Journal of Inequalities in Pure and Applied Mathematics

OSSTROWSKI TYPE INEQUALITIES FROM A LINEAR FUNCTIONAL POINT OF VIEW

BOGDAN GAVREA AND IOAN GAVREA

University Babeş-Bolyai Cluj-Napoca
Department of Mathematics and Computers
Str. Mihail Kogălniceanu 1
3400 Cluj-Napoca, ROMANIA
EMail: gb7581@math.ubbcluj.ro

Technical University Cluj-Napoca
Department of Mathematics
Str. C. Daicoviciu 15
3400 Cluj-Napoca, ROMANIA
EMail: Ioan.Gavrea@math.utcluj.ro

ISSN (electronic): 1443-5756
002-99

volume 1, issue 2, article 11, 2000.

Received 13 September, 1999; accepted 31 January, 2000.

Communicated by: P. Cerone

Abstract

Inequalities are obtained using P_0-simple functionals. Applications to Lipschitzian mappings are given.

2000 Mathematics Subject Classification: 26D15
Key words: P_0-simple functionals, inequalities, Lipschitz mappings

Contents

1. Introduction

Let I be a bounded interval of the real axis. We denote by $B(I)$ the set of all functions which are bounded on $[a,b]$.

Let A be a positive linear functional $A : B(I) \to \mathbb{R}$, such that $A(e_0) = 1$, where $e_i : I \to \mathbb{R}$, $e_i(x) = x^i$, $\forall\, x \in I$, $i \in \mathbb{N}$.

The following inequality is known in literature as the Grüss inequality for the functional A.

Theorem 1.1. *Let $f, g : I \to \mathbb{R}$ be two bounded functions such that $m_1 \leq f(x) \leq M_1$ and $m_2 \leq g(x) \leq M_2$ for all $x \in I$, m_1, M_1, m_2 and M_2 are constants. Then the inequality:*

$$|A(fg) - A(f)A(g)| \leq \frac{1}{4}(M_1 - m_1)(M_2 - m_2) \tag{1.1}$$

holds.

In 1938 Ostrowski (cf. for example [7, p. 468]) proved the following result:

Theorem 1.2. *Let $f : I \to \mathbb{R}$ be continuous on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e.*

$$\|f'\|_\infty := \sup_{t \in (a,b)} |f'(t)| < \infty.$$

Then

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)\|f'\|_\infty \tag{1.2}$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is best.

In the recent paper [4] S.S. Dragomir and S. Wang proved the following version of Ostrowski's inequality.

Theorem 1.3. *Let $f : I \to \mathbb{R}$ be a differentiable mapping in the interior of I and $a, b \in int(I)$ with $a < b$. If $f' \in L_1[a,b]$ and $\gamma \le f'(x) \le \Gamma$ for all $x \in [a,b]$ then we have the following inequality:*

(1.3)
$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right)\right| \le \frac{1}{4}(b-a)(\Gamma - \gamma)$$

for all $x \in [a,b]$.

The following inequality for mappings with bounded variation can be found in [1]:

Theorem 1.4. *Let $f : I \to \mathbb{R}$ be a mapping of bounded variation. Then for all $x \in [a,b]$ we have the inequality*

$$\left| \int_a^b f(t)dt - f(x)(b-a)\right| \le \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2}\right|\right] \bigvee_a^b f, \tag{1.4}$$

where $\bigvee_a^b f$ denotes the total variation of f.

The constant $\frac{1}{2}$ is the best possible one.

In [2] S.S. Dragomir gave the following result for Lipschitzian mappings:

Theorem 1.5. *Let $f : [a,b] \to \mathbb{R}$ be an L-Lipschitzian mapping on $[a,b]$, i.e.*

$$|f(x) - f(y)| \le L|x-y|, \text{ for all } x, y \in [a,b].$$

Then we have the inequality

$$\left|\int_a^b f(t)dt - f(x)(b-a)\right| \le L(b-a)^2\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right] \tag{1.5}$$

for all $x \in [a,b]$.
The constant $\frac{1}{4}$ *is the best possible one.*

S.S. Dragomir, P. Cerone, J. Roumeliotis and S. Wang in [3] proved the following theorem:

Theorem 1.6. *Let* $f, w : (a,b) \subseteq \mathbb{R} \to \mathbb{R}$ *be so that* $w(s) \ge 0$ *on* (a,b), w *is integrable on* (a,b) *and* $\int_a^b w(s)ds > 0$, f *is of* r*-Hölder type, i.e.*

$$|f(x)-f(y)| \le H|x-y|^r, \text{ for all } x,y \in (a,b) \tag{1.6}$$

where $H > 0$ *and* $r \in (0,1]$ *are given. If* $w, f \in L_1(a,b)$, *then we have the inequality:*

$$\left|f(x) - \frac{1}{\int_a^b w(s)ds}\int_a^b w(s)f(s)ds\right| \le H\frac{1}{\int_a^b w(s)ds}\int_a^b |x-s|^r w(s)ds \tag{1.7}$$

for all $x \in (a,b)$.

The constant factor 1 in the right hand side cannot be replaced by a smaller one.

The aim of this paper is to improve the results from Theorems 1.1 – 1.6 using a unitary method.

2. Auxiliary results

Let $X = (X, d)$ be a compact metric space and $C(X)$ the Banach lattice of real-valued continuous functions on the compact metric space $X = (X, d)$, endowed with the max norm $\|\cdot\|_X$.

For a function $f \in C(X)$, the modulus of continuity (with respect to the metric d) is defined by:

$$\omega(f;t) = \omega_d(f;t) = \sup_{d(x,y)\le t} |f(x) - f(y)|, \quad t \ge 0.$$

The least concave majorant of this modulus with respect to the variable t is given by

$$\widetilde{\omega}(f;t) = \begin{cases} \sup\limits_{\substack{0\le x\le t\le y \\ x\ne y}} \frac{(t-x)\omega(f;y)+(y-t)\omega(f;x)}{y-x} & \text{for } 0 \le t \le d(X); \\ \omega(f;d(X)) & \text{for } t > d(X), \end{cases}$$

where $d(X) < \infty$ is the diameter of the compact space X.

We denote by $Lip_M\alpha = Lip_M(\alpha; X)$ the set of all Lipschitzian functions of order α, $\alpha \in [0,1]$ having the same Lipschitz constant M. That is $f \in Lip_M\alpha$ iff for all $x, y \in X$

$$|f(x) - f(y)| \le M d^\alpha(x,y).$$

We see that

$$Lip_M(\alpha; X) = \{g \in C(X) : \omega(g;t) \le M t^\alpha\}.$$

Let $I = [a, b]$ be a compact interval of the real axis, S a subspace of $C(I)$, and A a linear functional defined on S. The following definition was given by T. Popoviciu in [8].

Definition 2.1. The linear functional A defined on the subspace S which contains all polynomials is P_n-simple $(n \geq -1)$ if

(i) $A(e_{n+1}) \neq 0$

(ii) for every $f \in S$ there are the distinct points $t_1, t_2, \ldots, t_{n+2}$ in $[a, b]$ such that

$$A(f) = A(e_{n+1})[t_1, t_2, \ldots, t_{n+2}; f],$$

where $[t_1, t_2, \ldots, t_{n+2}; f]$ is the divided difference of the function f on the points $t_1, t_2, \ldots, t_{n+2}$.

In [5] the following result is proved. The proof is reproduced here for completeness.

Theorem 2.1. *Let A be a bounded linear functional, $A : C(I) \to \mathbb{R}$. If A is P_0-simple then for all $f \in C(I)$ we have*

$$|A(f)| \leq \frac{\|A\|}{2} \widetilde{\omega}\left(f; \frac{2|A(e_1)|}{\|A\|}\right). \tag{2.1}$$

Proof. For $g \in C^1(I)$ we have

$$\begin{aligned} |A(f)| &= |A(f-g) + A(g)| \leq \|A\| \|f - g\| + |A(g)| \\ &\leq \|A\| \|f - g\| + |A(e_1)| \|g'\|. \end{aligned}$$

From this inequality we obtain

$$|A(f)| \le \inf_{g\in C^1(I)} (\|A\| \|f-g\| + |A(e_1)| \|g'\|)$$

and using the following result (see [10])

$$\inf_{g\in C^1(I)} \left(\|f-g\| + \frac{t}{2}\|g'\| \right) = \frac{1}{2}\widetilde{\omega}(f;t), \quad t \ge 0$$

we obtain the relation (2.1). □

The following result was proved by I. Raşa [9].

Theorem 2.2. *Let k be a natural number such that $0 \le k \le n$ and $A : C^{(k)}[a,b] \to \mathbb{R}$ a bounded linear functional, $A \ne 0$, $A(e_i) = 0$ for $i = 0, 1, \dots, n$ such that for every $f \in C^{(k)}[a,b]$ P_n-nonconcave $A(f) \ge 0$. Then A is P_n-simple.*

A function $f \in C^{(k)}[a,b]$ is called P_0-nonconcave if for any $n+2$ points $t_1, t_2, \dots, t_{n+2} \in [a,b]$ the inequality

$$[t_1, t_2, \dots, t_{n+2}; f] \ge 0$$

holds.

Another criterion for P_n-simple functionals was given by A. Lupaş in [6]. He proved that a bounded linear functional $A : C[a,b] \to \mathbb{R}$ for which $A(e_k) = 0$, $k = 0, 1, \dots, n$ and $A(e_{n+1}) \ne 0$ is P_n-simple if and only if A is P_n-simple on $C^{(n+1)}[a,b]$.

Now we can prove the following result (see also [5]):

Theorem 2.3. *Let A be a bounded linear functional, $A : C(I) \to \mathbb{R}$. If $A(e_1) \neq 0$ and the inequality (2.1) holds for any $f \in C(I)$ then A is P_0-simple.*

Proof. We can assume that $A(e_1) > 0$. Combining the results of I. Raşa and A. Lupaş, it is sufficient, for the proof of the theorem, to show that

$$A(f) \geq 0 \tag{2.2}$$

for every nondecreasing differentiable function f defined on I.

For such a function we have

$$|A(f)| \leq A(e_1)\|f'\|.$$

Let B be the linear functional defined by

$$B(f) = \frac{A(F)}{A(e_1)},$$

where

$$F(t) = \int_0^t f(u)du, \quad f \in C[0,1].$$

The functional B is bounded and for any $f \in C(I)$ we have

$$|B(f)| \leq \|f\|$$

with $B(e_0) = 1$.

Let f be a continuous function such that $f \geq 0$, $f \neq 0$.

From the inequalities

$$0 \le e_0 - \frac{f}{\|f\|} \le 1$$

we obtain

$$1 - \frac{B(f)}{\|f\|} \le \left| B\left(e_0 - \frac{f}{\|f\|} \right) \right| \le 1.$$

These inequalities imply that

$$B(f) \ge 0. \tag{2.3}$$

Further, let f be a differentiable function on I such that $f' \ge 0$, then, from (2.3) we obtain

$$B(f') \ge 0.$$

Since $B(f') = A(f)$, the inequality (2.2) is thus proved. □

3. An Integral Inequality of Ostrowski Type

The following inequality of Ostrowski type holds.

Theorem 3.1. *Let f be a continuous function on $[a,b]$ and $w:(a,b)\to\mathbb{R}_+$ an integrable function on (a,b) such that $\int_a^b \omega(s)ds = 1$. Then for any continuous function f the following inequality:*

(3.1)
$$\left| f(x) - \int_a^b w(s)f(s)ds \right| \le \left(\int_a^x w(t)dt\right) \widetilde{\omega}_{[a,x]}\left(f; \frac{\int_a^x w(t)(x-t)dt}{\int_a^x w(t)}\right) + \left(\int_x^b w(t)dt\right) \widetilde{\omega}_{[x,b]}\left(f; \frac{\int_x^b w(t)(t-x)dt}{\int_x^b w(t)dt}\right)$$

holds, where x is a fixed point in (a,b).

Proof. From Theorem 2.3 we get that the linear functionals

$$A_1 : C[a,x] \to \mathbb{R}, \quad A_2 : C[x,b] \to \mathbb{R}$$

defined by

$$A_1(f) = f(x)\int_a^x w(t)dt - \int_a^x f(t)w(t)dt$$

and

$$A_2(f) = f(x)\int_x^b w(t)dt - \int_x^b f(t)w(t)dt$$

are P_0-simple.

It is easy to see that:

$$\|A_1\| = 2\int_a^x w(t)dt \qquad \text{and} \qquad \|A_2\| = 2\int_x^b w(t)dt.$$

From the inequality:

$$\left|f(x) - \int_a^b w(s)f(s)ds\right| \le \left(\int_a^x w(t)dt\right)\widetilde{\omega}\left(f; \frac{|A_1(e_1)|}{\int_a^x w(t)dt}\right) + \left(\int_x^b w(t)dt\right)\widetilde{\omega}\left(f; \frac{A_2(e_1)}{\int_x^b w(t)dt}\right)$$

and from the results

$$|A_1(e_1)| = \int_a^x w(t)(x-t)dt \qquad \text{and} \qquad |A_2(e_1)| = \int_x^b w(t)(t-x)dt,$$

(3.1) follows. □

Corollary 3.2. *Let f be a continuous function on $[a,b]$, such that $f \in Lip_{M_1}(\alpha,[a,x])$ and $f \in Lip_{M_2}(\beta;[x,b])$. Then*

(3.2)
$$\left|f(x) - \int_a^b w(s)f(s)ds\right| \le M_1\left(\int_a^x w(t)dt\right)^{1-\alpha}\left[\int_a^x w(t)(x-t)dt\right]^{\alpha} + M_2\left(\int_x^b w(t)dt\right)^{1-\beta}\left[\int_x^b w(t)(t-x)dt\right]^{\beta}.$$

Proof. The proof follows from the inequality (3.1) and the fact that

$$\widetilde{\omega}_1(g;t) \leq Mt^r$$

for any function g, $g \in Lip_M(\alpha, [c,d])$, where $\widetilde{\omega}_1$ is taken on the interval $[c,d]$. □

Corollary 3.2 is an improvement of the result of Theorem 1.6.

Remark 3.1. In the particular case when $w(t) = \frac{1}{b-a}$ the inequality (3.2) becomes:

$$(3.3)\quad \left| f(x) - \frac{\int_a^b f(s)ds}{b-a} \right| \leq \left[M_1 \frac{(x-a)^{\alpha+1}}{2^\alpha} + M_2 \frac{(b-x)^{\beta+1}}{2^\beta} \right] \frac{1}{b-a}$$
$$\leq \max(M_1, M_2) \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a).$$

Inequality (3.3) improves the inequality (1.5).

Corollary 3.3. *Let $f : [a,b] \to \mathbb{R}$ be continuous on (a,b), whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b) and w a function as in Theorem 3.1. Then we have the following inequality:*

(3.4)
$$\left| f(x) - \int_a^b w(s)f(s)ds \right| \leq \left[\int_a^x w(t)(x-t)dt + \int_x^b w(t)(t-x)dt \right] \|f'\|_\infty.$$

Proof. The above inequality is a consequence of the inequality (3.1) and the fact that

$$\widetilde{\omega}(f;t) \leq \|f'\|_\infty t.$$

The inequality of Ostrowski follows from (3.4) if we consider

$$w(t) = \frac{1}{b-a}, \quad t \in [a,b].$$

□

Corollary 3.4. *Let $f : I \to \mathbb{R}$ be a mapping with bounded variation and w a function as in Theorem 3.1. Then for all $x \in [a,b]$ we have the inequalities*

$$\left| f(x) - \int_a^b w(s)f(s)ds \right| \le \bigvee_a^x f \int_a^x w(t)dt + \bigvee_x^b f \int_x^b w(t)dt \tag{3.5}$$

$$\left| f(x) - \int_a^b w(s)f(s)ds \right| \le \left(\frac{1}{2} + \frac{\left| \int_a^x w(t)dt - \int_x^b w(t)dt \right|}{2} \right) \bigvee_a^b f. \tag{3.6}$$

Proof. It is clear that

$$\widetilde{\omega}[a,x](f;t) \le \bigvee_a^x f \quad \text{and} \quad \widetilde{\omega}[x,b](f,t) \le \bigvee_x^b f \tag{3.7}$$

for every positive number t.

Thus, inequality (3.5) follows from (3.7).

For the proof of the inequality (3.6) we note that, if we suppose $\int_a^x w(t)dt \le \frac{1}{2}$ then $\int_x^b w(t)dt \ge \frac{1}{2}$ and vice versa.

For definiteness we assume that

$$\int_a^x w(t)dt \le \frac{1}{2} \qquad \text{and} \qquad \int_x^b w(t)dt \ge \frac{1}{2}.$$

We then have

$$\begin{aligned}\bigvee_a^x f \int_a^x w(t)dt + \bigvee_x^b f \int_x^b w(t)dt &\le \frac{1}{2}\bigvee_a^x f + \bigvee_x^b f \int_x^b w(t)dt \\ &= \frac{1}{2}\bigvee_a^b f + \bigvee_x^b f \left(\int_x^b w(t)dt - \frac{1}{2}\right)\end{aligned}$$

and so

(3.8)

$$\bigvee_a^x f \int_a^x w(t)dt + \bigvee_x^b f \int_x^b w(t)dt \le \left(\frac{1}{2} + \frac{\int_x^b w(t)dt - \int_a^x w(t)dt}{2}\right)\bigvee_a^b f.$$

From the inequalities (3.5) and (3.8), the inequality (3.6) follows. □

Remark 3.2. The inequality from Theorem 1.4 follows if we take in (3.6)

$$w(t) = \frac{1}{b-a}.$$

Theorem 3.5. *Let g be a continuous differentiable function on $[a,b]$ such that $g(a) = g(b) = 0$. Then the inequality*

(3.9)

$$\left|\frac{g(x)}{2} - \frac{1}{b-a}\int_a^b g(t)dt\right| \le \frac{(x-a)^2 + (b-x)^2}{8(b-a)}\widetilde{\omega}\left(g'; \frac{2}{3}\frac{(x-a)^3 + (y-b)^3}{(x-a)^2 + (y-b)^2}\right)$$

holds, where x is an arbitrary point in (a,b).

Proof. The following functional A defined on $C[a,b]$ by

$$A(f)=\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)f(t)dt$$

is a linear bounded functional having its norm equal to $\frac{b-a}{4}$. For every increasing function f we have:

$$A(f)\geq 0.$$

Using Theorem 2.3, we deduce that the functional A is P_0-simple with

$$A(e_1)=\frac{(b-a)^2}{12}.$$

From Theorem 2.1, we obtain the following inequality:

$$\left|\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)f(t)dt\right|\leq\frac{b-a}{8}\tilde{\omega}\left(f;\frac{2}{3}(b-a)\right). \tag{3.10}$$

Inequality (3.10) holds for every continuous function f.

Let us suppose that f is differentiable on $[a,b]$. From the inequality (3.10) (written for f') we obtain the following inequality:

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{f(a)+f(b)}{2}\right|\leq\frac{b-a}{8}\tilde{\omega}\left(f';\frac{2}{3}(b-a)\right). \tag{3.11}$$

Now, we can prove the inequality (3.9). We have the following identity:

(3.12)
$$-\frac{g(x)}{2}+\frac{1}{b-a}\int_a^b g(t)dt=\frac{x-a}{b-a}\left(\frac{1}{x-a}\int_a^x g(t)dt-\frac{g(a)+g(x)}{2}\right)+\frac{b-x}{b-a}\left(\frac{1}{b-x}\int_a^b g(t)dt-\frac{g(b)+g(x)}{2}\right).$$

Using the relations (3.11) and (3.12) we obtain

(3.13)
$$\left|\frac{g(x)}{2}-\frac{1}{b-a}\int_a^b g(t)dt\right| \leq \frac{(x-a)^2}{8(b-a)}\widetilde{\omega}\left(g';\frac{2}{3}(x-a)\right)+\frac{(b-x)^2}{8(b-a)}\widetilde{\omega}\left(g';\frac{2}{3}(b-x)\right).$$

As the function $\widetilde{\omega}(g';\cdot)$, is concave, then from (3.13) and using Jensen's inequality, we obtain the inequality (3.9). □

Corollary 3.6. *Let g be a continuous differentiable function on $[a,b]$ such that $g(a)=g(b)=0$, then the following inequality*

(3.14)
$$\left|\frac{g(x)}{2}-\frac{1}{b-a}\int_a^b g(t)dt\right|\leq\left[\frac{1}{8}+\frac{\left(x-\frac{a+b}{2}\right)^2}{2(b-a)}\right](b-a)\|g'\|_\infty$$

is valid for all $x\in[a,b]$.

Proof. It is well known that

$$\widetilde{\omega}(g';t) \le 2\|g'\|_\infty, \tag{3.15}$$

for every positive number t.

The inequality (3.15) then readily follows from the inequality (3.14). □

Remark 3.3. The result from the Theorem 1.3 can be written in terms of $\widetilde{\omega}$ using the inequality (3.13) for the function

$$g(x) = f(x) - \frac{x-a}{b-a} f(b) - \frac{b-x}{b-a} f(a).$$

In [5] the following result was proved:

Let A be a linear positive functional $A : C[0,1] \to \mathbb{R}$, $A(e_0) = 1$ and φ, $\varphi : [0,1] \to \mathbb{R}$ a continuous increasing function such that $A(e_1\varphi) - A(e_1)A(\varphi) > 0$. Then the following Grüss type inequality

$$|A(\varphi f) - A(\varphi)A(f)| \le \frac{A(|\varphi - A(\varphi)|)}{2} \widetilde{\omega}\left(f; \frac{2(A(e_1\varphi) - A(e_1)A(\varphi))}{A(|\varphi - A(\varphi)|)}\right) \tag{3.16}$$

holds.

We are interested in the following open problem:

Open problem. Let A be a linear positive functional defined on $C[0,1]$ and f, g be two continuous functions. Do positive numbers $\delta_1 = \delta_1(f) < 1$ and $\delta_2 = \delta_2(f) < 1$ exist such that

$$|A(fg) - A(f)A(g)| \le \frac{1}{4} \widetilde{\omega}(f;\delta_1) \widetilde{\omega}(f,\delta_2)?$$

References

[1] S.S. DRAGOMIR, On Ostrowski's integral inequality for mappings with bounded variation and applications, *Math. Inequal. Appl.* (in press).

[2] S.S. DRAGOMIR, On the Ostrowski integral inequality for Lipschitzian mappings and applications, *Comput. Math. Appl.*, **38** (1999), 33–37.

[3] S.S. DRAGOMIR, P. CERONE, J. ROUMELIOTIS AND S. WANG, A weighted version of Ostrowski inequality for mappings of Hölder type and applications in numerical analysis, *Bull. Math. Soc. Sc. Math. Roumanie*, **42(90)**(4), (1999), 301–314.

[4] S.S. DRAGOMIR AND S. WANG, An Inequality of Ostrowski-Grüss' type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules, *Comput. Math. Appl.*, **33**(11) (1997), 15–20.

[5] I. GAVREA, Preservation of Lipschitz constants by linear transformations and global smoothness preservation (submitted).

[6] A. LUPAŞ, *Contribuţii la teoria aproximării prin operatori liniari*, Ph. D. Thesis, Cluj-Napoca, Babeş-Bolyai University, 1975.

[7] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Inequalities for Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers.

[8] T. POPOVICIU, Sur le reste dans certains formules lineaires d'approximation de l'analyse, *Mathematica, Cluj.*, **1**(24), 95–142.

[9] I. RAŞA, Sur les fonctionnelles de la forme simple au sens de T. Popoviciu, *L'Anal. Num. et la Theorie de l'Approx.*, **9** (1980), 261–268.

[10] E.M. SEMENOV AND B.S. MITJAGIN, Lack of interpolation of linear operators in spaces of smooth functions, *Math. USSR-Izv.*, **11** (1977), 1229–1266.