2.1) \quad |A(f)| \leq \frac{\|A\|}{2} \tilde{\omega} \left(f; \frac{2|A(e_1)|}{\|A\|} \right).$$

Proof. For $g \in C^1(I)$ we have

$$\begin{aligned} |A(f)| &= |A(f - g) + A(g)| \leq \|A\| \|f - g\| + |A(g)| \\ &\leq \|A\| \|f - g\| + |A(e_1)| \|g'\|. \end{aligned}$$



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From this inequality we obtain

$$|A(f)| \leq \inf_{g \in C^1(I)} (\|A\| \|f - g\| + |A(e_1)| \|g'\|)$$

and using the following result (see [10])

$$\inf_{g \in C^1(I)} \left(\|f - g\| + \frac{t}{2} \|g'\| \right) = \frac{1}{2} \tilde{\omega}(f; t), \quad t \geq 0$$

we obtain the relation (2.1). □

The following result was proved by I. Raşa [9].

Theorem 2.2. *Let k be a natural number such that $0 \leq k \leq n$ and $A : C^{(k)}[a, b] \rightarrow \mathbb{R}$ a bounded linear functional, $A \neq 0$, $A(e_i) = 0$ for $i = 0, 1, \dots, n$ such that for every $f \in C^{(k)}[a, b]$ P_n -nonconcave $A(f) \geq 0$. Then A is P_n -simple.*

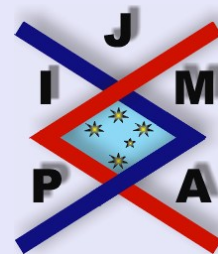
A function $f \in C^{(k)}[a, b]$ is called P_0 -nonconcave if for any $n + 2$ points $t_1, t_2, \dots, t_{n+2} \in [a, b]$ the inequality

$$[t_1, t_2, \dots, t_{n+2}; f] \geq 0$$

holds.

Another criterion for P_n -simple functionals was given by A. Lupaş in [6]. He proved that a bounded linear functional $A : C[a, b] \rightarrow \mathbb{R}$ for which $A(e_k) = 0$, $k = 0, 1, \dots, n$ and $A(e_{n+1}) \neq 0$ is P_n -simple if and only if A is P_n -simple on $C^{(n+1)}[a, b]$.

Now we can prove the following result (see also [5]):



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Theorem 2.3. Let A be a bounded linear functional, $A : C(I) \rightarrow \mathbb{R}$. If $A(e_1) \neq 0$ and the inequality (2.1) holds for any $f \in C(I)$ then A is P_0 -simple.

Proof. We can assume that $A(e_1) > 0$. Combining the results of I. Raşa and A. Lupaş, it is sufficient, for the proof of the theorem, to show that

$$(2.2) \quad A(f) \geq 0$$

for every nondecreasing differentiable function f defined on I .

For such a function we have

$$|A(f)| \leq A(e_1) \|f'\|.$$

Let B be the linear functional defined by

$$B(f) = \frac{A(F)}{A(e_1)},$$

where

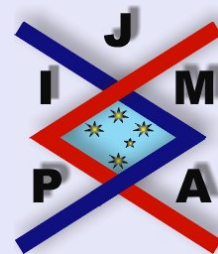
$$F(t) = \int_0^t f(u) du, \quad f \in C[0, 1].$$

The functional B is bounded and for any $f \in C(I)$ we have

$$|B(f)| \leq \|f\|$$

with $B(e_0) = 1$.

Let f be a continuous function such that $f \geq 0$, $f \neq 0$.



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From the inequalities

$$0 \leq e_0 - \frac{f}{\|f\|} \leq 1$$

we obtain

$$1 - \frac{B(f)}{\|f\|} \leq \left| B \left(e_0 - \frac{f}{\|f\|} \right) \right| \leq 1.$$

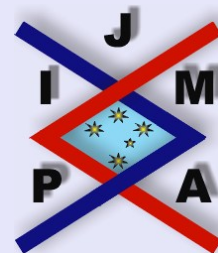
These inequalities imply that

$$(2.3) \quad B(f) \geq 0.$$

Further, let f be a differentiable function on I such that $f' \geq 0$, then, from (2.3) we obtain

$$B(f') \geq 0.$$

Since $B(f') = A(f)$, the inequality (2.2) is thus proved. \square



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3. An Integral Inequality of Ostrowski Type

The following inequality of Ostrowski type holds.

Theorem 3.1. *Let f be a continuous function on $[a, b]$ and $w : (a, b) \rightarrow \mathbb{R}_+$ an integrable function on (a, b) such that $\int_a^b w(s)ds = 1$. Then for any continuous function f the following inequality:*

$$(3.1) \quad \left| f(x) - \int_a^b w(s)f(s)ds \right| \leq \left(\int_a^x w(t)dt \right) \tilde{\omega}_{[a,x]} \left(f; \frac{\int_a^x w(t)(x-t)dt}{\int_a^x w(t)} \right) + \left(\int_x^b w(t)dt \right) \tilde{\omega}_{[x,b]} \left(f; \frac{\int_x^b w(t)(t-x)dt}{\int_x^b w(t)dt} \right)$$

holds, where x is a fixed point in (a, b) .

Proof. From Theorem 2.3 we get that the linear functionals

$$A_1 : C[a, x] \rightarrow \mathbb{R}, \quad A_2 : C[x, b] \rightarrow \mathbb{R}$$

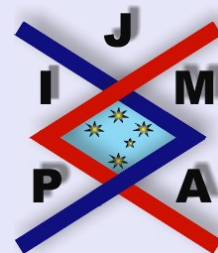
defined by

$$A_1(f) = f(x) \int_a^x w(t)dt - \int_a^x f(t)w(t)dt$$

and

$$A_2(f) = f(x) \int_x^b w(t)dt - \int_x^b f(t)w(t)dt$$

are P_0 -simple.



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It is easy to see that:

$$\|A_1\| = 2 \int_a^x w(t)dt \quad \text{and} \quad \|A_2\| = 2 \int_x^b w(t)dt.$$

From the inequality:

$$\left| f(x) - \int_a^b w(s)f(s)ds \right| \leq \left(\int_a^x w(t)dt \right) \tilde{\omega} \left(f; \frac{|A_1(e_1)|}{\int_a^x w(t)dt} \right) + \left(\int_x^b w(t)dt \right) \tilde{\omega} \left(f; \frac{A_2(e_1)}{\int_x^b w(t)dt} \right)$$

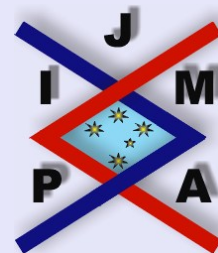
and from the results

$$|A_1(e_1)| = \int_a^x w(t)(x-t)dt \quad \text{and} \quad |A_2(e_1)| = \int_x^b w(t)(t-x)dt,$$

(3.1) follows. □

Corollary 3.2. *Let f be a continuous function on $[a, b]$, such that $f \in Lip_{M_1}(\alpha, [a, x])$ and $f \in Lip_{M_2}(\beta; [x, b])$. Then*

$$(3.2) \quad \left| f(x) - \int_a^b w(s)f(s)ds \right| \leq M_1 \left(\int_a^x w(t)dt \right)^{1-\alpha} \left[\int_a^x w(t)(x-t)dt \right]^\alpha + M_2 \left(\int_x^b w(t)dt \right)^{1-\beta} \left[\int_x^b w(t)(t-x)dt \right]^\beta.$$



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Proof. The proof follows from the inequality (3.1) and the fact that

$$\tilde{\omega}_1(g; t) \leq Mt^r$$

for any function g , $g \in Lip_M(\alpha, [c, d])$, where $\tilde{\omega}_1$ is taken on the interval $[c, d]$. \square

Corollary 3.2 is an improvement of the result of Theorem 1.6.

Remark 3.1. In the particular case when $w(t) = \frac{1}{b-a}$ the inequality (3.2) becomes:

$$(3.3) \quad \left| f(x) - \frac{\int_a^b f(s)ds}{b-a} \right| \leq \left[M_1 \frac{(x-a)^{\alpha+1}}{2^\alpha} + M_2 \frac{(b-x)^{\beta+1}}{2^\beta} \right] \frac{1}{b-a}$$

$$\leq \max(M_1, M_2) \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a).$$

Inequality (3.3) improves the inequality (1.5).

Corollary 3.3. *Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on (a, b) , whose derivative $f' : (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b) and w a function as in Theorem 3.1. Then we have the following inequality:*

$$(3.4) \quad \left| f(x) - \int_a^b w(s)f(s)ds \right| \leq \left[\int_a^x w(t)(x-t)dt + \int_x^b w(t)(t-x)dt \right] \|f'\|_\infty.$$

Proof. The above inequality is a consequence of the inequality (3.1) and the fact that

$$\tilde{\omega}(f; t) \leq \|f'\|_\infty t.$$



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The inequality of Ostrowski follows from (3.4) if we consider

$$w(t) = \frac{1}{b-a}, \quad t \in [a, b].$$

□

Corollary 3.4. *Let $f : I \rightarrow \mathbb{R}$ be a mapping with bounded variation and w a function as in Theorem 3.1. Then for all $x \in [a, b]$ we have the inequalities*

$$(3.5) \quad \left| f(x) - \int_a^b w(s)f(s)ds \right| \leq \bigvee_a^x f \int_a^x w(t)dt + \bigvee_x^b f \int_x^b w(t)dt$$

$$(3.6) \quad \left| f(x) - \int_a^b w(s)f(s)ds \right| \leq \left(\frac{1}{2} + \frac{\left| \int_a^x w(t)dt - \int_x^b w(t)dt \right|}{2} \right) \bigvee_a^b f.$$

Proof. It is clear that

$$(3.7) \quad \tilde{w}[a, x](f; t) \leq \bigvee_a^x f \quad \text{and} \quad \tilde{w}[x, b](f, t) \leq \bigvee_x^b f$$

for every positive number t .

Thus, inequality (3.5) follows from (3.7).

For the proof of the inequality (3.6) we note that, if we suppose $\int_a^x w(t)dt \leq \frac{1}{2}$ then $\int_x^b w(t)dt \geq \frac{1}{2}$ and vice versa.



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For definiteness we assume that

$$\int_a^x w(t)dt \leq \frac{1}{2} \quad \text{and} \quad \int_x^b w(t)dt \geq \frac{1}{2}.$$

We then have

$$\begin{aligned} \bigvee_a^x f \int_a^x w(t)dt + \bigvee_x^b f \int_x^b w(t)dt &\leq \frac{1}{2} \bigvee_a^x f + \bigvee_x^b f \int_x^b w(t)dt \\ &= \frac{1}{2} \bigvee_a^b f + \bigvee_x^b f \left(\int_x^b w(t)dt - \frac{1}{2} \right) \end{aligned}$$

and so

$$(3.8) \quad \bigvee_a^x f \int_a^x w(t)dt + \bigvee_x^b f \int_x^b w(t)dt \leq \left(\frac{1}{2} + \frac{\int_x^b w(t)dt - \int_a^x w(t)dt}{2} \right) \bigvee_a^b f.$$

From the inequalities (3.5) and (3.8), the inequality (3.6) follows. \square

Remark 3.2. The inequality from Theorem 1.4 follows if we take in (3.6)

$$w(t) = \frac{1}{b-a}.$$

Theorem 3.5. *Let g be a continuous differentiable function on $[a, b]$ such that $g(a) = g(b) = 0$. Then the inequality*

$$(3.9) \quad \left| \frac{g(x)}{2} - \frac{1}{b-a} \int_a^b g(t)dt \right| \leq \frac{(x-a)^2 + (b-x)^2}{8(b-a)} \tilde{\omega} \left(g'; \frac{2(x-a)^3 + (y-b)^3}{3(x-a)^2 + (y-b)^2} \right)$$



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holds, where x is an arbitrary point in (a, b) .

Proof. The following functional A defined on $C[a, b]$ by

$$A(f) = \frac{1}{b-a} \int_a^b \left(t - \frac{a+b}{2} \right) f(t) dt$$

is a linear bounded functional having its norm equal to $\frac{b-a}{4}$. For every increasing function f we have:

$$A(f) \geq 0.$$

Using Theorem 2.3, we deduce that the functional A is P_0 -simple with

$$A(e_1) = \frac{(b-a)^2}{12}.$$

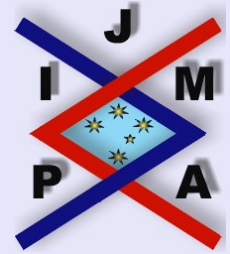
From Theorem 2.1, we obtain the following inequality:

$$(3.10) \quad \left| \frac{1}{b-a} \int_a^b \left(t - \frac{a+b}{2} \right) f(t) dt \right| \leq \frac{b-a}{8} \tilde{\omega} \left(f; \frac{2}{3}(b-a) \right).$$

Inequality (3.10) holds for every continuous function f .

Let us suppose that f is differentiable on $[a, b]$. From the inequality (3.10) (written for f') we obtain the following inequality:

$$(3.11) \quad \left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{f(a) + f(b)}{2} \right| \leq \frac{b-a}{8} \tilde{\omega} \left(f'; \frac{2}{3}(b-a) \right).$$



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Now, we can prove the inequality (3.9). We have the following identity:

$$(3.12) \quad -\frac{g(x)}{2} + \frac{1}{b-a} \int_a^b g(t) dt = \frac{x-a}{b-a} \left(\frac{1}{x-a} \int_a^x g(t) dt - \frac{g(a)+g(x)}{2} \right) + \frac{b-x}{b-a} \left(\frac{1}{b-x} \int_x^b g(t) dt - \frac{g(b)+g(x)}{2} \right).$$

Using the relations (3.11) and (3.12) we obtain

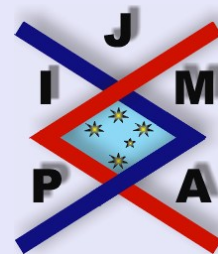
$$(3.13) \quad \left| \frac{g(x)}{2} - \frac{1}{b-a} \int_a^b g(t) dt \right| \leq \frac{(x-a)^2}{8(b-a)} \tilde{\omega} \left(g'; \frac{2}{3}(x-a) \right) + \frac{(b-x)^2}{8(b-a)} \tilde{\omega} \left(g'; \frac{2}{3}(b-x) \right).$$

As the function $\tilde{\omega}(g'; \cdot)$, is concave, then from (3.13) and using Jensen's inequality, we obtain the inequality (3.9). \square

Corollary 3.6. *Let g be a continuous differentiable function on $[a, b]$ such that $g(a) = g(b) = 0$, then the following inequality*

$$(3.14) \quad \left| \frac{g(x)}{2} - \frac{1}{b-a} \int_a^b g(t) dt \right| \leq \left[\frac{1}{8} + \frac{(x - \frac{a+b}{2})^2}{2(b-a)} \right] (b-a) \|g'\|_\infty$$

is valid for all $x \in [a, b]$.



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Proof. It is well known that

$$(3.15) \quad \tilde{\omega}(g'; t) \leq 2\|g'\|_{\infty},$$

for every positive number t .

The inequality (3.15) then readily follows from the inequality (3.14). \square

Remark 3.3. The result from the Theorem 1.3 can be written in terms of $\tilde{\omega}$ using the inequality (3.13) for the function

$$g(x) = f(x) - \frac{x-a}{b-a}f(b) - \frac{b-x}{b-a}f(a).$$

In [5] the following result was proved:

Let A be a linear positive functional $A : C[0, 1] \rightarrow \mathbb{R}$, $A(e_0) = 1$ and $\varphi, \psi : [0, 1] \rightarrow \mathbb{R}$ a continuous increasing function such that $A(e_1\varphi) - A(e_1)A(\varphi) > 0$. Then the following Grüss type inequality

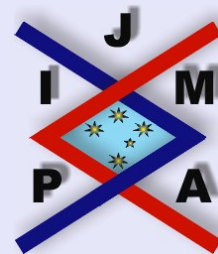
$$(3.16) \quad |A(\varphi\psi) - A(\varphi)A(\psi)| \leq \frac{A(|\varphi - A(\varphi)|)}{2} \tilde{\omega} \left(\psi; \frac{2(A(e_1\varphi) - A(e_1)A(\varphi))}{A(|\varphi - A(\varphi)|)} \right)$$

holds.

We are interested in the following open problem:

Open problem. Let A be a linear positive functional defined on $C[0, 1]$ and f, g be two continuous functions. Do positive numbers $\delta_1 = \delta_1(f) < 1$ and $\delta_2 = \delta_2(f) < 1$ exist such that

$$|A(fg) - A(f)A(g)| \leq \frac{1}{4} \tilde{\omega}(f; \delta_1) \tilde{\omega}(f, \delta_2)?$$



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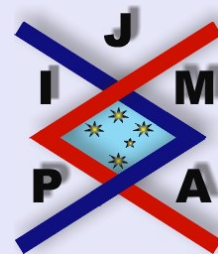
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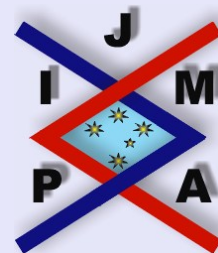
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