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# CERTAIN SECOND ORDER LINEAR DIFFERENTIAL SUBORDINATIONS

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## Abstract

In this present investigation, we obtain some results for certain second order linear differential subordination. We also discuss some applications of our results.

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## Contents

# 1. Introduction

Let $\mathcal{H}$ denote the class of all *analytic* functions in $\Delta := \{z \in \mathbb{C} : |z| < 1\}$. For a positive integer $n$ and $a \in \mathbb{C}$, let

$$\mathcal{H}[a,n] := \left\{ f \in \mathcal{H} : f(z) = a + \sum_{k=n}^{\infty} a_k z^k \quad (n \in \mathbb{N} := \{1,2,3,\ldots\}) \right\}$$

and

$$\mathcal{A}(p,n) := \left\{ f \in \mathcal{H} : f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k \quad (n,p \in \mathbb{N}) \right\}.$$

Set

$$\mathcal{A}_p := \mathcal{A}(p,1), \quad \mathcal{A} := \mathcal{A}_1.$$

For two functions $f$, $g \in \mathcal{H}$, we say that the function $f(z)$ is *subordinate* to $g(z)$ in $\Delta$ and write

$$f \prec g \qquad \text{or} \qquad f(z) \prec g(z),$$

if there exists a Schwarz function $w(z) \in \mathcal{H}$ with

$$w(0) = 0 \qquad \text{and} \qquad |w(z)| < 1 \quad (z \in \Delta),$$

such that

(1.1) $$f(z) = g(w(z)) \quad (z \in \Delta).$$

In particular, if the function $g$ is univalent in $\Delta$, the above subordination (1.1) is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

Miller and Mocanu [2] considered the *second order linear differential subordination*

$$A(z)z^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) \prec h(z),$$

where $A, B, C$ and $D$ are complex-valued functions defined on $\Delta$ and $h(z)$ is any convex function and in particular $h(z) = (1+z)/(1-z)$. In fact, they have proved the following:

**Theorem 1.1 (Miller and Mocanu [2, Theorem 4.1a, p.188]).** *Let $n$ be a positive integer and $A(z) = A \geq 0$. Suppose that the functions $B(z),\ C(z),\ D(z) : \Delta \to \mathbb{C}$ satisfy $\Re B(z) \geq A$ and*

$$[\Im C(z)]^2 \leq n[\Re B(z) - A]\Re(nB(z) - nA - 2D(z)). \tag{1.2}$$

*If $p \in \mathcal{H}[1, n]$ and if*

$$\Re\{Az^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z)\} > 0, \tag{1.3}$$

*then*

$$\Re p(z) > 0.$$

Also Miller and Mocanu [2] have proved the following:

**Theorem 1.2 (Miller and Mocanu [2, Theorem 4.1e, p.195]).** *Let* $h$ *be convex univalent in* $\Delta$ *with* $h(0) = 0$ *and let* $A \geq 0$*. Suppose that* $k > 4/|h'(0)|$ *and that* $B(z)$, $C(z)$ *and* $D(z)$ *are analytic in* $\Delta$ *and satisfy*

$$\Re\, B(z) \geq A + |C(z) - 1| - \Re(C(z) - 1) + k|D(z)|.$$

*If* $p \in \mathcal{H}[0, 1]$ *satisfies the differential subordination*

$$Az^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) \prec h(z)$$

*then* $p \prec h$*.*

In this paper, we extend Theorem 1.1 by assuming

$$\Re\{Az^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z)\} > \alpha, \quad (0 \leq \alpha < 1)$$

and Theorem 1.2 by assuming that the function $h(z)$ is convex of order $\alpha$. Certain results of Karunakaran and Ponnusamy [6], Juneja and Ponnusamy [7] and Owa and Srivastava [8] are obtained as special cases. Also we give application of our results to certain functions defined by the familiar Ruscheweyh derivatives.

For two functions $f(z)$ and $g(z)$ given by

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k, \quad g(z) = z^p + \sum_{k=n+p}^{\infty} b_k z^k \quad (n, p \in \mathbb{N}),$$

the *Hadamard product* (or *convolution*) of $f$ and $g$ is defined by

$$(f * g)(z) := z^p + \sum_{k=n+p}^{\infty} a_k b_k z^k =: (g * f)(z).$$

The *Ruscheweyh derivative* of $f(z)$ of order $\delta+p-1$ is defined by

$$D^{\delta+p-1} f(z) := \frac{z^p}{(1-z)^{\delta+p}} * f(z) \quad (f \in \mathcal{A}(p,n);\ \delta \in \mathbb{R}\setminus(-\infty,-p]) \tag{1.4}$$

or, equivalently, by

$$D^{\delta+p-1} f(z) := z^p + \sum_{k=p+1}^{\infty} \binom{\delta+k-1}{k-p} a_k\, z^k \tag{1.5}$$
$$(f \in \mathcal{A}(p,n);\ \delta \in \mathbb{R}\setminus(-\infty,-p]).$$

In particular, if $\delta = l$ $(l+p \in \mathbb{N})$, we find from the definition (1.4) or (1.5) that

$$D^{l+p-1} f(z) = \frac{z^p}{(l+p-1)!} \frac{d^{l+p-1}}{dz^{l+p-1}} \{z^{l-1} f(z)\}$$
$$(f \in \mathcal{A}(p,n);\ l+p \in \mathbb{N}).$$

In our present investigation of the second order linear differential subordination, we need the following definitions and results:

**Definition 1.1 (Miller and Mocanu [2, Definition 2.2b, p. 21]).** *Let $Q$ be the set of functions $q$ that are analytic and univalent on $\overline{\Delta} \setminus E(q)$, where*

$$E(q) = \{\zeta \in \partial\Delta : \lim_{z\to\zeta} q(z) = \infty\}$$

*and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\Delta \setminus E(q)$, where $\partial\Delta := \{z \in \mathbb{C} : |z| = 1\}$, $\overline{\Delta} := \Delta \cup \partial\Delta$.*

**Theorem 1.3 (Miller and Mocanu [2, Lemma 2.2d, p. 24]).** *Let $q \in Q$, with $q(0) = a$. Let $p(z) = a + p_n z^n + \cdots$ be analytic in $\Delta$ with $p(z) \not\equiv a$ and $n \geq 1$. If $p(z)$ is not subordinate to $q(z)$, then there exist points $z_0 = r_0 e^{\theta_0} \in \Delta$ and $\zeta_0 \in \partial\Delta - E(q)$, and an $m \geq n \geq 1$ for which $p(\Delta_{r_0}) \subset q(\Delta)$,*

$$\begin{aligned} &(i) && p(z_0) = q(\zeta_0) \\ (1.6)\qquad &(ii) && z_0 p'(z_0) = m\zeta_0 q'(\zeta_0), \text{ and} \\ &(iii) && \Re[z_0 p''(z_0)/p'(z_0) + 1] \geq m\Re[z_0 q''(z_0)/q'(z_0) + 1], \end{aligned}$$

*where $\Delta_r := \{z \in \mathbb{C} : \ |z| < r\}$.*

**Theorem 1.4 (cf. Miller and Mocanu [2, Theorem 2.3i (i), p. 35]).** *Let $\Omega$ be a simply connected domain and $\psi : \mathbb{C}^3 \times \Delta \to \mathbb{C}$ satisfies the condition*

$$\psi(i\sigma, \zeta, \mu + i\eta; z) \notin \Omega$$

*for $z \in \Delta$ and for real $\sigma, \zeta, \mu, \eta$ satisfying $\zeta \leq -n(1 + \sigma^2)/2$ and $\zeta + \mu \leq 0$. Let $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots$ be analytic in $\Delta$. If*

$$\psi(p(z), zp'(z), z^2 p''(z); z) \in \Omega,$$

*then $\Re p(z) > 0$.*

## 2. Differential Subordination with Convex Functions of Order $\alpha$

By appealing to Theorem 1.3, we first prove the following:

**Theorem 2.1.** *Let $h$ be a convex univalent function of order $\alpha$, $0 \leq \alpha < 1$, in $\Delta$ with $h(0) = 0$ and let $A \geq 0$. Suppose that*

$$k > 2^{2(1-\alpha)} / |h'(0)|$$

*and that $B(z)$, $C(z)$ and $D(z)$ are analytic in $\Delta$ and satisfy*

$$n\Re B(z) \geq n(1-\alpha n)A + \frac{1}{2\beta(\alpha)}[|C(z)-1| - \Re(C(z)-1)] + k|D(z)|, \tag{2.1}$$

*where*

$$\beta(\alpha) := \begin{cases} \dfrac{4^\alpha(1-2\alpha)}{4-2^{2\alpha+1}} & \alpha \neq \dfrac{1}{2} \\ (\log\ 4)^{-1} & \alpha = \dfrac{1}{2}. \end{cases} \tag{2.2}$$

*If $p \in \mathcal{H}[0,n]$ satisfies the differential subordination*

$$Az^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z) \prec h(z), \tag{2.3}$$

*then $p \prec h$.*

*Proof.* Our proof of Theorem 2.1 is essentially similar to Theorem 1.2 of Miller and Mocanu [2]. Let the subordination in (2.3) be satisfied so that $D(0) = 0$. Since

$$k|h'(0)| > 2^{2(1-\alpha)},$$

there is an $r_0$, $0 < r_0 < 1$ such that

$$\frac{(1+r_0)^{2(1-\alpha)}}{r_0} = k|h'(0)| \text{ and } 2^{2(1-\alpha)} < \frac{(1+r)^{2(1-\alpha)}}{r} < k|h'(0)|$$

for $r_0 < r < 1$. Since $h$ is convex of order $\alpha$ in $\Delta$, the function $h_r(z) = h(rz)$ is convex of order $\alpha$ in $\overline{\Delta}$ $(r_0 < r < 1)$. By setting $p_r(z) = p(rz)$ for $r_0 < r < 1$, we see that the subordination (2.3) becomes

$$u_r(z) := Az^2p_r''(z) + B(rz)zp_r'(z) + C(rz)p_r(z) + D(rz) \prec h_r(z) \quad (z \in \Delta;\ r_0 < r < 1). \tag{2.4}$$

Assume that $p_r$ is not subordinate to $h_r$, for some $r$ in $(r_0, 1)$. Then by Theorem 1.3 there exist points $z_0 \in \Delta$, $w_0 \in \partial\Delta$ and an $m \geq n \geq 1$ such that

$$p_r(z_0) = h_r(w_0), z_0p_r'(z_0) = mw_0h_r'(w_0), \tag{2.5}$$

$$\Re\left(1 + \frac{z_0p_r''(z_0)}{p_r'(z_0)}\right) \geq m\Re\left(1 + \frac{w_0h_r''(w_0)}{h_r'(w_0)}\right). \tag{2.6}$$

Therefore we have

$$\Re\left(1 + \frac{z_0^2p_r''(z_0)}{mw_0h'(w_0)}\right) \geq m\alpha. \tag{2.7}$$

From Equations (2.5), (2.6) and (2.7), it follows that

$$\Re\left(\frac{z_0^2 p_r''(z_0)}{w_0 h_r'(w_0)}\right) \geq m(m\alpha - 1). \tag{2.8}$$

Since $h_r(z)$ is convex of order $\alpha$ or equivalently

$$\Re\left(1 + \frac{z h_r''(z)}{h_r'(z)}\right) > \alpha \quad (z \in \overline{\Delta}),$$

by [2, Theorem 3.3f, p.115], we have

$$\Re\frac{z h_r'(z)}{h_r(z)} > \beta(\alpha) \quad (z \in \overline{\Delta})$$

where $\beta(\alpha)$ is given by Equation (2.2) and this condition is equivalent to

$$\left|\frac{h_r(z)}{z h_r'(z)} - \frac{1}{2\beta(\alpha)}\right| \leq \frac{1}{2\beta(\alpha)} \quad (z \in \overline{\Delta}).$$

Therefore,

$$\Re\left[(C(rz_0) - 1)\frac{h_r(w_0)}{w_0 h_r'(w_0)}\right] \geq \frac{1}{2\beta}\{\Re[C(rz_0) - 1] - |C(rz_0) - 1|\}. \tag{2.9}$$

Since $h$ is convex of order $\alpha$, we have the following well-known estimate:

$$|h'(z)| \geq \frac{|h'(0)|}{(1+r)^{2(1-\alpha)}} \quad (|z| = r < 1).$$

By setting $z = rw_0$, we see that

$$|w_0 h_r'(w_0)| \geq \frac{r|h'(0)|}{(1+r)^{2(1-\alpha)}} \quad (|w_0| = 1). \tag{2.10}$$

By setting

$$V := \frac{Az_0^2 p_r''(z_0)}{w_0 h_r'(w_0)} + \frac{B(rz_0) z_0 p_r'(z_0)}{w_0 h_r'(w_0)} + (C(rz_0) - 1)\frac{p_r(z_0)}{w_0 h_r'(w_0)} + \frac{D(rz_0)}{w_0 h_r'(w_0)}, \tag{2.11}$$

we see that

$$u_r(z_0) = h_r(w_0) + V w_0 h_r'(w_0). \tag{2.12}$$

From (2.8), (2.9), (2.10) and (2.11), we have

$$\begin{aligned}
\Re V &\geq m(m\alpha - 1)A + m\Re B(rz_0) + \frac{1}{2\beta(\alpha)}[\Re(C(rz_0) - 1) - |C(rz_0) - 1|] \\
&\quad - \frac{(1+r)^{2(1-\alpha)}}{r|h'(0)|}|D(rz_0)| \\
&\geq m[(n\alpha - 1)A + \Re B(rz_0)] \\
&\quad + \frac{1}{2\beta(\alpha)}[\Re(C(rz_0) - 1) - |C(rz_0) - 1|] - k|D(rz_0)| \\
&\geq n[(n\alpha - 1)A + \Re B(rz_0)] \\
&\quad - \frac{1}{2\beta(\alpha)}[|C(rz_0) - 1| - \Re(C(rz_0) - 1)] - k|D(rz_0)| \geq 0,
\end{aligned}$$

it follows that $u_r(z_0) \notin h_r(\Delta)$, a contradiction. Therefore, $p_r \prec h_r$ for $r \in (r_0, 1)$. By letting $r \to 1^-$, we obtain the desired conclusion $p \prec h$. □

**Remark 2.1.** *When $\alpha = 0, n = 1$, Theorem 2.1 reduces to Theorem 1.2 of Miller and Mocanu [2].*

From the proof of Theorem 2.1, it is clear that the condition $h(0) = 0$ in not necessary when $C(z) = 1$ and hence the following:

**Corollary 2.2.** *Let $h$ be a convex univalent function of order $\alpha$, $0 \leq \alpha < 1$, in $\Delta$, $h(0) = a$ and let $A \geq 0$. Suppose that*

$$k > 2^{2(1-\alpha)}/|h'(0)|$$

*and that $B(z)$ and $D(z)$ are analytic in $\Delta$ with $D(0) = 0$ and*

$$n\,\Re\, B(z) \geq n(1 - \alpha n)A + k|D(z)| \tag{2.13}$$

*for all $z \in \Delta$. If $p \in \mathcal{H}[a, n]$, $p(0) = h(0)$, satisfies the differential subordination*

$$Az^2 p''(z) + B(z)zp'(z) + p(z) + D(z) \prec h(z), \tag{2.14}$$

*then $p \prec h$.*

By taking $A = 0$ and $D(z) = 0$ in Theorem 2.1, we obtain the following:

**Corollary 2.3.** *Let $h$ be a convex univalent function of order $\alpha$, $0 \leq \alpha < 1$, in $\Delta$ with $h(0) = 0$. Let $B(z)$ and $C(z)$ be analytic functions on $\Delta$ satisfying*

$$\Re B(z) \geq \frac{1}{2n\beta(\alpha)}[|C(z) - 1| - \Re(C(z) - 1)],$$

*where $\beta(\alpha)$ is as given in Theorem 2.1. If $p \in \mathcal{H}[0,n]$ satisfies the subordination*

$$B(z)zp'(z) + C(z)p(z) \prec h(z),$$

*then $p(z) \prec h(z)$.*

By taking $B(z) = 1$, $\alpha = 0$, $n = 1$, in Corollary 2.3, we have the following:

**Corollary 2.4.** *Let $h$ be a convex univalent function in $\Delta$ with $h(0) = 0$. Let $C(z)$ be analytic functions on $\Delta$ satisfying*

$$\Re C(z) > |C(z) - 1|.$$

*If the analytic function $p(z)$ satisfies the subordination*

$$zp'(z) + C(z)p(z) \prec h(z),$$

*then $p(z) \prec h(z)$.*

## 3. Differential Subordination with Caratheodory Functions of Order $\alpha$

By appealing to Theorem 1.4, we now prove the following:

**Theorem 3.1.** *Let $n$ be a positive integer and $A(z) = A \geq 0$. Suppose that the functions $B(z), C(z), D(z) : \Delta \to \mathbb{C}$ satisfy $\Re B(z) \geq A$ and*

$$(3.1)\quad [\Im C(z)]^2 \leq n\,[\Re B(z) - A] \times \left[ n(\Re B(z) - A) - \frac{\delta + 2\alpha}{1-\alpha} \Re C(z) - \frac{2+\delta}{1-\alpha} \Re(D(z) - \alpha) \right].$$

*If $p \in \mathcal{H}[1, n]$ and*

$$(3.2)\qquad \Re\{Az^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z)\} > \alpha \quad (\alpha < 1),$$

*then*

$$\Re p(z) > \frac{\delta + 2\alpha}{\delta + 2}.$$

*Proof.* Define the function $P(z)$ by

$$P(z) := \frac{p(z) - \gamma}{1 - \gamma} \quad \text{where} \quad \gamma := \frac{\delta + 2\alpha}{\delta + 2}.$$

Then inequality (3.2) can be written as

$$\Re\{\psi(P(z),\ zP'(z),\ z^2P''(z);\ z)\} > 0,$$

where

$$\psi(r,\ s,\ t;\ z) = At + B(z)s + C(z)r + \frac{\gamma C(z) + D(z) - \alpha}{1-\gamma}.$$

In view of Theorem 1.4, it is enough to show that

$$\Re\psi(i\sigma,\ \zeta,\ \mu + i\eta;\ z) \le 0$$

for all real numbers $\sigma,\ \zeta,\ \mu$ and $\eta$ with $\zeta \le \frac{-n(1+\sigma^2)}{2}$, $\zeta + \mu \le 0$ and for all $z \in \Delta$. Now,

$$\begin{aligned}
&\Re\psi(i\sigma,\ \zeta,\ \mu + i\eta;\ z) \\
&\qquad = \mu A + \zeta \Re B(z) - \sigma \Im C(z) + \Re\left[\frac{\gamma C(z) + D(z) - \alpha}{1-\gamma}\right] \\
&\qquad \le \zeta(\Re B(z) - A) - \sigma \Im C(z) + \Re\left[\frac{\gamma C(z) + D(z) - \alpha}{1-\gamma}\right] \\
&\qquad \le -\frac{1}{2}\left\{n[\Re B(z) - A]\sigma^2 + 2\Im C(z)\sigma\right. \\
&\qquad\qquad \left. + n[\Re B(z) - A] - 2\Re\left[\frac{\gamma C(z) + D(z) - \alpha}{1-\gamma}\right]\right\} \le 0,
\end{aligned}$$

provided (3.1) holds. This completes the proof of our Theorem 3.1. □

For $\alpha = \delta = 0$, Theorem 3.1 reduces to Theorem 1.1.

By taking $D = 0$ and $C(z) = 1$ in Theorem 3.1, we have the following:

**Corollary 3.2.** *Let $A \geq 0$ and $\Re B(z) - A > \delta > 0$. If $p \in \mathcal{H}[1,n]$ satisfies*

$$\Re\{Az^2 p''(z) + B(z)zp'(z) + p(z)\} > \alpha \quad (\alpha < 1)$$

*then*

$$\Re p(z) > \frac{n\delta + 2\alpha}{n\delta + 2}.$$

**Corollary 3.3.** *Let $\lambda(z)$ and $R(z)$ be functions defined on $\Delta$ and*

$$\Re\lambda(z) > \delta + \frac{2+\delta}{(1-\alpha)n}\Re R(z) \geq 0.$$

*If $p \in \mathcal{H}[1,n]$ satisfies*

$$\Re\{\lambda(z)zp'(z) + p(z) + R(z)\} > \alpha \quad (\alpha < 1),$$

*then*

$$\Re p(z) > \frac{2\alpha + \delta n}{2 + \delta n}.$$

A special case of Corollary 3.3 is obtained by Owa and Srivastava [8, Lemma 2, p. 254].

The proof of the following theorem is similar and hence it is omitted.

**Theorem 3.4.** *Let $n$ be a positive integer and $A(z) = A \geq 0$. Suppose that the functions $B(z), C(z), D(z) : \Delta \to \mathbb{C}$ satisfy $\Re B(z) \geq A$ and*

$$(3.3) \quad [\Im C(z)]^2 \leq n[\Re B(z) - A]\left[n(\Re B(z) - A) - \frac{\delta + 2\alpha}{1-\alpha}\Re C(z) - \frac{2+\delta}{1-\alpha}\Re(D(z) - \alpha)\right].$$

*If* $p \in \mathcal{H}[1,n]$ *satisfies*

$$\Re\{Az^2p''(z) + B(z)zp'(z) + C(z)p(z) + D(z)\} < \alpha \quad (\alpha > 1), \tag{3.4}$$

*then*

$$\Re p(z) < \frac{\delta + 2\alpha}{\delta + 2}.$$

## 4. Applications

We now give certain applications of our results obtained in Section 2 and 3.

**Theorem 4.1.** *Let* $\gamma \in \mathbb{C}$ *with* $\gamma \neq -1, -2, -3, \dots$ *and let* $\phi, \Phi$ *be analytic functions on* $\Delta$ *with* $\phi(z)\Phi(z) \neq 0$ *for* $z \in \Delta$. *If*

$$\Re C(z) - |C(z) - 1| > 1 - 2n\beta(\alpha)\Re B(z),$$

*where*

$$B(z) := \frac{\Phi(z)}{\phi(z)} \text{ and } C(z) := \frac{\gamma\Phi(z) + z\Phi'(z)}{\phi(z)},$$

*then the integral operator defined by*

$$I(f)(z) := \frac{1}{z^\gamma \Phi(z)} \int_0^z t^{\gamma-1} f(t)\phi(t)dt$$

*satisfies* $I(f)(z) \prec h(z)$ *for every function* $f(z) \prec h(z)$ *where* $h(z)$ *is a convex function of order* $\alpha$.

*Proof.* The result follows immediately from Corollary 2.3. □

**Theorem 4.2.** *Let* $h$ *be a convex univalent function of order* $\alpha$ *in* $\Delta$, $0 \leq \alpha < 1$ *and* $h(0) = 1$. *Let* $M, N, R$ *be analytic in* $\Delta$ *with* $R(0) = 0$ *and*

$$M(z) = z^n + \dots, \text{ and } N(z) = z^n + \dots.$$

*Let*

$$\Re \frac{\beta N(z)}{zN'(z)} > k|R(z)| \quad \left(k > \frac{2^{2(1-\alpha)}}{|h'(0)|}\right).$$

*If*

$$\beta \frac{M'(z)}{N'(z)} + (1-\beta)\frac{M(z)}{N(z)} + R(z) \prec h(z), \tag{4.1}$$

*then*

$$\frac{M(z)}{N(z)} \prec h(z).$$

*Proof.* Let the function $p(z)$ be defined by

$$p(z) = M(z)/N(z).$$

Then $p(0) = 1 = h(0)$ and it follows that

$$p(z) + \frac{N(z)}{zN'(z)} zp'(z) = \frac{M'(z)}{N'(z)}.$$

Also, a computation shows that the subordination in (4.1) is equivalent to

$$p(z) + \frac{\beta N(z)}{zN'(z)} zp'(z) + R(z) \prec h(z).$$

The result now follows by an application of Corollary 2.2 □

**Remark 4.1.** *When $\beta = 1, \alpha = 0$, Theorem 4.2 reduces to [2, Theorem 4.1h, p. 199] of Miller and Mocanu. If $\alpha = 0$ and $R(z) = 0$, then Theorem 4.2 reduces to a result of Juneja and Ponnusamy [7, Corollary 1, p. 290].*

More generally, we have the following:

**Theorem 4.3.** *Let $\delta > -p$ be any real number, $\lambda \in \mathbb{C}$ with $\Re\lambda \geq 0$. Let $R(z)$ be a function defined on $\Delta$ with $R(0) = 0$ and $h(z)$ a convex function of order $\alpha$, $0 \leq \alpha < 1$, $h(0) = 1$. Let $g \in \mathcal{A}_p$ satisfy*

$$\Re\left\{\lambda\frac{D^{\delta+p-1}g(z)}{D^{\delta+p}g(z)}\right\} \geq \mu(\delta+p)|R(z)|, \quad \left(k > \frac{2^{2(1-\alpha)}}{|h'(0)|}\right).$$

*If $f \in \mathcal{A}_p$ satisfies*

$$(1-\lambda)\left[\frac{D^{\delta+p-1}f(z)}{D^{\delta+p-1}g(z)}\right]^{\mu} + \lambda\frac{D^{\delta+p}f(z)}{D^{\delta+p}g(z)}\left[\frac{D^{\delta+p-1}f(z)}{D^{\delta+p-1}g(z)}\right]^{\mu-1} + R(z) \prec h(z),$$

*then*

$$\left[\frac{D^{\delta+p-1}f(z)}{D^{\delta+p-1}g(z)}\right]^{\mu} \prec h(z).$$

*Proof.* Let the function $p(z)$ be defined by

$$p(z) := \left[\frac{D^{\delta+p-1}f(z)}{D^{\delta+p-1}g(z)}\right]^{\mu}.$$

Then a computation shows that the following subordination holds:

$$B(z)zp'(z) + p(z) + R(z) \prec h(z),$$

where

$$B(z) := \frac{\lambda}{\mu(\delta+p)}\frac{D^{\delta+p-1}g(z)}{D^{\delta+p}g(z)}.$$

The result follows by an application of Corollary 2.2. □

When $R(z) = 0$ and $\mu = 1$, the Theorem 4.3 reduces to Juneja and Ponnusamy [7, Theorem 1, p. 289].

**Theorem 4.4.** *Let $\alpha$ be a complex number $\Re\alpha > 0$ and $\beta < 1$. Let $M, N, R$ be analytic in $\Delta$ with $R(0) = 0$ and*

$$M(z) := z^n + c_1 z^{n+k} + \cdots, \quad N(z) := z^n + d_1 z^{n+k} + \cdots.$$

*Let*

$$\Re\frac{\alpha N(z)}{zN'(z)} > \delta + \frac{2+\delta k}{(1-\beta)k}\Re R(z).$$

*If*

$$\Re[\alpha\frac{M'(z)}{N'(z)} + (1-\alpha)\frac{M(z)}{N(z)} + R(z)] > \beta, \tag{4.2}$$

*then*

$$\Re\frac{M(z)}{N(z)} > \frac{2\beta + k\delta}{2 + k\delta}.$$

*Proof.* Let $p(z) := M(z)/N(z)$. Then $p(0) = 1 = h(0)$. It follows that

$$p(z) + \frac{N(z)}{zN'(z)}zp'(z) = \frac{M'(z)}{N'(z)}.$$

Then

$$\Re p(z) + \frac{\alpha N(z)}{zN'(z)}zp'(z) + R(z) = \Re[\alpha\frac{M'(z)}{N'(z)} + (1-\alpha)\frac{M(z)}{N(z)} + R(z)] > \beta.$$

If $B(z)$ is defined by $B(z) := \alpha N(z)/[zN'(z)]$, then it follows that

$$\Re B(z) > \delta + \frac{2+\delta k}{(1-\beta)k}\Re R(z).$$

The result now follows by an application of Corollary 3.3 □

**Remark 4.2.** *For $R(z) = 0$, $\beta = 0$, Theorem 4.4 is due to Karunakaran and Ponnusamy [6, Theorem B, p. 562].*

**Theorem 4.5.** *Let $\delta > -p$ be any real number, $\lambda \in \mathbb{C}$ with $\Re\lambda \geq 0$. Let $R(z)$ be a function defined on $\Delta$ with $R(0) = 0$, $0 \leq \alpha < 1$. Let $g \in \mathcal{A}_p$ satisfies*

$$\Re\left\{\lambda\frac{D^{\delta+p-1}g(z)}{D^{\delta+p}g(z)}\right\} > \mu(\delta+p)\delta + \frac{\mu(\delta+p)(2+\delta)}{1-\alpha}\Re R(z) \geq 0.$$

*If $f \in \mathcal{A}_p$ satisfies*

$$\Re\left\{(1-\lambda)\left[\frac{D^{\delta+p-1}f(z)}{D^{\delta+p-1}g(z)}\right]^{\mu} + \lambda\frac{D^{\delta+p}f(z)}{D^{\delta+p}g(z)}\left[\frac{D^{\delta+p-1}f(z)}{D^{\delta+p-1}g(z)}\right]^{\mu-1} + R(z)\right\} > \alpha,$$

*then*

$$\left[\frac{D^{\delta+p-1}f(z)}{D^{\delta+p-1}g(z)}\right]^{\mu} \geq \frac{2\alpha+\delta}{2+\delta}.$$

*Proof.* Let

$$p(z) := \left[\frac{D^{\delta+p-1}f(z)}{D^{\delta+p-1}g(z)}\right]^{\mu}.$$

Then a computation shows that

$$\Re\{B(z)zp'(z) + p(z) + R(z)\} > \alpha,$$

where

$$B(z) := \frac{\lambda}{\mu(\delta + p)} \frac{D^{\delta+p-1}g(z)}{D^{\delta+p}g(z)}.$$

The result follows easily. □

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