PROPERTIES OF NON POWERFUL NUMBERS

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Abstract: In this paper we study some properties of non powerful numbers. We evaluate the n-th non powerful number and prove for the sequence of non powerful numbers some theorems that are related to the sequence of primes: Landau, Mandl, Scherk. Related to the conjecture of Goldbach, we prove that every positive integer ≥ 3 is the sum between a prime and a non powerful number.

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1. Introduction

A positive integer v is called non powerful if there exists a prime p such that $p|v$ and $p^2 \nmid v$.

Otherwise, if v has the canonical decomposition $v = q_1^{\alpha_1} \cdot \dots \cdot q_r^{\alpha_r}$, there exists $j \in \{1, 2, \dots, r\}$ such that $\alpha_j = 1$.

It results that v can be written uniquely as $v = f \cdot u$, where f is squarefree, u is powerful and $(f, u) = 1$.

In this paper we use the following notations:

- $K(x)$= the number of powerful numbers less than or equal to x
- $C(x)$= the number of non powerful numbers less than or equal to x
- v_n is the n-th non powerful number

We use a special case of a classical formula:

Theorem A. *If $h \in C^1$, g is continuous, a is powerful and*

$$G(x) = \sum_{\substack{a \le v \le x \\ v \text{ non powerful}}} g(v),$$

then

$$\sum_{\substack{a \le v \le x \\ v \text{ non powerful}}} h(v)g(v) = h(x)G(x) - \int_a^x h'(t)G(t)dt.$$

G. Mincu and L. Panaitopol proved [5] the following.

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Theorem B.

$$K(x) \geq c\sqrt{x} - 1.83522\sqrt[3]{x} \quad \textit{for } x \geq 961$$

and

$$K(x) \leq c\sqrt{x} - 1.207684\sqrt[3]{x} \quad \textit{for } x \geq 4.$$

As $C(x) = [x] - K(x)$ it results that

$$[x] - c\sqrt{x} + 1.207684\sqrt[3]{x} \leq C(x) \leq [x] - c\sqrt{x} + 1.83522\sqrt[3]{x} \tag{1.1}$$

the first inequality being true for $x \geq 4$, while the second one is true for $x \geq 961$.

We also use

Theorem C. *We have the relation*

$$K(x) = \frac{\zeta(3/2)}{\zeta(3)}\sqrt{x} + \frac{\zeta(2/3)}{\zeta(2)}\sqrt[3]{x} + O\left(x^{\frac{1}{6}}\exp(-c_1 \log^{\frac{3}{5}}(\log\log x)^{-\frac{1}{5}}\right).$$

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2. Inequalities for v_n

Theorem 2.1. *We have the relation*

$$v_n > n + c\sqrt{n} - a\sqrt[3]{n} \quad \textit{for } n \geq 88,$$

where $a = 1.83522$.

Proof. If we put $x = v_n$ in the second inequality from (1.1), it results that

$$n \leq v_n - c\sqrt{v_n} + a\sqrt[3]{v_n}$$

for $n \geq 4$.

Let $f(x) = x - c\sqrt{x} + a\sqrt[3]{x} - n$ and $x'_n = n + c\sqrt{n} - k\sqrt[3]{n}$. As $f(v_n) > 0$, and f is increasing, if we prove that $f(x'_n) < 0$, it results that $v_n > x'_n$.

Denote $g(n) = f(x'_n)$. Proving that $f(x'_n) < 0$ is equivalent with proving that $g(n) < 0$. Therefore we intend to prove that

$$g(n) = c\sqrt{n} - k\sqrt[3]{n} - c\sqrt{n + c\sqrt{n} - k\sqrt[3]{n}} + a\sqrt[3]{n + c\sqrt{n} - k\sqrt[3]{n}} < 0.$$

We use the following relations for $x > 0$:

$$1 + \frac{x}{2} > \sqrt{1+x} > 1 + \frac{x}{2} - \frac{x^2}{8} \tag{2.1}$$

and

$$1 + \frac{x}{3} > \sqrt[3]{1+x} > 1 + \frac{x}{3} - \frac{x^2}{9}. \tag{2.2}$$

Putting $x = x'_n$ in (2.1) gives

$$\sqrt{n} + \frac{c}{2} - \frac{k}{2\sqrt[6]{n}} > \sqrt{n + c\sqrt{n} - k\sqrt[3]{n}} > \sqrt{n} + \frac{c}{2} - \frac{k}{2\sqrt[6]{n}} - \frac{\sqrt{n}}{8}\left(\frac{c}{\sqrt{n}} - \frac{k}{\sqrt[3]{n^2}}\right)^2,$$

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while $x = x'_n$ gives from (2.2)

$$\sqrt[3]{n}+\frac{c}{3\sqrt[6]{n}}-\frac{k}{3\sqrt[3]{n}} > \sqrt[3]{n+c\sqrt{n}-k\sqrt[3]{n}} > \sqrt[3]{n}+\frac{c}{3\sqrt[6]{n}}-\frac{k}{3\sqrt[3]{n}}-\frac{\sqrt[3]{n}}{9}\left(\frac{c}{\sqrt{n}}-\frac{k}{\sqrt[3]{n^2}}\right)^2.$$

Using the previous relations in the expression of $g(n)$ yields

$$g(n) < c\sqrt{n}-k\sqrt[3]{n}-c\sqrt{n}-\frac{c^2}{2}+\frac{ck}{2\sqrt[6]{n}}+\frac{c\sqrt{n}}{8}\left(\frac{c}{\sqrt{n}}-\frac{k}{\sqrt[3]{n^2}}\right)^2+a\sqrt[3]{n}+\frac{ac}{3\sqrt[6]{n}}-\frac{ak}{3\sqrt[3]{n}}.$$

In order to prove $g(n) > 0$ it is enough to prove that

$$(a-k)\sqrt[3]{n}-\frac{c^2}{2}+\left(\frac{ck}{2}+\frac{ac}{3}\right)\frac{1}{\sqrt[6]{n}}-\frac{ak}{3\sqrt[3]{n}}+\frac{c\sqrt{n}}{8}\left(\frac{c}{\sqrt{n}}-\frac{k}{\sqrt[3]{n^2}}\right)^2 < 0.$$

The best result is obtained by taking $k = a$, therefore

$$-\frac{c^2}{2}+\frac{5}{6}\frac{ac}{\sqrt[6]{n}}-\frac{a^2}{3\sqrt[3]{n}}+\frac{c\sqrt{n}}{8}\left(\frac{c}{\sqrt{n}}-\frac{a}{\sqrt[3]{n^2}}\right)^2 < 0.$$

As $\frac{c}{\sqrt{n}} > \frac{a}{\sqrt[3]{n^2}}$ for $n \geq 1$, it is enough to prove that

$$\frac{5ac}{6\sqrt[6]{n}}+\frac{c\sqrt{n}}{8}\cdot\frac{c^2}{n} < \frac{c^2}{2}+\frac{a^2}{3\sqrt[3]{n}}.$$

The last relation is true because

$$\frac{c^3}{8\sqrt{n}} < \frac{a^2}{3\sqrt[3]{n}} \Leftrightarrow \left(\frac{3c^3}{8a^2}\right)^6 < n \quad \text{that holds for } n \geq 816$$

and

$$\frac{5ac}{6\sqrt[6]{n}} < \frac{c^2}{2} \Leftrightarrow \left(\frac{5a}{3c}\right)^6 < n \quad \text{that holds for } n \geq 8.$$

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In conclusion, we have

$$v_n > n + c\sqrt{n} - a\sqrt[3]{n}$$

for $n \geq 816$. Verifications done using the computer allow us to lower the bound to $n \geq 88$. $\square$

Theorem 2.2. *We have the relation*

$$v_n < n + c\sqrt{n} - \sqrt[3]{n} \quad \textit{for } n \geq 1.$$

Proof. If we put $x = v_n$ in the first inequality from (1.1), it results that

$$n > v_n - c\sqrt{v_n} + \alpha\sqrt[3]{v_n},$$

where $\alpha = 1.207684$.

Let $f(x) = x - c\sqrt{x} + \alpha\sqrt[3]{x} - n$ and $x_n'' = n + c\sqrt{n} - h\sqrt[3]{n}$. We have $f(v_n) < 0$, f is increasing, so if we prove that $f(x_n'') > 0$, it results that $v_n < x_n''$.

Denote $g(n) = f(x_n'')$. Proving that $f(x_n'') > 0$ is equivalent to proving that $g(n) > 0$. Therefore we have to prove that

$$g(n) = n + c\sqrt{n} - h\sqrt[3]{n} - c\sqrt{n + c\sqrt{n} - h\sqrt[3]{n}} + \alpha\sqrt[3]{n + c\sqrt{n} - h\sqrt[3]{n}} - n < 0.$$

Using the relations (2.1) and (2.2) as we did in the proof of Theorem 2.1, gives

$$c\sqrt{n} - h\sqrt[3]{n} - c\sqrt{n} - \frac{c^2}{2} + \frac{ch}{2\sqrt[6]{n}} + \alpha\sqrt[3]{n} + \frac{\alpha c}{3\sqrt[6]{n}} - \frac{\alpha h}{3\sqrt[3]{n}} - \frac{\alpha\sqrt[3]{n}}{9}\left(\frac{c}{\sqrt{n}} - \frac{h}{\sqrt[3]{n^2}}\right)^2 > 0.$$

The previous relation is equivalent to

$$\sqrt[3]{n}(\alpha - h) + \left(\frac{ch}{2} + \frac{\alpha c}{3}\right)\frac{1}{\sqrt[6]{n}} > \frac{c^2}{2} + \frac{\alpha\sqrt[3]{n}}{9}\left(\frac{c}{\sqrt{n}} - \frac{h}{\sqrt[3]{n^2}}\right)^2 + \frac{\alpha h}{3\sqrt[3]{n}}.$$

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Thus, it is enough to prove that for $h < \alpha$

$$\sqrt[3]{n}(\alpha - h) + \frac{c}{\sqrt[6]{n}}\left(\frac{h}{2} + \frac{\alpha}{3}\right) > \frac{c^2}{2} + \frac{\alpha h}{3\sqrt[3]{n}} + \frac{\alpha\sqrt[3]{n}}{9} \cdot \frac{c^2}{n}.$$

We have $\sqrt[3]{n}(\alpha - h) > \frac{c^2}{2}$, if

$$n > \left(\frac{c^2}{2(\alpha - h)}\right)^3. \tag{2.3}$$

It remains to prove that

$$\frac{c}{\sqrt[6]{n}}\left(\frac{h}{2} + \frac{\alpha}{3}\right) > \frac{\alpha h}{3\sqrt[3]{n}} + \frac{\alpha c^2}{9\sqrt[3]{n^2}}.$$

Therefore it is enough to prove that $c\frac{h}{2} > \frac{\alpha h}{3\sqrt[6]{n}}$ and that $c\frac{\alpha}{3} > \frac{\alpha c^2}{9\sqrt{n}}$; both the relations are true for $n \geq 1$.

In conclusion, the condition (2.3) gives the lower bound for realizing the inequality from Theorem 2.2: we take $h = 1$ so $n > 1471$. Verification using the computer allows us to take $n \geq 1$. □

Theorem 2.3. *There exists* $c_2 > 0$ *such that*

$$v_n = n + \frac{\zeta(3/2)}{\zeta(3)}\sqrt{n} + \frac{\zeta(2/3)}{\zeta(2)}\sqrt[3]{n} + O\left(\exp(-c_2 \log^{\frac{3}{5}} n (\log\log n)^{-\frac{1}{5}}\right).$$

Proof. We have $C(x) = [x] - K(x)$, and put $x = v_n$. It results that $n = v_n - K(v_n)$; we use Theorem C to evaluate K, and obtain

$$n = v_n - c\sqrt{v_n} - b\sqrt[3]{v_n} + O\left(n^{\frac{1}{6}} g(n)\right),$$

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where $c=\zeta(3/2)/\zeta(3)$, $b=\zeta(2/3)/\zeta(2)$ and $g(n)=\exp\left(-c_2(\log n)^{\frac{3}{5}}(\log\log n)^{\frac{-1}{5}}\right)$ with $c_2>0$ and $g(n)\to\infty$ as $n\to\infty$.

So

$$-n+v_n-c\sqrt{v_n}-b\sqrt[3]{v_n}=O\left(n^{\frac{1}{6}}g(n)\right). \tag{2.4}$$

From Theorem 2.1 and 2.2 we have

$$n+c\sqrt{n}-1.83522\sqrt[3]{n}<v_n<n+c\sqrt{n}-\sqrt[3]{n},$$

therefore

$$v_n=n+c\sqrt{n}-x_n\sqrt[3]{n},\qquad \text{with } (x_n)_{n\geq 1} \text{ bounded.} \tag{2.5}$$

It is known that

$$\sqrt{1+x}=1+\frac{x}{2}-\frac{x^2}{8}+\cdots$$

and

$$\sqrt[3]{1+x}=1+\frac{x}{3}-\frac{x^2}{9}+\cdots.$$

Therefore

$$\sqrt{v_n}=\sqrt{n}\left(\sqrt{1+\frac{c}{\sqrt{n}}-\frac{x_n}{\sqrt[3]{n^2}}}\right)$$
$$=\sqrt{n}\left(1+\frac{c}{2\sqrt{n}}-\frac{x_n}{2\sqrt[3]{n^2}}-\frac{1}{8}\left(\frac{c}{\sqrt{n}}-\frac{x_n}{\sqrt[3]{n^2}}\right)^2+\cdots\right),$$

so

$$\sqrt{v_n}=\sqrt{n}+\frac{c}{2}-\frac{x_n}{2\sqrt[6]{n}}-\frac{\sqrt{n}}{8}\left(\frac{c}{\sqrt{n}}-\frac{x_n}{\sqrt[3]{n^2}}\right)^2+\cdots. \tag{2.6}$$

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In a similar manner, we get

$$\sqrt[3]{v_n} = \sqrt[3]{n} + \frac{c}{3\sqrt[6]{n}} - \frac{x_n}{3\sqrt[3]{n}} - \frac{\sqrt[3]{n}}{9}\left(\frac{c}{\sqrt{n}} - \frac{x_n}{\sqrt[3]{n^2}}\right)^2 + \cdots . \tag{2.7}$$

From (2.4), (2.6) and (2.7) it results that

$$c\sqrt{n} - x_n\sqrt[3]{n} - c\sqrt{n} - \frac{c^2}{2} + \frac{cx_n}{2\sqrt[6]{n}} + \frac{c\sqrt{n}}{8}\left(\frac{c}{\sqrt{n}} - \frac{x_n}{\sqrt[3]{n^2}}\right)^2$$
$$- b\sqrt[3]{n} - \frac{bc}{3\sqrt[6]{n}} + \frac{bx_n}{3\sqrt[3]{n}} + \frac{b\sqrt[3]{n}}{9}\left(\frac{c}{\sqrt{n}} - \frac{x_n}{\sqrt[3]{n^2}}\right)^2 + \cdots = O\left(n^{\frac{1}{6}}g(n)\right).$$

Therefore

$$-\sqrt[3]{n}(x_n + b) = O\left(n^{\frac{1}{6}}g(n)\right),$$

which yields

$$x_n = -b + O\left(\frac{g(n)}{\sqrt[6]{n}}\right). \tag{2.8}$$

From (2.5) and (2.8) we obtain

$$v_n = n + c\sqrt{n} + b\sqrt[3]{n} + O\left(g(n)\sqrt[3]{n}\right).$$

In conclusion, there exists $c_2 > 0$ such that

$$v_n = n + c\sqrt{n} + b\sqrt[3]{n} + O\left(\exp(-c_2 \log^{\frac{3}{5}} n (\log\log n)^{\frac{1}{5}})\right).$$

□

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3. Some Properties of the Sequence of Non Powerful Numbers

In relation to the prime number distribution function, E. Landau [4] proved in 1909 that

$$\pi(2x) < 2\pi(x) \quad \text{for } x \geq x_0.$$

Afterwards J.B. Rosser and L. Schoenfeld proved [6] that

$$\pi(2x) < 2\pi(x) \text{ for all } x > 2.$$

In relation to this problem we can state the following result.

Theorem 3.1. *We have the relation*

$$C(2x) \geq 2C(x) \quad \textit{for all integers } x \geq 7. \tag{3.1}$$

Proof. Using Theorem B we obtain:

$$[x] - c\sqrt{x} + 1.207684\sqrt[3]{x} \leq C(x) \leq [x] - c\sqrt{x} + 1.83522\sqrt[3]{x},$$

for $x \geq 961$.

In order to prove (3.1) it is therefore sufficient to show that

$$[2x] - c\sqrt{2x} + 1.207864\sqrt[3]{2x} \geq 2[x] - 2c\sqrt{x} + 3.67044\sqrt[3]{x}.$$

As $[2x] \geq 2[x]$, it is sufficient to show that

$$c\sqrt{x}(2 - \sqrt{2}) > 2.14885307\sqrt[3]{x},$$

which is true if $\sqrt[6]{x} \geq 1.687939$, more precisely for $x \geq 24$. Verifications done using the computer show that Theorem 3.1 is true for every integer $8 \leq x \leq 961$, which concludes our proof. □

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Remark 1. From Theorem 3.1 it follows that $v_{n+1} < 2v_n$ for every $n \geq 1$.

The Mandl inequality [2] states that, for $n \geq 9$

$$p_1 + p_2 + \ldots + p_n < \frac{1}{2} n p_n,$$

where p_n is the n-the prime.

Related to this inequality, we prove that for non powerful numbers

Theorem 3.2. *We have for* $n \geq 7$ *that*

$$v_1 + v_2 + \ldots + v_n > \frac{1}{2} n v_n. \tag{3.2}$$

Proof. Let $n > C(961) + 1 = 912$. In order to evaluate the sum $\sum_{i=1}^{n} v_i$, we use Theorem A with $h(t) = t$, $g(t) = 1$ and $a = 961$. It follows that $G(x) = C(x) - C(961)$ and then we obtain

$$\sum_{i=C(961)+1}^{n} v_i = v_n(n - C(961)) - \int_{961}^{v_n} (C(t) - C(961))dt.$$

Then

$$\sum_{i=1}^{n} v_i = \sum_{i=1}^{C(961)} v_i + n v_n - v_n C(961) - \int_{961}^{v_n} C(t)dt + C(961)(v_n - 961).$$

Using Theorem B, we get a better upper bound for $k'(x)$, namely

$$k'(x) \leq x - c\sqrt{x} + 1.83522\sqrt[3]{x} \text{ for } x \geq 961.$$

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Therefore, it is enough to prove that

$$\sum_{i=1}^{C(961)} v_i + nv_n - 961C(961) - \int_{961}^{v_n} \left(t + c\sqrt{t} + 1.83522\sqrt[3]{t}\right) dt > \frac{nv_n}{2}.$$

Integrating and making some further numerical calculus ($C(961) = 911$, $\sum_{i=1}^{911} v_i = 445213$) lead us to

$$v_n \left(\frac{n}{2} - \frac{v_n}{2} + \frac{2c}{3}\sqrt{v_n} - \frac{3}{4} \cdot 1.83522\sqrt[3]{v_n} \right) > -463153.9136.$$

So, in order to prove (3.2), it is enough to prove that

$$\frac{n}{2} - \frac{v_n}{2} + \frac{2c}{3}\sqrt{v_n} - \frac{3}{4} \cdot 1.83522\sqrt[3]{v_n} > 0.$$

This is equivalent with proving that

$$v_n < n + \frac{4c}{3}\sqrt{v_n} - \frac{3}{2} \cdot 1.83522\sqrt[3]{v_n}.$$

Taking into account Theorem 2.2 and the fact that for $n > C(961) + 1$ we have $n < v_n < 2n$, it is enough to prove that

$$n + c\sqrt{n} - \sqrt[3]{n} < n + \frac{4c}{3}\sqrt{n} - \frac{3}{2} \cdot 1.83522 \cdot \sqrt[3]{2} \cdot \sqrt[3]{n},$$

which is true for $n \geq 1565$.

Verifications done with the computer, lead us to state that the theorem is true for every $n \geq 1$, excepting the case $n = 7$. □

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The well known conjecture of Goldbach states that every even number is the sum of two odd primes. Related to this problem, Chen Jing-Run has shown [1] using the Large Sieve, that all large enough even numbers are the sum of a prime and the product of at most two primes.

We present a weaker result, that has the advantage that is easily obtained and the proof is true for every integer $n \geq 3$.

Theorem 3.3. *Every integer $n \geq 3$ is the sum between a prime and a non powerful number.*

Proof. Let $n \geq 3$ and p_i the largest prime that does not exceed n. Thus $p_i < n \leq p_{i+1}$ and

$$i = \begin{cases} \pi(n) - 1, \text{ if n is prime,} \\ \pi(n), \text{ otherwise} \end{cases}$$

Then we consider the numbers $n - p_1, n - p_2, \ldots, n - p_i$. We prove that one of these i numbers is non powerful.

Suppose that all these i numbers are powerful. It results that

$$c\sqrt{n-2} \geq k(n-2) \geq i \geq \pi(n) - 1.$$

Taking into account that $\pi(x) > \frac{x}{\log x}$ for $x \geq 59$, we obtain

$$c\sqrt{n-2} \geq \frac{n}{\log n} - 1 \text{ for } n \geq 59.$$

For $n \geq 4$ we have $c\sqrt{n-2} > 2\sqrt{n} - 1$, therefore it is enough to prove that

$$2\sqrt{n} \geq \frac{n}{\log n}.$$

But for $n \geq 75$ we have $2 \log n < \sqrt{n}$.

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Therefore the supposition we made (that $n - p_1, n - p_2, \dots, n - p_i$ are all powerful) is certainly false for $n \geq 75$ and it results that every integer greater than 75 is the sum between a prime and a non powerful number. Direct computation leads us to state that every integer $n \geq 3$ is the sum between a prime and a non powerful number. □

In 1830, H. F. Scherk found that

$$p_{2n} = 1 \pm p_1 \pm p_2 \pm \dots \pm p_{2n-2} + p_{2n-1}$$

and

$$p_{2n+1} = 1 \pm p_1 \pm p_2 \pm \cdots \pm p_{2n-1} + 2p_{2n}.$$

The proof of these relations was first given by S. Pillai in 1928. W. Sierpinski gave a proof of Scherk's formulae in 1952, [7].

In relation to Scherk's formulae, we have the following.

Theorem 3.4. *For $n \geq 6$, we have*

$$v_n = \pm\varepsilon_n \pm v_1 \pm v_2 \pm \dots \pm v_{n-2} + v_{n-1}$$

where ε_n is 0 or 1.

Proof. Following the method Sierpinski used in [7], we make an induction proof of this theorem.

If $n = 6$, we have $v_6 = 10$ and

$$\begin{aligned} 1 &= \quad -2 - 3 + 5 - 6 + 7, \\ 2 &= 1 - 2 - 3 + 5 - 6 + 7, \\ 3 &= \quad -2 - 3 - 5 + 6 + 7, \end{aligned}$$

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$$\begin{aligned}4 &= 1-2-3-5+6+7,\\ 5 &= \quad\;\; 2-3+5-6+7,\\ 6 &= 1+2-3+5-6+7,\\ 7 &= \quad\;\; 2-3-5+6+7,\\ 8 &= 1+2-3-5+6+7,\\ 9 &= \quad -2+3-5+6+7,\\ 10 &= 1-2+3-5+6+7.\end{aligned}$$

Therefore every natural number less than or equal to 10 can be expressed in the desired form.

We suppose the theorem is true for n and prove it for $n+1$.

Let k be a positive integer less than or equal to v_{n+1}. Then, because $v_{i+1} < 2v_i$ for every natural number i, we have

$$k \le v_{n+1} < 2v_n,$$

so

$$-v_n < k - v_n < v_n.$$

It follows that $0 \le \pm(k - v_n) < v_n$; we can apply the induction hypothesis and write $\pm(k-v_n) = \pm\varepsilon_n \pm v_1 \pm v_2 \pm \ldots \pm v_{n-2} + v_{n-1}$. It will immediately follow that there exist a choice of the signs $+$ and $-$ such that

$$k = \pm\varepsilon_n \pm v_1 \pm v_2 \pm \ldots \pm v_{n-1} + v_n.$$

As $v_n \le v_{n+1}$, we get

$$v_n = \pm\varepsilon_n \pm v_1 \pm v_2 \pm \ldots \pm v_{n-2} + v_{n-1}.$$

□

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